RATES OF GROWTH IN A CLASS OF HOMOGENEOUS MULTIDIMENSIONAL MARKOV CHAINS

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Abstract

We investigate the asymptotic behaviour of homogeneous multidimensional Markov chains whose states have non-negative integer components. We obtain growth rates for these models in a situation similar to the near-critical case for branching processes, provided they converge to infinity with positive probability. Finally, the general theoretical results are applied to a class of controlled multitype branching processes where random control is allowed.

1. Introduction

One of the problems that is approached in the scientific literature on branching processes is the study of the growth rate of certain biological (human, animal, cell, ...) or physical (particle, cosmic ray, ...) populations. In the simplest models, such as the Bienaymé-Galton-Watson process, only geometric growth is possible when extinction does not occur. The classical non-decomposable multitype Galton-Watson process somewhat inherits this dual behaviour of the one-dimensional model. Nonetheless, in some homogeneous modifications of these processes non-exponential rates of growth are also possible, particularly in the case known as critical or near-critical.

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In the present work, we deal with the problem of determining the rate of growth for a class of processes more general than (homogeneous) branching processes: homogeneous multidimensional Markov chains in discrete time, taking values in the space of vectors with non-negative integer components. The aim is to investigate what conditions must be imposed on such models in order to obtain non-geometric rates of growth, provided there exists a positive probability of convergence to infinity. A detailed study of the indefinite growth of these chains has been considered in [5], whereas conditions for their geometric growth can be found in [4].

We shall try to maintain the branching process and population dynamics perspective, and shall use their special terminology. An entire section of the paper will deal with the controlled multitype branching processes with random control, a topic which has not previously been investigated.

Mathematically, we consider an $m$-dimensional homogeneous Markov chain $\{Z(n)\}_{n\geq 0}$, whose states have non-negative integer components, i.e. $S \subseteq \mathbb{N}_0^m$ where $S$ is the set of states, which we call by the acronym HMMC. This chain can model the evolution of a population of $m$ different types of coexisting individuals. More specifically, the $i$th component of $Z(n)$ might represent the number of $i$-type individuals $n$ generations after the process was started. The event explosion of the chain, denoted by $\mathcal{D}_\infty := \{\|Z(n)\| \to \infty\}$, with $\| \cdot \|$ an arbitrary norm on $\mathbb{R}^m$, will play a fundamental role in our study and must be assumed to have positive probability.

In Section 2 of the paper, we investigate the limiting behaviour of some sequences of linear functionals associated to HMMCs. After providing conditions for the event $\mathcal{D}_\infty$ to have positive probability, we show that, under certain conditions, they can be normalized on the explosion set by a sequence of constants with the same order as $\{n^\alpha\}_{n\geq 0}$ for some $\alpha > 0$. In Section 3 we come back to the $m$-dimensional process $\{Z(n)\}_{n\geq 0}$ and prove that it is also possible to find the same growth rate for such a process, again on the explosion set. Finally, in Section 4, we apply the results of Sections 2 and 3 to a class of controlled multitype branching processes.

As indicated in the previous paragraph, for each $\mu \in \mathbb{R}_+^m$ we will consider, associated to the chain $\{Z(n)\}_{n\geq 0}$, the sequence of linear functionals $\{Z(n)\mu\}_{n\geq 0}$. This process is not a Markov chain, but it has some remarkable properties. Indeed $\mathcal{D}_\infty = \{Z(n)\mu \to \infty\}$, so that the explosion of the chain is equivalent to the unlimited growth of the
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sequence of functionals. In relation with this sequence of linear functionals we can introduce the variables \( \xi_n^\mu, n \geq 0 \), and the functions \( g_\mu(z) \) and \( \sigma_\mu^2(z) \), defined for every non-null vector \( z \in \mathbb{N}_0^m \) by:

\[
\begin{align*}
\xi_{n+1}^\mu & := Z(n+1)\mu - E[Z(n+1)\mu|Z(n)], \\
g_\mu(z) & := E[Z(n+1)\mu|Z(n) = z] - z\mu, \\
\sigma_\mu^2(z) & := \text{Var}[Z(n+1)\mu|Z(n) = z].
\end{align*}
\]

Notice that they depend on the choice of the vector \( \mu \), although in the rest of the paper, whenever there is no chance of ambiguity, we shall drop the use of \( \mu \) in the notation and write \( \xi_n, g(z) \) and \( \sigma^2(z) \).

In order to determine non-geometric growth, we will consider vectors \( \mu \in \mathbb{R}_+^m \), such that

\[
\lim_{\|z\| \to \infty} \frac{g(z)}{z\mu} = 0,
\]

which can be interpreted as that the mean growth rate of the process \( \{Z(n)\mu\}_{n \geq 0} \), i.e., \( (z\mu)^{-1}E[Z(n+1)\mu|Z(n) = z] \), is close to unity for \( \|z\| \) large enough. Notice that (2) is an assumption on the Markov chain \( \{Z(n)\}_{n \geq 0} \). This situation corresponds to the critical or near-critical case in branching processes.

2. Asymptotic behaviour of \( \{Z(n)\mu\}_{n \geq 0} \)

In this section, we search for sequences of constants with non-geometric growth that suitably normalize the sequence of linear functionals \( \{Z(n)\mu\}_{n \geq 0} \), with \( \mu \in \mathbb{R}_+^m \), on the set \( D_\infty \). First we provide conditions to guarantee that \( \mathbb{P}[D_\infty] > 0 \), as can be found in [5].

**Theorem 2.1.** Let \( z^{(0)} \in S \) be a vector such that

for every \( C > 0 \) there exists \( n \geq 1 \) such that \( \mathbb{P}[Z(n)1 > C|Z(0) = z^{(0)}] > 0 \),

with \( 1 \) being the \( m \)-dimensional vector with all its components equal to unity. Assume that there exists a vector \( \mu \in \mathbb{R}_+^m \) for which

\[
\lim_{\|z\| \to \infty} \frac{g(z)}{z\mu} = 0 \quad \text{and} \quad \liminf_{\|z\| \to \infty} \frac{2(z\mu)g(z)}{\sigma^2(z)} > 1.
\]
Suppose further that for some $\delta$, $0 < \delta \leq 1$, and $\gamma > 0$, the following equality holds:

$$E \left[ \left| \xi_{n+1} \right|^{2+\delta} | Z(n) = z \right] = o \left( (z\mu)^{2+\delta} / (\log(z\mu))^{1+\gamma} \right).$$

Then $P[\|Z(n)\| \to \infty | Z(0) = z^{(0)}] > 0$.

**Remark 2.1.** Under condition (2), [5] obtained that $P[D_\infty] = 0$ if

$$\limsup_{\|z\| \to \infty} \frac{2(z\mu)g(z)}{\sigma^2(z)} < 1,$$

and for some $0 < \delta < 1$, it is satisfied that

$$E \left[ \left| \xi_{n+1} \right|^{2+\delta} | Z(n) = z \right] = o \left( (z\mu)^{1+\delta} g(z) \right).$$

**Remark 2.2.** For the classical multitype branching process with irreducible matrix of means it is satisfied that $(z\mu)^{-1} g(z) = \rho - 1$ for every non-null $z \in \mathbb{N}_0^m$, with $\rho$ being the Perron-Frobenius eigenvalue associated to the matrix of means and $\mu \in \mathbb{R}^m_+$ a right eigenvector associated to $\rho$ (see [14]). In this case, $P[D_\infty] > 0$ if and only if $\rho > 1$.

