ALGEBRA SCHEMES AND THEIR REPRESENTATIONS

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INTRODUCTION

The equivalence (Cartier duality) between the category of topologically flat formal \( k \)-groups and the category of flat bialgebras has been treated as a duality of continuous vector spaces (of functions) [G, Exposé VII\( B \) by P. Gabriel, 2.2.1]. This is owing to the fact that the reflexivity of vector spaces of infinite dimension does not hold if one does not provide them with a certain topology and does not consider the continuous dual. In this paper we obtain this duality without providing the vector spaces of functions with a topology.

Let \( R \) be a commutative ring with unit. It is natural to consider \( R \)-modules as \( R \)-module functors in the following way: if \( M \) is an \( R \)-module, let \( \mathcal{M}(S) := M \otimes_R S \) for every \( R \)-algebra \( S \) which belongs to the category \( \mathcal{C}_R \) of \( R \)-algebras. Now, if \( M \) is a functor of \( R \)-modules, its dual \( M^* \) can be defined in a natural way as the functor of \( R \)-modules defined \( M^*(S) := \text{Hom}_S(M_S, S) \). In this work we will prove that the functor defined by an \( R \)-module is reflexive: \( M \sim M^{**} \), even in the case of \( R \) being a ring.

We call the functors \( M^* \) \( R \)-module schemes and if they are \( R \)-algebra functors too, we will say they are \( R \)-algebra schemes. In section 2 we study and characterize the vector space schemes (2.3, 2.17) and we characterize when the module scheme closure of an \( R \)-module functor \( M \) is equal to \( M^{**} \) (2.8, 2.9).

P. Gabriel [G, Exposé VII\( B \), 1.3.5] proved that the category of topologically flat formal \( R \)-varieties is equivalent to the category of flat cocommutative \( R \)-coalgebras, where \( R \) is a pseudocompact ring. We prove (4.2) that the category of \( R \)-algebra schemes is equivalent to the category of \( R \)-coalgebras, where \( R \) is a ring.

From this perspective, on the theory of algebraic groups and their representations \( R \)-module schemes appear in a necessary way, as also do \( R \)-algebra schemes as linear envelopes of groups. Let \( G = \text{Spec} A \) be an \( R \)-group and let \( G^* \) be the functor of points of \( G \), i.e., \( G(S) = \text{Hom}_{R-\text{sch}}(\text{Spec} S, G) \) for all \( S \in \mathcal{C}_R \), and let \( RG^* \) be the “linear envelope of \( G^* \)” (see section 3). We prove that the \( R \)-algebra scheme closure of \( RG^* \) is the \( R \)-algebra scheme \( A^* \) (3.3, 5.4) and the category of \( G \)-modules is equal to the category of \( A^* \)-modules (5.5). So, the theory of linear representations of a group \( G = \text{Spec} A \) is a particular case of the theory of \( A^* \)-modules (5.7, 5.8, 6.4, etc). Moreover, there is a bijective correspondence between the \( R \)-rational points of \( A^* \) and the multiplicative characters of \( G \) (5.6). When \( R \) is an algebraically closed field and \( G \) is smooth we prove that the completion of \( RG^* \) by its ideal functors of finite codimension is also \( A^* \) (3.5, 5.9).

Date: October, 2004.
Finally we prove that every $\mathcal{R}$-algebra scheme $\mathcal{A}^*$ is an inverse limit of finite $\mathcal{R}$-algebra schemes (4.12). We characterize the separable algebra schemes (7.4) and we prove the theorem of Wedderburn-Malcev (8.8) in the context of algebra schemes.

This paper is essentially self-contained.

1. $\mathcal{R}$-module schemes. Reflexivity theorem.

Let $R$ be a commutative ring with unit, let $\mathcal{C}_R$ be the category of commutative $R$-algebras and let $\mathcal{R} : \mathcal{C}_R \to \mathcal{C}_R$ be the algebra functor that assigns the $R$-algebra $\mathcal{R}(S) := S$ to $S$. Let $\mathcal{C}_{Ab}$ be the category of commutative groups.

**Definition 1.1.** A functor $\mathcal{M} : \mathcal{C}_R \to \mathcal{C}_{Ab}$ with a morphism of functors $\mathcal{R} \times \mathcal{M} \to \mathcal{M}$ is said to be an $\mathcal{R}$-module functor if $\mathcal{M}(S)$ with the morphism $S \times \mathcal{M}(S) \to \mathcal{M}(S)$ is an $S$-module for each $S \in \mathcal{C}_R$.

Given an $R$-module $M$, the functor $\mathcal{M}$ defined by $\mathcal{M}(S) := M \otimes_R S$ is an $\mathcal{R}$-module functor.

