BISEXUAL BRANCHING PROCESSES WITH IMMIGRATION DEPENDENT ON THE NUMBER OF COUPLES

MANUEL MOLINA, INÉS DEL PUERTO and ALFONSO RAMOS

Department of Mathematics
University of Extremadura
06071 Badajoz, Spain
e-mail: aramos@unex.es

Abstract

In this work, we introduce a discrete-time bisexual branching process which considers mating, offspring, and immigration dependent on the number of couples in the population at the previous generation. For such a process, we establish some relationships among the probability generating functions involved in the probabilistic model and determine some stochastic monotony properties.

1. Introduction

In order to describe the probabilistic evolution of two-sex populations, where females and males coexist and form couples (female-male mating units), some bisexual branching processes have been investigated. We refer the reader to Hull [5], or Haccou et al. [4], for surveys about these

2000 Mathematics Subject Classification: 60J80.

Keywords and phrases: branching processes, bisexual processes, immigration processes, population-size dependent processes.

Research supported by the Ministerio de Ciencia y Tecnología and the FEDER through the Plan Nacional de Investigación Científica, Desarrollo e Innovación Tecnológica, Grant No. BFM 2003-06074.

Received February 27, 2006

© 2007 Pushpa Publishing House
processes. However, the range of bisexual models studied is not large enough to get an optimum mathematical modelling in some populations with sexual reproduction. It is worth noticing that in certain two-sex populations, it is reasonable to assume an individual's mating behaviour dependent on the number of their progenitor couples. It might seem conceivable that by environmental, social, or other factors, the same number of females and males gives rise to different number of couples in different generations. Furthermore, by similar reasons, the offspring and the immigration laws may be influenced for the number of reproductive couples. In an attempt to improve the mathematical description about the probabilistic evolution in such populations, in this work we introduce a general discrete-time bisexual branching process which allows immigration of females, males and couples, in each generation. Moreover, the function governing the mating, the offspring and the immigration; distributions may change depending on the number of couples in the population.

The paper is structured as follows: In Section 2, we provide its mathematical formal description and its intuitive interpretation. We also introduce and discuss some working assumptions. Section 3 is devoted to determining some relationships among the probability generating functions involved in the probabilistic model. As a consequence, relationships among the main moments are derived. Section 4 deals with the study of stochastic monotony properties.

2. The Probability Model

We introduce the bisexual process with immigration dependent on the number of couples as a stochastic model \( \{(F_n, M_n)\}_{n \geq 1} \) defined, for \( n = 0, 1, \ldots \) in the recursive form:

\[
(F_{n+1}, M_{n+1}) = \sum_{i=1}^{Z_n} (f_{n,i}(Z_n), m_{n,i}(Z_n)) + (f^I_{n+1}(Z_n), m^I_{n+1}(Z_n)),
\]

\[
Z_0 = N_0 \geq 1, \quad Z_{n+1} = L_{Z_n}(F_{n+1}, M_{n+1}) + I_{n+1}(Z_n), \tag{1}
\]

where, for \( N \in Z^+ \), \( \{(f_{n,i}(N), m_{n,i}(N))\}_{n \geq 0; i \geq 1} \). \( \{(f^I_{n}(N), m^I_{n}(N))\}_{n \geq 1} \) and
\{I_n(N)\}_{n \geq 1}$ are independent sequences. The random variables of these sequences are independent, identically distributed, non-negative, and integer-valued. \{L_N\}_{N \geq 0}$ is a sequence of non-negative real functions on $\mathbb{R}^+ \times \mathbb{R}^+$ such that each $L_N$ is assumed to be monotonic non-decreasing in each argument, integer-valued on the integers, and satisfying that $L_N(x, 0) = L_N(0, y) = 0$, $x, y \in \mathbb{R}^+$, with $\mathbb{Z}^+$ and $\mathbb{R}^+$ denoting the non-negative integers and real numbers, respectively.