Moreover $\{Z(n)\}_{n \geq 0}$ presents geometric growth on $D_\infty$ with rate $\rho$ (see [13]). Notice that, for this process, condition (2) (critical case) implies $P[D_\infty] = 0$.

Let us now consider the sequence $\{Z(n)\}_{n \geq 0}$, with $\mu \in \mathbb{R}_+^m$. Suppose that there exist positive real functions $g(x)$ and $\sigma^2(x)$ such that $g(z) = \mathcal{g}(z\mu)$ and $\sigma^2(z) = \mathcal{g}^2(z\mu)$ for every vector $z \in \mathbb{N}_0^m$, satisfying the following assumptions:

(A1) $\mathcal{g}(x) = cx^\alpha + o(x^\alpha)$ for all $x > 0$ and some $\alpha < 1$ and $c > 0$.

(A2) $\mathcal{g}^2(x) = vx^\beta + o(x^\beta)$ for all $x > 0$ and some $\beta \leq 1 + \alpha$ and $v > 0$.

(A3) $E[|\xi_{n+1}|^{2+\delta} | Z(n) = z] = O(\sigma^{2+\delta}(z))$ for some $0 < \delta \leq 1$.

Since $\alpha < 1$, assumption (A1) implies that condition (2) holds. For mathematical reasons, we consider $\mathcal{g}(x)$ to be twice continuously differentiable and $\mathcal{g}^2(x)$ continuously differentiable.

**Remark 2.3.** If (A1)–(A3) are satisfied and $z^{(0)} \in S$ is a vector such that (3) holds, then $D_\infty$ has positive probability.

Indeed, since

$$\liminf_{\|z\| \to \infty} \frac{2(z\mu)g(z)}{\sigma^2(z)} = \liminf_{\|z\| \to \infty} \frac{2c(z\mu)^{1+\alpha-\beta}(1 + o(1))}{v + o(1)} = \begin{cases} \frac{2c}{v} & \text{if } \beta = 1 + \alpha, \\ \infty & \text{if } \beta < 1 + \alpha, \end{cases}$$
For the notation introduced in (1), we decompose the process as the following stochastic difference equation:

$$Z(n + 1) = Z(n) + g(Z(n)) + \xi_{n+1} = Z(n) + \mathcal{g}(Z(n)) + \xi_{n+1}, \quad n \geq 0 \quad (4)$$

and

$$E[|\xi_{n+1}|^{2+\delta} | Z(n) = z] (\log(z \mu))^{1+\gamma} = O((z \mu)^{(\beta-2)\delta/2} (\log(z \mu))^{1+\gamma}),$$

for some constants $\delta, \gamma > 0$, from Theorem 2.1 we deduce that $P[\mathcal{D}_\infty | Z(0) = z^{(0)}] > 0$ if either $\beta < 1 + \alpha$ or $\beta = 1 + \alpha$ and $v < 2c$.

The next result summarizes the asymptotic behaviour of the process $\{Z(n)\}_{n \geq 0}$ under conditions (A1)–(A3) and assuming that $\mathcal{D}_\infty$ has positive probability. We denote by $\{a_n\}_{n \geq 0}$ the solution of the difference equation:

$$a_0 = 1, \quad a_{n+1} = a_n + \mathcal{g}(a_n), \quad n \geq 0.$$ 

It is a matter of straightforward computation to verify that the sequence $\{a_n\}_{n \geq 0}$ is asymptotically equivalent to $(1 - \alpha)cn^{1/(1-\alpha)}$.

**Theorem 2.2.** Assume (A1)–(A3) and $P[\mathcal{D}_\infty] > 0$. Then

(a) If $\beta = 1 + \alpha$ and $v < 2c$, for all $x \in \mathbb{R}$,

$$\lim_{n \to \infty} P \left[ \frac{Z(n) \mu^{1-\alpha}}{n} \leq x | \mathcal{D}_\infty \right] = \Gamma_{a,b}(x),$$

where $\Gamma_{a,b}(x)$ denotes the gamma distribution function with parameters $a := (v(1-\alpha))^{-1}(2c-v\alpha)$, $b := 2^{-1}v(1-\alpha)^2$.

(b) If $0 < \alpha < 1$, $\beta < \alpha + 1$, we have that

(b.1) For $\beta < 3\alpha - 1$, on $\mathcal{D}_\infty$, $a_n^{-1}Z(n)\mu$ converges almost surely and in $L^1$ to 1 and $\mathcal{g}(a_n)^{-1}(Z(n)\mu - a_n)$ converges almost surely.

(b.2) For $\beta \geq 3\alpha - 1$, on $\mathcal{D}_\infty$, $a_n^{-1}Z(n)\mu$ converges to 1 in $L^1$ and for all $x \in \mathbb{R}$

$$\lim_{n \to \infty} P \left[ \Delta_n^{1/2} \frac{Z(n)\mu - a_n}{\mathcal{g}(a_n)} \leq x | \mathcal{D}_\infty \right] = \Phi(x),$$

with $\Phi(x)$ being the standard normal distribution function and

$$\Delta_n := \begin{cases} \frac{v}{e(1-\alpha) \log n} & \text{if } \beta = 3\alpha - 1 \\ \frac{v}{\beta - 3\alpha + 1} e^{\frac{1-\alpha}{\beta-3\alpha+1} (1-\alpha) n^{\frac{\beta-2}{1-\alpha}}} & \text{if } \beta > 3\alpha - 1. \end{cases}$$

**Proof.** With the notation introduced in (1), we decompose the process $\{Z(n)\mu\}_{n \geq 0}$ as the following stochastic difference equation:

$$Z(n + 1) = Z(n) + g(Z(n)) + \xi_{n+1} = Z(n) + \mathcal{g}(Z(n)) + \xi_{n+1}, \quad n \geq 0 \quad (4)$$

and
Let us define the function $G(x)$

$$G(x) := \int_1^x \frac{dy}{\overline{g}(y)},$$

and check that the assumptions of Theorem 1 in [11] are fulfilled, namely

$$\lim_{x \to \infty} \frac{\overline{g}'(x)G(x)}{\overline{g}(x)G(x)} = \frac{v(1-\alpha)}{1 - \alpha - \frac{1}{3}}$$

and

$$\lim_{x \to \infty} \frac{\overline{g}'(x)G(x)}{\overline{g}(x)G(x)} = \frac{v(1-\alpha)}{c}.$$

But this is immediate given that conditions (A1)–(A3) hold with $\beta = 1 + \alpha$ and the equivalence

$$G(x) \sim c(1-\alpha)^{-1}x^{1-\alpha}.$$  \hspace{1cm} (5)

Then, from a direct application of Theorem 1 in [11], we obtain

$$\lim_{n \to \infty} P\left[\frac{G(Z(n)\mu)}{n} \leq x|D_\infty\right] = \Gamma_{a,b}(x),$$

with $a = (v(1-\alpha))^{-1}(2c - va)$ and $b := (2c)^{-1}v(1-\alpha)$. Now, applying again (5) and Slutsky’s Theorem, the proof of (a) is complete.