Unless otherwise stated, we assume that all functors considered in this article are functors from the category $\mathcal{C}_R$ to another one.

**Definition 1.2.** Given a pair of $\mathcal{R}$-module functors $\mathcal{M}$ and $\mathcal{M}'$, we will denote by $\mathcal{Hom}_\mathcal{R}(\mathcal{M}, \mathcal{M}')$ the functor of all $\mathcal{R}$-linear morphisms from $\mathcal{M}$ to $\mathcal{M}'$, i.e.,

$$\mathcal{Hom}_\mathcal{R}(\mathcal{M}, \mathcal{M}')(S) = \mathcal{Hom}_\mathcal{S}(\mathcal{M}_{|\mathcal{S}}, \mathcal{M}'_{|\mathcal{S}})$$

where $\mathcal{M}_{|\mathcal{S}}$ denotes the functor $\mathcal{M}$ restricted to the category of commutative $S$-algebras $\mathcal{C}_S$. An element of $\mathcal{Hom}_\mathcal{S}(\mathcal{M}_{|\mathcal{S}}, \mathcal{M}'_{|\mathcal{S}})$ consists of assigning a morphism of $T$-modules $\mathcal{M}(T) \to \mathcal{M}'(T)$ to each $S$-algebra $T$.

We denote by $\mathcal{M}^* = \mathcal{Hom}_\mathcal{R}(\mathcal{M}, \mathcal{R})$ the dual functor of $\mathcal{M}$.

**Proposition 1.3.** For every $\mathcal{R}$-module functor $\mathcal{M}$ and every $R$-module $M$, it holds that

$$\mathcal{Hom}_\mathcal{R}(\mathcal{M}, M) = \mathcal{Hom}_\mathcal{R}(M, \mathcal{M}(R))$$

**Proof.** Given an $\mathcal{R}$-linear morphism $f : \mathcal{M} \to \mathcal{M}'$, we have for every $R$-algebra $S$ a morphism of $S$-modules $f_S : M \otimes_R S \to \mathcal{M}(S)$ and a commutative diagram

$$
\begin{array}{ccc}
M \otimes_R S & \xrightarrow{f_S} & \mathcal{M}(S) \\
\uparrow & & \uparrow \\
M & \xrightarrow{f_R} & \mathcal{M}(R)
\end{array}
$$

Hence, the morphism of $S$-modules $f_S$ is determined by $f_R$. \qed

**Lemma 1.4.** Let $S$ be an $R$-algebra, let $M$ be an $R$-module and let $\mathcal{M}$, $\mathcal{M}'$ be $R$-module functors. Then

1. $\mathcal{M}_{|S}$ is the functor associated to $M \otimes_R S$ on $\mathcal{C}_S$.
2. $\mathcal{Hom}_\mathcal{R}(\mathcal{M}, \mathcal{M}')_{|S} = \mathcal{Hom}_\mathcal{S}(\mathcal{M}_{|S}, \mathcal{M}'_{|S})$.

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3If $\mathcal{M} = \mathcal{M}$ or $\mathcal{M} = \mathcal{M}^*$ then $\mathcal{Hom}_\mathcal{R}(\mathcal{M}, \mathcal{M}')(S)$ is a set (see 1.3, 1.6 and Yoneda’s lemma) and $\mathcal{Hom}_\mathcal{R}(\mathcal{M}, \mathcal{M}')$ is a functor. When we write $\mathcal{M}^*$ or $\mathcal{M}^{**}$ we will suppose that they are well-defined functors. However, for any $M$ and $M'$, in order for $\mathcal{Hom}_\mathcal{S}(\mathcal{M}_{|\mathcal{S}}, \mathcal{M}'_{|\mathcal{S}})$ to be a set (it will be necessary in 2.2, 2.3 and 8.1), instead of taking into account the category of commutative algebras, we consider an infinite set $X$ and the category of commutative algebras whose cardinal is less than or equal to $\text{card}(X^2)$. See [D, General conventions].
Definition 1.5. Given a commutative $R$-algebra $A$, we define the functor $(\text{Spec } A)^{\ast}$ to be $(\text{Spec } A)^{\ast} (S) = \text{Hom}_{R-\text{alg}} (A, S)$ for each commutative $R$-algebra $S$. This functor will be called the functor of points of $\text{Spec } A$.

By Yoneda’s lemma (see [E, Appendix A5.3]), $\text{Hom}_{\text{func}}((\text{Spec } A), M) = M(A)$.

Given an $R$-module $M$, we will denote by $S_{R}M$ the symmetric algebra of $M$. Let us recall the next well-known lemma (see [D, II, §1, 2.1] or [G, Exposé VII$_{R}$, 1.2.4]).