The process starts with $N_0$ couples and from an intuitive viewpoint, if $Z_n = N$, then $(f_{n,i}(N), m_{n,i}(N))$ represents the number of females and males descending from the $i$th couple of the $n$th generation. Obviously $P(f_{n,i}(0) = 0, m_{n,i}(0) = 0) = 1$. Furthermore, $(f_{n+1}^I(N), m_{n+1}^I(N))$ and $I_{n+1}(N)$ denote, respectively, the number of immigrant females and males, and immigrant couples, in the generation $(n + 1)$. Their respective probability laws will be referred as the offspring, the female and mate immigration, and the couple immigration, distributions. From (1), it follows that $(F_{n+1}, M_{n+1})$ is the number of females and males in the $(n + 1)$th generation, which form $Z_{n+1}$ couples according to the mating function $L_N$. It is easy to verify that $\{(Z_{n-1}, F_n, M_n)\}_{n \geq 1}$ and $\{Z_n\}_{n \geq 0}$ are homogeneous Markov chains.

It is worth pointing out that in addition to its theoretical interest, the bisexual model introduced in this work has several practical implications in population dynamics. In particular, in phenomena concerning to inhabit or re-inhabit environments with animal species which have sexual reproduction, the probabilistic evolution of the number of females, males, and couples in the population may be described in term of this bisexual model. Indeed, the motivation behind the process introduced in (1) is the interest in developing mathematical models to describe such situations.

**Remark 2.1.** The process (1) includes several models studied in the bisexual branching process literature; see for example Daley [1], González et al. [2, 3], Molina et al. [6, 7], or Xing and Wang [8].
In order to develop some theoretical results about the bisexual process (1), we shall introduce some working assumptions on the mating functions, the offspring law and the immigration distributions.

(A1) \( \{L_N\}_{N \geq 0} \) is such that each \( L_N \) is a superadditive function namely, for \( x_i, y_i \in \mathbb{R}^+ \), \( i = 1, 2 \), \( L_N(x_1 + x_2, y_1 + y_2) \geq L_N(x_1, y_1) + L_N(x_2, y_2) \).

(A2) For \( x, y \in \mathbb{R}^+ \) fixed, \( \{L_N(x, y)\}_{N \geq 0} \) is non-decreasing.

(A3) The sequences \( \{f_{0,1}(N)\}_{N \geq 0} \) and \( \{m_{0,1}(N)\}_{N \geq 0} \) are non-decreasing.

(A4) \( f_1\left(\sum_{j=1}^{k} N_j\right) \geq \sum_{j=1}^{k} f_1(N_j) \) and \( m_1\left(\sum_{j=1}^{k} N_j\right) \geq \sum_{j=1}^{k} m_1(N_j), k = 1, 2, \ldots \).

(A5) \( I_1\left(\sum_{j=1}^{k} N_j\right) \geq \sum_{j=1}^{k} I_1(N_j), k = 1, 2, \ldots \).

**Remark 2.2.** Assumption (A1) expresses the following intuitive behaviour: \( x_1 + x_2 \) females and \( y_1 + y_2 \) males coexisting together will form a number of couples greater than or equal to the total number of couples produced from \( x_1 \) females and \( y_1 \) males, and from \( x_2 \) females and \( y_2 \) males, living separately. Superadditivity is not a serious restriction, most of the mating functions considered in the bisexual branching process theory are superadditive. Assumption (A2) represents the usual fact in many biological situations which the number of matings in certain generation depends on the number of couples in the previous one in such a way that the mating is promoted when the number of progenitor couples grows. Some sequences \( \{L_N\}_{N \geq 0} \) verifying assumptions (A1) and (A2) are for example: (a) \( L_N(x, y) = x \min\{N, y\} \); (b) \( L_N(x, y) = \min\{x, Ny\} \); or (c) \( L_N(x, y) = \min\{x, y\} \) if \( N \leq k_0 \) or \( x \min\{N, y\} \) if \( N > k_0 \), (restricted to non-negative integers) where \( k_0 \) is a fixed positive integer. Finally, assumptions (A3)-(A5) consider reproduction and immigration conducts such that the offspring per couple and the immigration of females, males, and couples, are encouraged when the number of reproductive couples grows.
3. Probability Generating Functions and Moments

In this section we shall establish some relationships among the underlying probability generating functions of the model (1) and, as a consequence, we shall derive expressions for the main moments involved in the model. To this end, only it will be necessary to use Assumption (A1).

For $n, N \in \mathbb{Z}^+$ and $s, t \in [0, 1]$ let us denote by $h_{n+1}(s, t) = E[s^{F_{n+1}}t^{M_{n+1}}]$, $\varphi_N(s, t) = E[s^{f_{0,1}(N)}t^{m_{0,1}(N)}]$ and $\varphi_N^I(s, t) = E[s^{f^I_{0,1}(N)}t^{m^I_{0,1}(N)}]$. Clearly $\varphi_0(s, t) = 1$.