We must here introduce some additional notation needed for the proof of (b). Let us rewrite (4) as:

$$Z(n+1)\mu = Z(n)\mu + \overline{g}(Z(n)\mu)(1 + \eta_{n+1}), \hspace{0.5cm} n \geq 0,$$  \hspace{1cm} (6)

where $\eta_{n+1} := \xi_{n+1}/\overline{g}(Z(n)\mu)$, at least on $\{g(Z(n)) \neq 0\}$. Defining the function $\overline{g}'(z) := \overline{g}(z)\overline{g}'(z)$, it is immediate that $\overline{g}'(x) \sim vc^{-2}x^{3-2\alpha}$. Also

$$E[\eta_{n+1}|Z(n) = z] = 0, \hspace{0.5cm} E[\eta_{n+1}^2|Z(n) = z] = \overline{g}'(z\mu),$$

and, from (A3),

$$E[|\eta_{n+1}|^2+4|Z(n) = z] = O(\overline{g}'^2(z\mu)).$$

Defining also the function

$$\psi(x) := \int_1^x \frac{\overline{g}'(y)}{\overline{g}(y)} dy,$$

one easily derives from (A1) and (A2) that $\beta \geq 3\alpha - 1$ is equivalent to $\psi(\infty) = \infty$, and more specifically

$$\psi(x) \sim \begin{cases} \frac{v}{c^\alpha} \frac{1}{\beta - 3\alpha + 1} x^{\beta - 3\alpha + 1} & \text{if } \beta > 3\alpha - 1 \\ v \log x & \text{if } \beta = 3\alpha - 1. \end{cases}$$  \hspace{1cm} (7)

Now, since (b) is a direct consequence of Theorem 3 in [10], we only need to check the hypotheses of that theorem:
(A) From condition (2), \( g(x) = o(x) \). Moreover, since \( 0 < \alpha < 1 \), \( g(x) \) is ultimately concave and \( g'(x) \) ultimately convex.

(B) From (5), \( \tilde{G}(x) \sim (c(1 - \alpha)x)^{1/(1 - \alpha)} \), \( \tilde{G} \) being the inverse function of \( G \). Therefore, since \( \beta < 1 + \alpha \), \((\tilde{G}' \circ \tilde{G})(x)\) is ultimately concave and 

\[
\lim_{t \to \infty} \int_1^t x^{-2}(\tilde{G}' \circ \tilde{G})(x)dx = \lim_{t \to \infty} \frac{\mathcal{V}}{c^2}(c(1 - \alpha))^{\frac{2}{1-\alpha}} \int_1^t x^{\frac{2-2}{1-\alpha}} dx < \infty.
\]

(C) The function \(|g''(x)g(x)\phi^{-2}(x)|\) is equivalent to a positive multiple of \( x^{4\alpha - \beta - 2} \), so that it is ultimately decreasing if \( \beta \geq 3\alpha - 1 \) or equivalently if \( \psi(\infty) = \infty \). Also, if \( \psi(\infty) < \infty \), then \(|g''(x)g(x)|\) is equivalent to a positive multiple of \( x^{2(\alpha - 1)} \) and, since \( \alpha < 1 \), is ultimately decreasing.

(D) Taking into account that \( g'(x) \sim c\alpha x^{\alpha - 1} \) and (7), \( g'(x)\phi^{1/2}(x) = o(1) \).

Let \( \tilde{\psi}(x) := \psi \circ \tilde{G}(x) \).

If, on the one hand, \( \beta < 3\alpha - 1 \) then \( \tilde{\psi}(\infty) < \infty \), and we obtain (b.1) by applying Theorem 3(a) in [10] since \( \{Z(n)\mu\}_{n \geq 0} \) is a sequence of non-negative random variables.

If, on the other hand, \( \beta \geq 3\alpha - 1 \) then

\[
\tilde{\psi}(x) \sim \begin{cases} 
\frac{\mathcal{V}}{c^2(1 - \alpha)} \frac{x^{\beta - 2}}{\beta - 3\alpha + 1} ((1 - \alpha)x)^{\frac{\beta - 3\alpha + 1}{1-\alpha}} & \text{if } \beta > 3\alpha - 1 \\
\frac{\mathcal{V}}{c^2(1 - \alpha)} \log x & \text{if } \beta = 3\alpha - 1,
\end{cases}
\]

and consequently \( \tilde{\psi}(\infty) = \infty \). Then, applying Theorem 3(b) of [10] we derive

\[
\lim_{n \to \infty} P \left[ \hat{\psi}(n)^{-1/2} (\frac{Z(n)\mu - a_n}{\mathcal{G}(a_n)}) \leq x | D_\infty \right] = \Phi(x),
\]

and, using again (8) and Slutsky’s Theorem, (b.2) is proven.

**Remark 2.4.** Figure 1 shows, depending on the values of \( \alpha \) and \( \beta \), the different kinds of limiting behaviour that Theorem 2.2 predicts for the process \( \{Z(n)\mu\}_{n \geq 0} \) suitably normalized. Notice that if either \( \beta = 1 + \alpha \) and \( 2c < \mathcal{V} \) or \( \beta > 1 + \alpha \) then \( P[D_\infty] = 0 \) (see Remark 2.1).
Remark 2.5. If \( \mu \in \mathbb{R}_m^+ \) satisfies the assumptions (A1)–(A3), so does any other vector \( \pi \in \mathbb{R}_m^+ \) proportional to \( \mu \), and consequently Theorem 2.2 remains true for the sequence of linear functionals \( \{Z(n)\pi\}_{n \geq 0} \) with the parameters of the limit distributions replaced by those corresponding to the vector \( \pi \).

In order to establish the next result, which is very important from a practical point of view, we require also

\[
P[Z(n) \to 0] + P[\|Z(n)\| \to \infty] = 1,
\]

with 0 being the null state. This behaviour, typical of some homogeneous branching processes, is known as the extinction-explosion duality, i.e., almost surely, either the population becomes extinct or the total number of individuals grows indefinitely. Using Markov chain theory (see [2]), it is easy to verify that if the null state is absorbing and every non-null state is transient then the chain satisfies (9). We observe that the condition \( P[Z(1) = 0 | Z(0) = z] > 0 \) for all \( z \in \mathbb{N}_0^m \) is sufficient for every non-null state to be transient, obviously under the consideration of 0 being absorbing.

Proposition 2.1. Assume (A1)–(A3), \( 0 < P[\mathcal{D}_\infty] < 1 \), and equation (9). Then

(a) Under the conditions of Theorem 2.2(a), the following statements hold for all \( x \in \mathbb{R} \):

(a.1)

\[
\lim_{n \to \infty} P \left[ \frac{(Z(n)\mu)^{1-\alpha}}{\overline{G}_{n}} \leq x \right] = P[Z(n) \to 0]I_{\mathbb{R}_+}(x) + P[\mathcal{D}_\infty]\Gamma_{a,b}(x),
\]
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(a.2) 
\[
\lim_{n \to \infty} P \left[ \frac{(Z(n) \mu)^{1-\alpha}}{n} \leq x \|Z(n)\| > 0 \right] = \Gamma_{a,b}(x),
\]
with \( a = (v(1-\alpha))^{-1}(2c-v\alpha) \) and \( b = 2^{-1}v(1-\alpha)^2 \).

(b) Under the conditions of Theorem 2.2(b.2), the following statements hold for all \( x \in \mathbb{R} \):

(b.1) 
\[
\lim_{n \to \infty} P \left[ \Delta^{-1} \frac{(Z(n) \mu - a_n)^2}{\sigma^2(a_n)} \leq x \right] = P[D_\infty | \chi^2_1(x)],
\]
(b.2) 
\[
\lim_{n \to \infty} P \left[ \Delta^{-1/2} \frac{Z(n) \mu - a_n}{\sigma(a_n)} \leq x \|Z(n)\| > 0 \right] = \Phi(x),
\]
with \( \chi^2_1(x) \) being the chi-squared distribution function with 1 degree of freedom.

**Proof.** Since \( 0 < P[D_\infty] < 1 \) and (9) holds, we deduce that \( P[Z(n) \to 0] > 0 \), which implies that the null state is absorbing. Indeed, since

\[
P[Z(n) \to 0] = \lim_{n \to \infty} P \left[ \bigcap_{k=n}^{\infty} \{Z(k) = 0\} \right],
\]
and for each \( n \geq 1 \)

\[
P \left[ \bigcap_{k=n}^{\infty} \{Z(k) = 0\} \right] = \lim_{s \to \infty} P \left[ \bigcap_{k=n}^{n+s} \{Z(k) = 0\} \right] = P[Z(n) = 0] \lim_{s \to \infty} p_{00},
\]
with \( p_{00} := P[Z(1) = 0|Z(0) = 0] \), then \( p_{00} = 1 \), and therefore \( 0 \) is an absorbing state.

Taking this into account, let us now prove the result.

(a) For simplicity, define \( Y(n) := n^{-1}(Z(n) \mu)^{1-\alpha}, n \geq 1 \). Since for all \( x > 0 \),

\[
\lim_{n \to \infty} P[Y(n) \leq x|Z(r) \to 0] = 1,
\]
we deduce that

\[
\lim_{n \to \infty} P[Y(n) \leq x] = P[Z(r) \to 0] + P[D_\infty] \lim_{n \to \infty} P[Y(n) \leq x|D_\infty], \tag{10}
\]
and, applying Theorem 2.2(a), the proof of (a.1) is completed.

Also, since \( P[D_\infty] = \lim_{n \to \infty} P[\|Z(n)\| > 0] \),

\[
\lim_{n \to \infty} P[\|Z(n)\| > 0, Z(r) \to 0] = 0,
\]
and

\[
P[Y(n) \leq x, \|Z(n)\| > 0, D_\infty] = P[Y(n) \leq x, D_\infty],
\]
we deduce, for all \( x \in \mathbb{R} \), that
\[
\lim_{n \to \infty} P[Y(n) \leq x\|Z(n)\| > 0] = \lim_{n \to \infty} \frac{P[Y(n) \leq x|D_{\infty}]P[D_{\infty}]}{P[\|Z(n)\| > 0]},
\]
and, again using Theorem 2.2(a), we obtain (a.2).

(b) In order to prove (b.1), define
\[
Y(n) := \Delta_n^{-1/2} \frac{Z(n)\mu - a_n}{g(a_n)},
\]
and use a decomposition similar to (10) and the fact that \( \Delta_n^{-1/2} g^{-1}(a_n)a_n \) converges to \( \infty \). Applying Theorem 2.2(b), we deduce the result.

The proof of (b.2) follows the same steps, now defining
\[
Y(n) := \left( \Delta_n^{-1/2} \frac{Z(n)\mu - a_n}{g(a_n)} \right)^2.
\]

**Remark 2.6.** Notice that the limit in (b.1) is an improper distribution function. Moreover, from part (b.2), we deduce that
\[
\lim_{n \to \infty} P\left[ \Delta_n^{-1} \frac{(Z(n)\mu - a_n)^2}{g^2(a_n)} \leq x\|Z(n)\| > 0 \right] = \chi^2_1(x),
\]
i.e., the chi-squared distribution function with 1 degree of freedom.

### 3. Asymptotic behaviour of \( \{Z(n)\}_{n \geq 0} \)

In the previous section, for every \( \mu \in \mathbb{R}^m_+ \) under assumptions (A1)–(A3), we found sequences, \( \{b_n\}_{n \geq 0} \), such that \( \{b_n^{-1}Z(n)\mu\}_{n \geq 0} \) converges to a non-null random variable \( W \) on \( D_{\infty} \), provided this set has positive probability. As a consequence, we now prove the convergence of \( \{b_n^{-1}Z(n)\}_{n \geq 0} \) to a random vector \( \tilde{W} \) concentrated in a one-dimensional subspace of \( \mathbb{R}^m \) and whose magnitude is given by \( W \).

We first need to introduce new notation. Let us impose the following condition on the transition vector of means of the chain:
\[
E[Z(n+1)|Z(n) = z] = z\tilde{M} + \tilde{h}(z), \quad z \in \mathbb{N}_0^m,
\]
where \( \tilde{M} \) is a square matrix of order \( m \) with non-negative coefficients and \( \tilde{h}(z) \) a function from \( \mathbb{R}^m \) to \( \mathbb{R}^m \) such that \( \tilde{h}_j(z) = o(\|z\|) \) for all \( j \in \{1, \ldots, m\} \). We also assume the matrix \( \tilde{M} \) to be positively regular, so that if \( \tilde{\rho} \) is its Perron-Frobenius
eigenvalue and $\tilde{\mu} \in \mathbb{R}_+^m$ one of its associated right eigenvectors (see [14]), then $g_{\tilde{\mu}}(z) = (z \tilde{\mu})(\tilde{\rho} - 1) + \tilde{h}(z)\tilde{\mu}$. Consequently (2) is equivalent to $\tilde{\rho} = 1$.

Let $\tilde{\mu}^{(1)}, \ldots, \tilde{\mu}^{(m)}$ be a basis of right eigenvectors and right generalized eigenvectors of $\tilde{M}$ such that $\tilde{\mu}^{(1)} = \tilde{\mu}$ and $\tilde{\nu} \in \mathbb{R}_+^m$ is the left eigenvector associated to $\tilde{\rho} = 1$ satisfying that $\tilde{\nu} \tilde{\mu} = 1$ and consequently $\tilde{\nu} \tilde{\mu}^{(i)} = 0$ for each $i \in \{2, \ldots, m\}$.

Finally define $G^{(i)}(z)$ for each $z \in \mathbb{N}_0^m$ and $i \in \{1, \ldots, m\}$ by

$$G^{(i)}(z) := E \left[ \left| \xi^{(i)}_{n+1} \right| \left| Z(n) = z \right| \right] = E \left[ \left| Z(n+1)\tilde{\mu}^{(i)} - E[Z(n+1)\tilde{\mu}^{(i)}] \right| \left| Z(n) = z \right| \right]. \quad (12)$$

We can now formulate the following result:

**Theorem 3.1.** Assume that equation (11) holds and it is satisfied that $E[Z(0)\tilde{\mu}] < \infty$, $P[\mathcal{D}_{\infty}] > 0$, and (A1)–(A3) for the vector $\tilde{\mu}$. Suppose further that there exist constants $\delta_1, \delta_2 < 1$ such that

(i) $|\tilde{h}(z)\tilde{\mu}^{(i)}| = O((z \tilde{\mu})^{\delta_1})$ for all $i \in \{2, \ldots, m\}$.