**Proposition 3.1.** For $n \in \mathbb{Z}^+$ and $s, t \in [0, 1]$

$$h_{n+1}(s, t) = E[(\varphi_{Z_n}(s, t))^{Z_n} \varphi_{Z_n}^I(s, t)].$$

**Proof.** For $n \in \mathbb{Z}^+$ and $s, t \in [0, 1]$

$$h_{n+1}(s, t) = E[s^{F_{n+1}}t^{M_{n+1}}] = E[E[s^{F_{n+1}}t^{M_{n+1}} | Z_n]]$$

$$= \sum_{N=0}^{\infty} E_{Z_n=N} \left[ \sum_{i=1}^{N} f_{n,i}(N) + f_{n+1,i}(N) \sum_{i=1}^{N} m_{n,i}(N) + m_{n+1,i}(N) \right] P(Z_n = N)$$

$$= \sum_{N=0}^{\infty} \prod_{i=1}^{N} E[s^{f_{n,i}(N)}t^{m_{n,i}(N)}] E[s^{f^I_{n+1,i}(N)}t^{m^I_{n+1,i}(N)}] P(Z_n = N)$$

$$= \sum_{N=0}^{\infty} (\varphi_N(s, t))^N \varphi_N^I(s, t) P(Z_n = N)$$

$$= E[(\varphi_{Z_n}(s, t))^{Z_n} \varphi_{Z_n}^I(s, t)].$$

For $N \in \mathbb{Z}^+$, let us write

$$\mu(N) = E[(f_{0,1}(N), m_{0,1}(N))], \quad \Sigma(N) = Cov[(f_{0,1}(N), m_{0,1}(N))],$$

$$\mu^I(N) = E[(f^I_{1}(N), m^I_{1}(N))], \quad \Sigma^I(N) = Cov[(f^I_{1}(N), m^I_{1}(N))].$$
As a direct consequence of Proposition 3.1, one deduces the following expressions for the mean vector and the covariance matrix of $(F_{n+1}, M_{n+1})$, $n \in \mathbb{Z}^+$

(i) $E[(F_{n+1}, M_{n+1})] = E[Z_n^\mu(Z_n) + \mu^I(Z_n)]$

(ii) $Cov[(F_{n+1}, M_{n+1})] = E[Z_n^\Sigma(Z_n)] + V[Z_n^\mu(Z_n)^I \mu(Z_n)] + E[\Sigma^I(Z_n)]$

where $V[Z_n^\mu(Z_n)^I \mu(Z_n)] = (Var[Z_n^\mu_i(Z_n) \mu_j(Z_n)])_{i,j=1,2}$.

Let us consider the random variables

$L_{n,i}(N) = L_N(f_{n,i}(N), m_{n,i}(N))$

and

$L^I_{n}(N) = L_N(f^I_{n}(N), m^I_{n}(N))$.

Let

$\phi_N(s) = E[s^{L_{0,1}(N)}]$

$\phi_N^I(s) = E[s^{L^I_{1}(N)}]$

and

$\pi_N(s) = E[s^{I_1(N)}]$.

Also, for $n \in \mathbb{Z}^+$, denote by $g_n(s) = E[s^{Z_n}]$ and $g_n^*(s, t) = E[s^{Z_n^* t} Z_n]$, where $Z_n^* = \sum_{k=0}^n Z_k$, namely the number of couples accumulated until generation $n$. Clearly $g_0(s) = s^{N_0}$ and $g_n^*(s, t) = (st)^{N_0}$, $s, t \in [0, 1]$.

Proposition 3.2. Assume (A1). Then, for $s, t \in [0, 1]$ and $n \in \mathbb{Z}^+$

(i) $g_{n+1}(s) \leq E[(\phi_{Z_n}(s))^{Z_n} \phi_{Z_n}^I(s) \pi_{Z_n}(s)]$

(ii) $g_{n+1}^*(s, t) \leq E[s^{Z_n^*} (\phi_{Z_n}(st))^{Z_n} \phi_{Z_n}^I(st) \pi_{Z_n}(st)]$. 
Proof. (i) Using (A1), one has for \( n \in \mathbb{Z}^+ \) and \( s \in [0, 1] \)

\[
g_{n+1}(s) = E[s^{Z_{n+1}}] = E[E[s^{Z_{n+1}} | Z_n]]
\]

\[
= \sum_{N=0}^{\infty} E \left[ s^{L_N} \sum_{i=1}^{N} f_{n,i}(N)+f_{n+1}^I(N), \sum_{i=1}^{N} m_{n,i}(N)+m_{n+1}^I(N)+I_{n+1}(N) \right] P(Z_n = N)
\]

\[
\leq \sum_{N=0}^{\infty} E \left[ s^{\sum_{i=1}^{N} L_{n,i}(N)} \right] E[s^{L_{n+1}^I(N)}] \pi_N(s) P(Z_n = N)
\]

\[
= \sum_{N=0}^{\infty} (\phi_N(s))^N \phi_N(s) \pi_N(s) P(Z_n = N) = E[(\phi_{Z_n}(s))^{Z_n} \phi_{Z_n}^I(s) \pi_{Z_n}(s)].
\]

(ii) Using again (A1), one deduces for \( n \in \mathbb{Z}^+ \) and \( s, t \in [0, 1] \)

\[
E[(st)^{Z_{n+1}} | Z_n] \leq (\phi_{Z_n}(st))^{Z_n} \phi_{Z_n}^I(st) \pi_{Z_n}(st) \text{ a.s.}
\]

Hence, denoting by \( F_n = \sigma(Z_0, ..., Z_n) \), one obtains

\[
g_{n+1}^{*}(s, t) = E[s^{Z_{n+1}^{*} t^{Z_{n+1}}}] = E[E[s^{Z_{n+1}} (st) | F_n]]
\]

\[
= E[s^{Z_n} E[(st)^{Z_{n+1}} | Z_n]] \leq E[s^{Z_n} (\phi_{Z_n}(st))^{Z_n} \phi_{Z_n}^I(st) \pi_{Z_n}(st)].
\]

By using of Proposition 3.2, one derives that

\[
E[Z_{n+1}] \geq E[Z_n \lambda(Z_n) + \lambda^I(Z_n) + \rho(Z_n)], \quad n \in \mathbb{Z}^+,
\]

where, for \( N \in \mathbb{Z}^+ \), \( \lambda(N) = E[L_{0,1}(N)] \), \( \lambda^I(N) = E[L^I_1(N)] \), and \( \rho(N) = E[I_1(N)] \).

Proposition 3.3. Assume (A1). If \( L_{0,1}(N) \geq L_{0,1}(1), \ L^I_1(N) \geq L_1(0) \), and \( I_1(N) \geq I_1(0) \), \( N = 1, 2, ..., \) then, for \( s, t \in [0, 1] \) and \( n \in \mathbb{Z}^+ \)

(i) \( g_{n+1}(s) \leq g_n(\phi_1(s)) \phi^I_0(s) \pi_0(s) \).

(ii) \( g_{n+1}^{*}(s, t) \leq g_n^{*}(s, \phi_1(st)) \phi^I_0(st) \pi_0(st) \).
Proof. (i) Note that, from the hypotheses, one deduces that \( \phi_N(s) \leq \phi_1(s), \phi_N^I(s) \leq \phi_0^I(s) \), and \( \pi_N(s) \leq \pi_0(s) \), \( s \in [0, 1] \). Hence, taking into account Proposition 3.2 (i) and the fact that \( \phi_0(s) = 1 \), one obtains for \( n \in \mathbb{Z}^+ \) and \( s, t \in [0, 1] \)

\[
\mathbb{g}_{n+1}(s) \leq \sum_{N=0}^{\infty} (\phi_N(s))^N \phi_N^I(s) \pi_N(s) P(Z_n = N) \\
= \sum_{N=1}^{\infty} (\phi_N(s))^N \phi_N^I(s) \pi_N(s) P(Z_n = N) + \phi_0^I(s) \pi_0(s) P(Z_n = 0) \\
\leq \phi_0^I(s) \pi_0(s) \sum_{N=0}^{\infty} (\phi_1(s))^N P(Z_n = N) = \phi_0^I(s) \pi_0(s) g_n(\phi_1(s)).
\]

(ii) It is proved in a similar manner using the fact that

\[
\mathbb{g}_{n+1}^*(s, t) \leq \sum_{N=0}^{\infty} s^{Z_{n-1}^*+N}(\phi_N(st))^N \phi_N^I(st) \pi_N(st) P(Z_n = N).
\]