(ii) $G^{(i)}(z) = O((z \tilde{\mu})^{\delta_2})$ for all $i \in \{2, \ldots, m\}$.

Then,

(a) If $\beta = 1 + \alpha$ and $\nu < 2c$, then for every vector $\pi = (\pi_1, \ldots, \pi_m) \in \mathbb{R}^m$

$$\lim_{n \to \infty} P \left[ \frac{Z(n)}{n^{1/(1-\alpha)}} \leq \pi | \mathcal{D}_{\infty} \right] = F_{\pi Z}(\pi),$$

with $F_{\pi Z}(\pi)$ being the distribution function associated to the random vector $\tilde{\nu}Z$, where $Z$ is a random variable such that $Z^{1-\alpha}$ follows a gamma distribution with parameters $(\nu(1 - \alpha))^{-1}(2c - \alpha)$ and $2^{-1}v(1 - \alpha)^2$.

(b) If $0 < \alpha < 1$ and $\beta < \alpha + 1$, then, on $\mathcal{D}_{\infty}$, $n^{-1/(1-\alpha)}Z(n)$ converges in $L^1$ to $(c(1-\alpha))^{1/(1-\alpha)}\tilde{\nu}$.

Moreover, if $\beta \geq 3\alpha - 1$ and $2\max\{\delta_1, \delta_2\} < (\beta - \alpha + 1)$, then for every vector $\pi \in \mathbb{R}^m$ it is verified that

$$\lim_{n \to \infty} P \left[ \frac{Z(n) - \tilde{\nu}a_n}{\Lambda_n} \leq \pi | \mathcal{D}_{\infty} \right] = F_{\pi U}(\pi),$$
with $F_{\tilde{\nu}U}(\pi)$ being the distribution function associated to the random vector $\tilde{\nu}U$, where $U$ is a random variable with standard normal distribution and

$$
\Lambda_n := \begin{cases} 
\nu^{1/2}((1-\alpha))^{(3\alpha-1)/(2(1-\alpha))}n^{\alpha/(1-\alpha)}(\log n)^{1/2} & \text{if } \beta = 3\alpha - 1 \\
(v(\beta - 3\alpha + 1)^{-1}c\beta/(1-\alpha)((1-\alpha)n^{\beta-\alpha+1)/(1-\alpha)})^{1/2} & \text{if } \beta > 3\alpha - 1.
\end{cases}
$$

**Proof.** To prove the result, we apply a reasoning similar to that used by [12] in the context of population-size-dependent multitype branching processes.

Since $\tilde{M}$ is positively regular, the eigenvalue $\tilde{\rho} = 1$ has multiplicity one, and any other eigenvalue of $\tilde{M}$, say $r$, satisfies $|r| < \tilde{\rho}$. Suppose that $r$ is an eigenvalue with multiplicity $s \geq 1$ and with right generalized eigenvectors $\tilde{\mu}^{(i_1)}, \ldots, \tilde{\mu}^{(i_s)}$, i.e.,

$$
\tilde{M}\tilde{\mu}^{(i_1)} = r\tilde{\mu}^{(i_1)}, \quad \tilde{M}\tilde{\mu}^{(i_j)} = r\tilde{\mu}^{(i_j)} + \tilde{\mu}^{(i_{j-1})} \quad \text{for } j \in \{2, \ldots, s\}.
$$

Let us prove, by induction on $j$, that for each $j \in \{1, \ldots, s\}$

$$
\lim_{n \to \infty} \frac{Z(n)\tilde{\mu}^{(i_j)}}{n^{1/(1-\alpha)}} = 0 \quad \text{in } L^1. \quad (13)
$$

For $j = 1$, using (11), it is satisfied almost surely that

$$
Z(n + 1)\tilde{\mu}^{(i_1)} = E[Z(n + 1)\tilde{\mu}^{(i_1)}|Z(n)] + Z(n + 1)\tilde{\mu}^{(i_1)} - E[Z(n + 1)\tilde{\mu}^{(i_1)}|Z(n)]
$$

$$
= rZ(n)\tilde{\mu}^{(i_1)} + \tilde{h}(Z(n))\tilde{\mu}^{(i_1)} + Z(n + 1)\tilde{\mu}^{(i_1)} - E[Z(n + 1)\tilde{\mu}^{(i_1)}|Z(n)],
$$

and hence

$$
E[|Z(n + 1)\tilde{\mu}^{(i_1)}|] \leq |r|E[|Z(n)\tilde{\mu}^{(i_1)}|] + E[|\tilde{h}(Z(n))\tilde{\mu}^{(i_1)}|] + E|[G^{(i_1)}(Z(n))]|. \quad (14)
$$

From (i) and (ii), and taking into account that, on $D_\infty$, $n^{-1/(1-\alpha)}Z(n)\tilde{\mu}$ converges in distribution to a non-negative random variable (see Theorem 2.2), we conclude that, for some constants $\delta_1, \delta_2 < 1$,

$$
E[|\tilde{h}(Z(n))\tilde{\mu}^{(i_1)}|] = O\left(E \left[|Z(n)\tilde{\mu}|^{\delta_1}\right]\right) = O\left(n^{\delta_1/(1-\alpha)}\right),
$$

and

$$
E[|G^{(i_1)}(Z(n))|] = O\left(E \left[|Z(n)\tilde{\mu}|^{\delta_2}\right]\right) = O\left(n^{\delta_2/(1-\alpha)}\right).
$$

Then, from (14), we obtain that for all $n \geq 0$

$$
E[|Z(n + 1)\tilde{\mu}^{(i_1)}|] \leq |r|E[|Z(n)\tilde{\mu}^{(i_1)}|] + O\left(n^{\max\{\delta_1, \delta_2\}/(1-\alpha)}\right).
$$
Applying an iterative process, for all \( n \geq 0 \) it is verified that

\[
E[|Z(n)\tilde{\mu}^{(i)}|] \leq \sum_{k=1}^{n} |r|^{n-k} O \left( k^{\max\{\delta_1, \delta_2\}/(1-\alpha)} \right) + |r|^{n+1} E[|Z(0)\tilde{\mu}^{(i)}|],
\]

(15)

and therefore, since \( |r| < 1 \) and \( \max\{\delta_1, \delta_2\} < 1 \), one deduces (13) for \( j = 1 \). Also, if \( j \in \{2, \ldots, s\} \) and assuming that (13) holds for \( 1, \ldots, j - 1 \), we get, through a decomposition similar to (14),

\[
E[|Z(n+1)\tilde{\mu}^{(i)}|] \leq |r|E[|Z(n)\tilde{\mu}^{(i)}|] + E[|Z(n)\tilde{\mu}^{(j-1)}|] + E[\tilde{h}(\tilde{\mu}^{(i)})] + E[G^{(i)}(Z(n))],
\]

and, reasoning analogously to the case \( j = 1 \), we deduce (13) for \( j \).