As a consequence of Proposition 3.3 and (2), it is matter of straightforward calculation to determine that, for \( n \in \mathbb{Z}^+ \)

(i) \( E[Z_{n+1}] \geq B_n(\lambda(1)) \) if \( \lambda(1) = 1 \) or \( B_n^2(\lambda(1)) \) if \( \lambda(1) \neq 1 \), where

\[
B_n^1(a) = N_0 + (\lambda^I(0) + \rho(0))(n + 1)
\]

and

\[
B_n^2(a) = N_0 a^{n+1} + (\lambda^I(0) + \rho(0))(1 - a^{n+1})(1 - a)^{-1}.
\]

(ii) \( E[Z_{n+1}^*] \geq C_n(\lambda(1)) \) if \( \lambda(1) = 1 \) or \( C_n^2(\lambda(1)) \) if \( \lambda(1) \neq 1 \), where

\[
C_n^1(a) = N_0(n + 2) + \frac{1}{2}(\lambda^I(0) + \rho(0))(n + 1)(n + 2)
\]

and

\[
C_n^2(a) = \frac{N_0(1 - a^{n+2})(1 - a) + (\lambda^I(0) + \rho(0))(1 - a)(n + 1) - a(1 - a^{n+1})}{(1 - a)^2}.
\]

We recall that \( \lambda(1) = E[L_{0,1}(1)], \lambda^I(0) = E[L^I(0)] \) and \( \rho(0) = E[I_1(0)] \).
4. Stochastic Monotony

In this section we shall provide some results about the stochastic monotony of the sequences \( \{Z_n\}_{n \geq 0} \), \( \{F_n\}_{n \geq 1} \), and \( \{M_n\}_{n \geq 1} \). To this end, we shall use assumptions (A1)-(A5). The first result establishes that \( \{Z_n\}_{n \geq 0} \) is a stochastically monotone sequence.

**Proposition 4.1.** Assume (A2)-(A5). Then, given \( N_1, N_2 \in \mathbb{Z}^+ \) with \( N_1 < N_2 \),

\[
P(Z_{n+1} \leq y | Z_n = N_2) \leq P(Z_{n+1} \leq y | Z_n = N_1), \quad y \in \mathbb{R}, \ n \in \mathbb{Z}^+.
\]

**Proof.** Using (A2), the fact that \( L_{N_2} \) is monotonic non-decreasing, and (A3)-(A5), one deduces for \( y \in \mathbb{R} \)

\[
P(Z_{n+1} > y | Z_n = N_2)
\]

\[
= P\left( L_{N_2} \left( \sum_{i=1}^{N_2} f_{n,i}(N_2) + f_{n+1}(N_2) + m_{n,i}(N_2) + m_{n+1}(N_2) \right) + I_{n+1}(N_2) > y \right)
\]

\[
\geq P\left( L_{N_1} \left( \sum_{i=1}^{N_1} f_{n,i}(N_2) + f_{n+1}(N_2) + m_{n,i}(N_2) + m_{n+1}(N_2) \right) + I_{n+1}(N_2) > y \right)
\]

\[
\geq P\left( L_{N_1} \left( \sum_{i=1}^{N_1} f_{n,i}(N_1) + f_{n+1}(N_1) + m_{n,i}(N_1) + m_{n+1}(N_1) \right) + I_{n+1}(N_1) > y \right)
\]

\[
= P(Z_{n+1} > y | Z_n = N_1).
\]

**Remark 4.1.** Note that assumptions (A4) and (A5) in Proposition 4.1 may be relaxed requiring that the sequences \( \{f^I_1(N)\}_{N \geq 0} \), \( \{m^I_1(N)\}_{N \geq 0} \) and \( \{I_1(N)\}_{N \geq 0} \) are non-decreasing.

Let \( \{(F^{(i)}_n, M^{(i)}_n)_{n \geq 1}\}_{i \geq 1} \) and \( \{(Z^{(i)}_n)_{n \geq 0}\}_{i \geq 1} \), be with \( Z^{(i)}_0 = 1, \ i = 1, 2, ..., \)

independent versions of \( \{(F_n, M_n)_{n \geq 1}\} \) and \( \{Z_n\}_{n \geq 0} \), respectively.
Proposition 4.2. Assume (A1)-(A5). Then, for \( n, k \in \mathbb{Z}^+ \) and \( y \in \mathbb{R} \)

(i) \[
P(Z_{k+n+1} \leq y) \leq P \left( \sum_{i=1}^{Z_k} Z_{n+1}^{(i)} \leq y \right)
\]

(ii) \[
P(F_{k+n+1} \leq y) \leq P \left( \sum_{i=1}^{Z_k} F_{n+1}^{(i)} \leq y \right)
\]

and

\[
P(M_{k+n+1} \leq y) \leq P \left( \sum_{i=1}^{Z_k} M_{n+1}^{(i)} \leq y \right).
\]

Proof. (i) We shall use the simplified notation \( F_n(N) = \sum_{j=1}^{N} f_{n,j}(N) \)

+ \( f_{n+1}^I(N) \) and \( M_n(N) = \sum_{j=1}^{N} m_{n,j}(N) + m_{n+1}^I(N) \). First, we shall prove that for \( n, k, N \in \mathbb{Z}^+ \)

\[
P(Z_{k+n+1} \leq y | Z_{n+k} = N) \leq P \left( \sum_{i=1}^{Z_k} Z_{n+1}^{(i)} \leq y \left| \sum_{i=1}^{Z_k} Z_n^{(i)} = N \right. \right).
\]  

(3)

In fact, by using of (A2), (A1), and (A3)-(A5), in this order, one deduces

\[
P \left( \sum_{i=1}^{Z_k} Z_{n+1}^{(i)} \leq y \left| \sum_{i=1}^{Z_k} Z_n^{(i)} = N \right. \right)
\]

\[
= P \left( \sum_{i=1}^{Z_k} (L_{Z_n^{(i)}}(F_n(Z_n^{(i)}), M_n(Z_n^{(i)})) + I_{n+1}^{(i)}(Z_n^{(i)})) \leq y \left| \sum_{i=1}^{Z_k} Z_n^{(i)} = N \right. \right)
\]

\[
\geq P \left( \sum_{i=1}^{Z_k} (L_{Z_n^{(i)}}(F_n(Z_n^{(i)}), M_n(Z_n^{(i)})) + I_{n+1}^{(i)}(Z_n^{(i)})) \leq y \left| \sum_{i=1}^{Z_k} Z_n^{(i)} = N \right. \right)
\]
\[ \geq P\left( \sum_{i=1}^{Z_n} F_n(Z^{(i)}_n), \sum_{i=1}^{Z_n} M_n(Z^{(i)}_n) \right) \]
\[ + \sum_{i=1}^{Z_n} I_{n+1}^{(i)}(Z^{(i)}_n) \leq y \left| \sum_{i=1}^{Z_n} Z^{(i)}_n = N \right. \]
\[ \geq P\left( L_{N}(F_{n+k}(N), M_{n+k}(N)) + I_{n+k+1}(N) \leq y \right) \]
\[ = P(Z_{n+k+1} \leq y \mid Z_{n+k} = N). \]

We now prove the result by induction on \( n \). For \( n = 0 \), using that \( Z_0^{(i)} = 1 \), one obtains that
\[ P(Z_k \leq y) = P\left( \sum_{i=1}^{Z_k} Z_0^{(i)} \leq y \right). \]
Suppose that
\[ P(Z_{n+k} \leq y) \leq P\left( \sum_{i=1}^{Z_k} Z_n^{(i)} \leq y \right). \]
Then, considering that \( \{ P(Z_{n+k+1} \leq y \mid Z_{n+k} = N) \}_{N \geq 0} \) is non-increasing, the induction hypothesis, (3) and Lemma A1 (see Appendix), one obtains
\[ P(Z_{n+k+1} \leq y) = \sum_{N=0}^{\infty} P(Z_{n+k+1} \leq y \mid Z_{n+k} = N)P(Z_{n+k} = N) \]
\[ \leq \sum_{N=0}^{\infty} P(Z_{n+k+1} \leq y \mid Z_{n+k} = N)P\left( \sum_{i=1}^{Z_k} Z_n^{(i)} = N \right) \]
\[ \leq \sum_{N=0}^{\infty} P \left( \sum_{i=1}^{Z_k} Z_n^{(i)} \leq y \left| \sum_{i=1}^{Z_k} Z_n^{(i)} = N \right. \right) P\left( \sum_{i=1}^{Z_k} Z_n^{(i)} = N \right) \]
\[ = P \left( \sum_{i=1}^{Z_k} Z_n^{(i)} \leq y \right). \]