In order to finish the proof, let us consider any vector \( \eta \in \mathbb{R}^m \) and denote by \( \eta_1, \ldots, \eta_m \in \mathbb{C} \) its components in the basis \( \tilde{\mu}^{(1)}, \ldots, \tilde{\mu}^{(m)} \). Since \( \tilde{\nu}\tilde{\mu} = 1 \) and it is verified that \( \tilde{\nu}\tilde{\mu} = 0 \) for all \( i \in \{2, \ldots, m\} \), then \( \eta_1 = \tilde{\nu}\eta \). Moreover

\[
\frac{Z(n)\eta}{n^{1/(1-\alpha)}} = \sum_{i=1}^{m} \eta_i \frac{Z(n)\tilde{\mu}^{(i)}}{n^{1/(1-\alpha)}}.
\]

(16)

If \( \alpha < 1 \) and \( \beta = 1 + \alpha \), from Theorem 2.2(a) we deduce, that for all \( x \in \mathbb{R} \),

\[
\lim_{n \to \infty} P \left[ \frac{(Z(n)\tilde{\mu})^{1-\alpha}}{n} \leq x | \mathcal{D}_\infty \right] = \Gamma_{a,b}(x),
\]

with \( a = (v(1-\alpha))^{-1}(2c - va) \) and \( b = 2^{-1}v(1-\alpha)^2 \). Hence, applying (13), (16), and Slutsky’s Theorem, we obtain that, for all \( x \in \mathbb{R} \),

\[
\lim_{n \to \infty} P \left[ \frac{Z(n)\eta}{n^{1/(1-\alpha)}} \leq x | \mathcal{D}_\infty \right] = F_Z(x/\tilde{\nu}\eta),
\]

with \( F_Z(x) \) being the distribution function of a random variable \( Z \) such that \( Z^{1-\alpha} \) follows the gamma model with parameters \( a, b \). Taking the Cramèr-Wold device into account, we conclude the proof of (a).

If \( 0 < \alpha < 1 \) and \( \beta < 1 + \alpha \), we deduce from Theorem 2.2(b) that, conditioned on \( \mathcal{D}_\infty \),

\[
\lim_{n \to \infty} \frac{Z(n)\tilde{\mu}}{a_n} = 1 \quad \text{in } L^1.
\]

Given that \( a_n \sim (c(1-\alpha)n)^{1/(1-\alpha)} \) and again applying equations (13) and (16), we obtain that, for any \( \eta \in \mathbb{R}^m \), conditioned on \( \mathcal{D}_\infty \),

\[
\lim_{n \to \infty} \frac{Z(n)\eta}{n^{1/(1-\alpha)}} = (c(1-\alpha))^{1/(1-\alpha)}\tilde{\nu}\eta \quad \text{in } L^1.
\]
and therefore the first part of (b) is proved by appropriately choosing the vectors \( \eta \).

Moreover, if \( \beta \geq 3\alpha - 1 \) we deduce from Theorem 2.2(b.2) that for all \( x \in \mathbb{R} \)
\[
\lim_{n \to \infty} P \left[ \Delta_n^{1/2} \frac{Z(n)\bar{\mu} - a_n}{\bar{g}(a_n)} \leq x|D_\infty \right] = \Phi(x). \tag{17}
\]

Since \( 2\max\{\delta_1, \delta_2\} < \beta - \alpha + 1 \), by a similar argument to that used to prove (13), we obtain for each \( i \in \{2, \ldots, m\} \)
\[
\lim_{n \to \infty} \frac{Z(n)\tilde{\mu}(i)}{n^{\beta - \alpha + 1/(2(1 - \alpha))}} = 0 \quad \text{in } L^1. \tag{18}
\]

By (16), it is satisfied for any \( \eta \in \mathbb{R}^m \) that
\[
\frac{(Z(n) - a_n\tilde{\nu})\eta}{\Lambda_n} = \tilde{\nu} \eta \left( \frac{Z(n)\bar{\mu} - a_n}{\Lambda_n} \right) + \sum_{i=2}^{m} \eta_i Z(n)\tilde{\mu}(i) \Lambda_n. \tag{19}
\]

Since \( a_n \sim ((1 - \alpha)cn)^{1/(1 - \alpha)} \) and \( \bar{g}(x) \sim cx^{\alpha} \), using equations (17), (18) and (19) we deduce the second part of (b) by applying again the Cramèr-Wold device and Slutsky’s Theorem.

**Remark 3.1.** As we indicated above, notice that the limit vector obtained has a fixed direction given by \( \tilde{\nu} \) and a random magnitude given by the limit of the sequence \( \{Z(n)\tilde{\mu}\}_{n \geq 0} \), suitably normalized.

**Remark 3.2.** Taking Remark 2.5 into consideration, we deduce that, under the assumptions of Theorem 3.1, the behaviour of an HMMC does not depend on the choice of the right eigenvector \( \tilde{\mu} \in \mathbb{R}^m_+ \).

The following result is more precise from a practical point of view. For the proof, omitted because it is similar to that of Proposition 2.1, it is necessary to again assume the chain’s dual behaviour given by equation (9).

**Corollary 3.1.** If equation (9) holds and \( 0 < P[D_\infty] < 1 \) then, under the hypotheses of Theorem 3.1 (a), it is verified for every vector \( \pi \in \mathbb{R}^m \) that:
\[
(a) \quad \lim_{n \to \infty} P \left[ \frac{Z(n)}{n^{1/(1 - \alpha)}} \leq \pi \right] = P[Z(n) \to 0|I_{R_n^+}(\pi)] + P[D_\infty]F_{\tilde{\nu}Z}(\pi),
\]
\[
(b) \quad \lim_{n \to \infty} P \left[ \frac{Z(n)}{n^{1/(1 - \alpha)}} \leq \pi \|Z(n)\| > 0 \right] = F_{\tilde{\nu}Z}(\pi),
\]
with $F_{\tilde{\nu}Z}(\pi)$ being the distribution function of the random vector $\tilde{\nu}Z$, where $Z$ is a random variable such that $Z^{1-\alpha}$ follows a gamma model with parameters $(v(1-\alpha))^{-1}(2c-v\alpha)$ and $2^{-1}v(1-\alpha)^2$.

**Remark 3.3.** If $Z$ is a random variable with distribution function $F_Z(x)$ and $\tilde{\nu} \in \mathbb{R}^m_+$, then the distribution function of the random vector $\tilde{\nu}Z$ is given by

$$F_{\tilde{\nu}Z}(\pi) = F_Z\left(\min_{1 \leq i \leq m} \{\pi_i/\tilde{\nu}_i\}\right)$$

for $\pi \in \mathbb{R}^m$.

### 4. On controlled multitype branching processes

Unlike the one-dimensional version, the controlled multitype branching process has received sparse attention in the scientific literature. Historically, the possibility of controlling the number of potential progenitors in the population was proposed by [15] deterministically in both the univariate and the multidimensional cases. Random control has been considered by [6], [7], [8], [9], and [17], for the univariate case only. In this section we shall apply the results obtained for the HMMCs to a new multitype branching model called the controlled multitype branching process with random control, in which the number of progenitors of each type is controlled by means of a random mechanism. Furthermore, dependence among the individuals of the same generation at reproduction time is allowed. This is a major novel feature with respect to the classical branching models. Mathematically, we consider a sequence of $m$-dimensional random vectors $\{Z(n)\}_{n \geq 0}$, defined recursively by:

$$Z(0) = z \in \mathbb{N}_0^m; \quad Z(n+1) = \sum_{i=1}^m \sum_{j=1}^m \phi^n_i(Z(n)) X^{i,n,j}, \quad n \geq 0,$$

where $\{X^{i,n,j} : i = 1, \ldots, m; n = 0, 1, \ldots; j = 1, 2, \ldots\}$ and $\{\phi^n(z) : n = 0, 1, \ldots; z \in \mathbb{N}_0^m\}$ are two independent sequences of $m$-dimensional, non-negative, integer-valued random vectors defined on the same probability triple $(\Omega, A, P)$, such that:

(i) The stochastic processes $\{\phi^n(z) : z \in \mathbb{N}_0^m, n = 0, 1, \ldots\}$ are independent, and for each $z \in \mathbb{N}_0^m$ the vectors $\{\phi^n(z) : n = 0, 1, \ldots\}$ are identically distributed.

(ii) The stochastic processes $\{X^{i,n,j} : i = 1, \ldots, m; j = 1, 2, \ldots\}$, $n = 0, 1, \ldots$ are independent and identically distributed. Moreover, for each $i = 1, \ldots, m$ the vectors $\{X^{i,n,j} : n = 0, 1, \ldots; j = 1, 2, \ldots\}$ are identically distributed.
The sequence \( \{Z(n)\}_{n \geq 0} \) is denominated controlled multitype branching process with random control and called by the acronym CMP.

The controlled branching processes proposed by [15] and [17] can be deduced as particular cases of the CMP. Moreover, a CMP is an HMMC and, taking into account the independence of control and reproduction, it is verified that for every \( z \in \mathbb{N}_0^m \):

\[
E[Z(n+1)|Z(n)=z] = E[\phi_0^0(z)] R_i
\]

with \( R := (r_{ij})_{1 \leq i,j \leq m} \) being the square matrix of order \( m \) with elements \( r_{ij} := E[X_{j,0,1}^{i,n}] \).

In the present study, we assume that for each type \( i = 1, \ldots, m \) there exists \( \phi_i \geq 0 \) and \( h_i(z) \) such that

\[
E[\phi_0^0(z)] = z_i \phi_i + h_i(z) \quad \text{and} \quad h_i(z) = o(||z||).
\]

Then, by equation (20), (11) holds with the matrix \( \tilde{M} \) given by the elements \( \tilde{m}_{ij} := \phi_i r_{ij} \), \( i, j = 1, \ldots, m \) and with \( \tilde{h}_j(z) := \sum_{i=1}^m h_i(z) r_{ij}, \quad j = 1, \ldots, m \). This condition means that the average number of potential progenitors of each type at a generation is proportional to the number of individuals of this type plus/minus a certain quantity of individuals that is negligible compared to the total population size. Notice that, under (21), immigration/emigration of progenitors of each type is allowed. Immigration is possible even if there are no individuals of a given type. This could not occur if \( h_i(z) = z_i o(1) \). In this case, however, we could determine \( \phi_i \) explicitly as:

\[
\phi_i = \lim_{||z|| \to \infty, z_i \neq 0} \frac{E[\phi_0^0(z)]}{z_i}.
\]

The matrix \( (\phi_i r_{ij})_{1 \leq i,j \leq m} \) is irreducible if and only if \( \phi_i \) is non-null for all \( i = 1, \ldots, m \) and the matrix \( R \) is irreducible. In this case, if \( \tilde{\rho} \) is the Perron-Frobenius eigenvalue of \( (\phi_i r_{ij})_{1 \leq i,j \leq m} \), (2) holds if and only if \( \tilde{\rho} = 1 \), taking \( \mu = \tilde{\mu} \in \mathbb{R}_+^m \) to be one of this matrix’s right eigenvectors associated to \( \tilde{\rho} \).

In order to apply the results proven in the previous section, let us bound conveniently \( E[|\xi_{n+1}|^{2+\delta}|Z(n) = z] \) and \( G^{(i)}(z) \) for \( \delta > 0, \quad i = 1, \ldots, m \) and \( z \in \mathbb{N}_0^m \). It is verified that

\[
E[|\xi_{n+1}|^{2+\delta}|Z(n) = z] = E \left[ \sum_{i=1}^m \left( \phi_i^n(z) \sum_{j=1}^m X_{i,n,j} \bar{\mu} - E[\phi_0^0(z)] E[X_{i,0,1}^{i,n}] \bar{\mu} \right) \right]^{2+\delta}.
\]
Taking into account that

\[ \sum_{j=1}^{n} X_{i,n,j}^{1} \tilde{\mu} - E[\phi_i^n(z)] E[X_{i,n,1}^{1}] \tilde{\mu} = \sum_{j=1}^{n} (X_{i,n,j}^{1} - E[X_{i,n,1}^{1}]) \tilde{\mu} + (\phi_i^n(z) - E[\phi_i^n(z)]) E[X_{i,n,1}^{1}] \tilde{\mu}, \] (22)

from the \( C_r \)-inequality we get that

\[ E[|\xi_{n+1}|^{2+\delta}|Z(n) = z] \leq A_1 \sum_{i=1}^{m} E \left[ \sum_{j=1}^{m} (X_{i,n,j}^{1} - E[X_{i,n,1}^{1}]) \tilde{\mu} \right]^{2+\delta} + A_2 \sum_{i=1}^{m} E[|\phi_i^n(z)| - E[\phi_i^n(z)]|^{2+\delta}] (E[X_{i,n,1}^{1}] \tilde{\mu})^{2+\delta}, \]

for certain constants \( A_1, A_2 > 0 \). Moreover, in the general case, i.e. when the random vectors \( X_{i,n,j}^{1}, j = 1, 2, \ldots; i = 1, \ldots, m \) are not necessarily independent for each fixed \( n \geq 0 \), it can be shown that

\[ E \left[ \sum_{j=1}^{m} (X_{i,n,j}^{1} - E[X_{i,n,1}^{1}]) \tilde{\mu} \right]^{2+\delta} \leq E[\phi_i^n(z)]^{2+\delta} E \left[ ((X_{i,n,1}^{1} - E[X_{i,n,1}^{1}]) \tilde{\mu})^{2+\delta} \right] \] (23)

If, on the other hand, these vectors are independent, then, using Marcinkiewicz-Zygmund inequality (see [1]), (23) can be bounded by

\[ E[\phi_i^n(z)]^{1+\delta/2} E \left[ ((X_{i,n,1}^{1} - E[X_{i,n,1}^{1}]) \tilde{\mu})^{2+\delta} \right]. \]

Taking the definition of \( G^{(i)}(z) \) into account, we bound it as

\[ G^{(i)}(z) \leq \sum_{i=1}^{m} \sum_{j=1}^{m} |\tilde{\mu}^{(1)}_{i,j}| \left( E \left[ \sum_{k=1}^{m} X_{j,n,k}^{1} - E[\phi_i^n(z)] r_{ij} \right] \right). \]