(ii) Note that, for \( n, k, N \in \mathbb{Z}^+ \)
\[ P(F_{k+n+1} \leq y \mid Z_{n+k} = N) \leq P\left( \sum_{i=1}^{Z_k} F_{n+1}^{(i)} \leq y \left| \sum_{i=1}^{Z_k} Z_n^{(i)} = N \right. \right). \] (4)
In fact, by using of (A4) and (A3), one has that

\[
P \left( \sum_{i=1}^{Z_k} F_{n+1}^{(i)} \leq y \left| \sum_{i=1}^{Z_k} Z_n^{(i)} = N \right. \right) 
= P \left( \sum_{i=1}^{Z_k} Z_n^{(i)} + \sum_{i=1}^{Z_k} f_{n+1}^I (Z_n^{(i)}) \leq y \left| \sum_{i=1}^{Z_k} Z_n^{(i)} = N \right. \right) 
\geq P \left( \sum_{i=1}^{N} f_{n+1}^I (Z_n^{(i)}) \leq y \right) 
= P \left( \sum_{i=1}^{N} f_{n+1}^I (N) \leq y \right) 
= P(F_{k+n+1} \leq y | Z_{n+k} = N) = P(F_{k+n+1} \leq y | Z_{n+k} = N). 
\]

Now, considering (i), the fact that \( \{P(F_{k+n+1} \leq y | Z_{n+k} = N)\}_{N \geq 0} \) is non-increasing, Lemma A1, and (4), one follows that

\[
P(F_{k+n+1} \leq y) = \sum_{N=0}^{\infty} P(F_{k+n+1} \leq y | Z_{n+k} = N) P(Z_{n+k} = N) 
\leq \sum_{N=0}^{\infty} P(F_{k+n+1} \leq y | Z_{n+k} = N) \left( \sum_{i=1}^{Z_k} Z_n^{(i)} = N \right) 
\leq \sum_{N=0}^{\infty} \left( \sum_{i=1}^{Z_k} F_{n+1}^{(i)} \leq y \left| \sum_{i=1}^{Z_k} Z_n^{(i)} = N \right. \right) P \left( \sum_{i=1}^{Z_k} Z_n^{(i)} = N \right) 
= P \left( \sum_{i=1}^{Z_k} F_{n+1}^{(i)} \leq y \right) .
\]

Analogously it is proved that \( P(M_{k+n+1} \leq y) \leq P \left( \sum_{i=1}^{Z_k} M_{n+1}^{(i)} \leq y \right) . \)
5. Conclusions

In this work, we have introduced a bisexual branching process which allows mating functions, offspring, and immigration distributions dependent on the number of couples in the population, and we have derived some theoretical results about it. In particular, relationships among their underlying probability generating functions and stochastic monotony properties have been established. In addition its theoretical interest, we remark that the bisexual model introduced in this work also has practical implications in population dynamics. Some open questions for research are, for example, to study its limiting behaviour, its inferential theory, and to develop its potential applications.

Appendix

Lemma A1. Let \((x_1, \ldots, x_n), (y_1, \ldots, y_n), (u_1, \ldots, u_n) \in \mathbb{R}^n\) such that
\[
\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i, \quad k = 1, \ldots, n \quad \text{and} \quad u_1 \geq \ldots \geq u_n \geq 0.
\]
Then
\[
\sum_{i=1}^{n} u_i x_i \leq \sum_{i=1}^{n} u_i y_i.
\]

Proof. Let \(t_i = \sum_{j=1}^{i} x_j\) and \(s_i = \sum_{j=1}^{i} y_j, \quad i = 1, \ldots, n\). It is clear that \(t_i \leq s_i, \quad i = 1, \ldots, n\). It is sufficient to prove that
\[
\sum_{i=1}^{n-1} (u_i - u_{i+1}) t_i + u_n t_n \leq \sum_{i=1}^{n-1} (u_i - u_{i+1}) s_i + u_n s_n.
\]
Now, this inequality holds because \(u_i - u_{i+1} \geq 0, \quad i = 1, \ldots, n\) and \(u_n \geq 0\).

References