Under the independence assumption and proceeding in the same way as in (22), we can obtain the following bound from the von Bahr-Esseen inequality (see [16])

\[ E \left[ \sum_{k=1}^{m} X_{j,n,k}^{1} - E[\phi_i^n(z)] r_{ij} \right] \leq (2E[\phi_i^n(z)] E[|X_{j,n,1}^{1} - r_{ij}|^{\tilde{\alpha}}])^{1/\tilde{\alpha}} + (E[|\phi_i^n(z)| - E[\phi_i^n(z)]|^{\tilde{\alpha}}] E[|X_{j,n,1}^{1} - r_{ij}|^{\tilde{\alpha}}])^{1/\tilde{\alpha}}, \]

for some \( 1 \leq \tilde{\alpha} \leq 2 \).
In order to guarantee (3), it is sufficient that for every non-null \( z \in \mathbb{N}_0^m \)
\[
P[\phi_i^0(z) > z_i, X^{i,0,j} \neq 0, i \in I(z), j = 1, \ldots, \phi_i^0(z)] > 0
\] (24)
with \( I(z) = \{ i \in \{1, \ldots, m \} : z_i \neq 0 \} \). To summarize, we establish the following result for the CMP

**Corollary 4.1.** Let \( \{Z(n)\}_{n \geq 0} \) be a CMP satisfying (21) where \((\phi_i r_{ij})_{1 \leq i, j \leq m}\) is a positively regular matrix with Perron-Frobenius eigenvalue \( \tilde{\rho} = 1 \) and associated right eigenvector \( \tilde{\mu} \in \mathbb{R}_+^m \). Suppose further that (24) holds and for every non-null vector \( z \)

(i) \( h_i(z) = c_i(z\tilde{\mu})^\alpha + o((z\tilde{\mu})^\alpha) \) for each \( i = 1, \ldots, m \) and for some \( \alpha < 1 \) and \( c_i \in \mathbb{R} \) such that \( \sum_{i=1}^m \sum_{j=1}^m c_i r_{ij} \tilde{\mu}_j > 0 \).

(ii) \( \tilde{\sigma}^2(z\tilde{\mu}) := \text{Var}[Z(n+1)\tilde{\mu} | Z(n) = z] = v(z\tilde{\mu})^\beta + o((z\tilde{\mu})^\beta) \) for some \( \beta \leq 1 + \alpha \) and \( v > 0 \).

(iii) \( \max_{1 \leq i \leq m} \{ E[|\phi_i^0(z)| - E[\phi_i^0(z)]|^{2+\delta}, E[|\phi_i^0(z)|^{2+\delta}] \} = O((\tilde{\sigma}(z\tilde{\mu}))^{2+\delta}) \) for some \( 0 < \delta \leq 1 \).

(iv) \( \max_{1 \leq i \leq m} \{ |h_i(z)| \} = O((z\tilde{\mu})^{\delta_1}) \) for some \( \delta_1 < 1 \).

(v) \( \max_{1 \leq i \leq m} \left\{ E \left[ \left| \sum_{k=1}^n X_k^{i,n,k} - E[\phi_i^0(z)] r_{ij} \right| \right] \right\} = O((z\tilde{\mu})^{\delta_2}) \) for some \( \delta_2 < 1 \).

Then the statements of Theorem 3.1 hold.

**Remark 4.1.** Notice that, under the independence assumption, (iii) and (v) can be replaced by

\[
\max_{1 \leq i \leq m} \left\{ E[|\phi_i^0(z)| - E[\phi_i^0(z)]|^{2+\delta}, E[\phi_i^0(z)]^{1+\delta/2}] \right\} = O((\tilde{\sigma}(z\tilde{\mu}))^{2+\delta}),
\] (25)

and

\[
\max_{1 \leq i \leq m} \left\{ E[|\phi_i^0(z)|, E[|\phi_i^0(z)| - E[\phi_i^0(z)]|^{\delta}] \right\} = O((z\tilde{\mu})^{\tilde{\alpha} \delta}),
\] (26)

for some \( 1 < \tilde{\alpha} \leq 2 \), respectively.

**Remark 4.2.** Under the conditions of this last corollary and the dual extinction-explosion behaviour, we can apply Proposition 2.1 and Corollary 3.1 to obtain results that are more precise from a practical point of view.
As we indicated before, in order to guarantee (9), it is sufficient to check that the null state is absorbing and every non-null state is transient. For the CMP, it is easy to prove that

$$P[\phi^0(0) = 0] = 1,$$

and every non-null vector $$z \in \mathbb{N}_0^m$$ is transient if

$$P\left[ \bigcap_{i=1}^m \left( \{\phi^0(z) = 0\} \cup \{\phi^0(z) > 0, X^{i,0,j} = 0, j = 1, \ldots, \phi^0(z)\} \right) \right] > 0.$$  

By way of example, we consider a CMP with $$m = 2$$, such that the random variables $$\{X^i_{j,n,k} : i, j = 1, \ldots, m; n = 0, 1, \ldots; k = 1, 2, \ldots\}$$ are independent with mean and variance equal to 1, i.e., $$r_{ij} = \text{Var}[X^i_{j,1}] = 1$$. We also assume that the random variables $$\{\phi^n_i(z) : i = 1, \ldots, m; n = 0, 1, \ldots; z \in \mathbb{N}_0^m\}$$ are also independent and follow Poisson distributions where $$E[\phi^0_i(z)] = 0.5z_i + 1$$.

It is not hard to prove that (11) holds, with $$\tilde{\rho} = 1, \tilde{\mu} = (1, 1)$$, and $$\tilde{\nu} = (0.5, 0.5)$$. Furthermore, the conditions (i), (ii) and (iv) of the last corollary are satisfied with $$\alpha = 0, \beta = 1, c_i = 1$$, and $$v = 3$$. Also, from the properties of the Poisson distribution we get conditions (25) and (26). Therefore, applying Theorem 3.1(a) with $$c = 4$$, we deduce that for all $$x \in \mathbb{R}^2$$

$$\lim_{n \to \infty} P\left[ \frac{Z(n)}{n} \leq x | D_\infty \right] = \Gamma_{8/3,3/2}(2 \min\{x_1, x_2\}) = \Gamma_{8/3,3/4}(\min\{x_1, x_2\}).$$

To illustrate this type of behaviour, we simulated a total of 20000 processes until generation 500 of the above model with $$Z(0) = (1, 2)$$ and a reproduction law following independent Poisson marginal distribution. Figure 2 shows the empirical distribution of $$Z_1(500)/500$$ (left plot) and $$Z_2(500)/500$$ (right plot), together with the density function of the limit variable $$\Gamma_{8/3,3/4}$$ (solid line) and a kernel density estimate for the positive distribution (dotted line).
Finally, we illustrate the behaviour of the vector $Z(500)/500$. Figure 3 (left) shows the sample space. One observes the strong linear dependency given by the eigenvector $\tilde{\nu} = (0.5, 0.5)$. This is related to Figure 3 (right) which shows a histogram for $Z_1(500)/Z_2(500)$.

Figure 4 shows a kernel density estimate of the joint density function of the vector $Z(500)/500$. One observes the limiting behaviour of the process described above.
Remark 4.3. For the computer simulation, we used the language and environment for statistical computing and graphics R (“GNU S”) (see [3]).

References