Lemma 1.6. If $M$ is an $R$-module, then $\mathcal{M}^{\ast} = (\text{Spec } S_{R}M)$ as $\mathcal{R}$-module functors.

Proof. For every $R$-algebra $S$, it holds that

\[ \mathcal{M}^{\ast} (S) = \text{Hom}_{S} (\mathcal{M}|_{S}, R|_{S}) \overset{1.4}{=} \text{Hom}_{S} (\mathcal{M} \otimes_{R} S, S) \overset{1.4}{=} \text{Hom}_{S} (\mathcal{M} \otimes_{R} S, S) = \]

\[ = \text{Hom}_{R} (M, S) = \text{Hom}_{R-\text{alg}} (S_{R}M, S) = (\text{Spec } S_{R}M) (S) \]

\[ \square \]

Definition 1.7. The tensorial product of two functors $\mathcal{M}$, $\mathcal{N}$ in the category of $\mathcal{R}$-module functors is defined to be $(\mathcal{M} \otimes_{R} \mathcal{N})(S) = \mathcal{M}(S) \otimes_{S} \mathcal{N}(S)$.

Proposition 1.8. Let $M$, $M'$ be $R$-modules. Then

\[ \text{Hom}_{R} (\mathcal{M}^{\ast}, \mathcal{M}') = \mathcal{M} \otimes_{R} \mathcal{M}' \]

Proof. We know that $\mathcal{M}^{\ast}$ is represented by $\text{Spec } S_{R}M$, therefore

\[ \text{Hom}_{R} (\mathcal{M}^{\ast}, \mathcal{M}') \subseteq \text{Hom}_{\text{func}} (\mathcal{M}^{\ast}, \mathcal{M}') = \mathcal{M}' (S_{R}M) = S_{R}M \otimes_{R} M' \]

However, in order for $w \in S_{R}M \otimes_{R} M'$ to be a linear application, it must be $w \in M \otimes_{R} M'$. Hence, $\text{Hom}_{R} (\mathcal{M}^{\ast}, \mathcal{M}') = M \otimes_{R} M'$.

For every $R$-algebra $S$, we have that

\[ \text{Hom}_{R} (\mathcal{M}^{\ast}, \mathcal{M}') (S) = \text{Hom}_{S} (\mathcal{M}^{\ast}|_{S}, \mathcal{M}'|_{S}) = \text{Hom}_{S} ((\mathcal{M} \otimes_{R} S)^{\ast}, \mathcal{M}' \otimes_{R} S) \]

\[ = (M \otimes_{R} S) \otimes_{S} (M' \otimes_{R} S) = (M \otimes_{R} M') (S) \]

\[ \square \]

Remark 1.9. If $\mathcal{C}_{R}$ is the category of commutative $R$-algebras whose cardinal is less than or equal to $m$, then we have to suppose that $S_{R}M \in \mathcal{C}_{R}$, i.e., that $\text{card } S_{R}M \leq m$. If $M$ is a free $R$-module of whichever cardinal, we obtain the proposition again: Let $\{ M_{i} \}$ be the set of quotients of $M$ whose cardinal is less than or equal to $m$. It is easily seen that $\mathcal{M}^{\ast} = \lim_{i} M_{i}^{\ast}$. Then

\[ \text{Hom}_{R} (\mathcal{M}^{\ast}, \mathcal{M}') = \text{Hom}_{R} (\lim_{i} M_{i}^{\ast}, \mathcal{M}') = \lim_{i} (M_{i} \otimes_{R} M') = M \otimes_{R} M' \]

where $\lim$ is a consequence of the equality $M \otimes_{R} M' \otimes_{R} S = \lim_{i} (M_{i} \otimes_{R} M' \otimes_{R} S)$ for every $R$-algebra $S \in \mathcal{C}_{R}$. Even more, we can assume $M$ is a projective $R$-module, i.e., a direct sum of a free $R$-module.

As a corollary we obtain the following

Theorem 1.10. Let $M$ be an $R$-module. Then

\[ \mathcal{M}^{**} = \mathcal{M} \]
Definition 1.11. Quasi-coherent $\mathcal{R}$-modules are defined to be $\mathcal{R}$-module functors of the type $\mathcal{M}$, where $M$ is any $\mathcal{R}$-module. We shall say that $M$ is a coherent $\mathcal{R}$-module if $M$ is a finitely generated $\mathcal{R}$-module.

$\mathcal{R}$-module schemes are defined to be $\mathcal{R}$-module functors of the type $\mathcal{M}^\ast$.

If $M$ is a free finitely generated $\mathcal{R}$-module then $\mathcal{M}$ is a quasi-coherent $\mathcal{R}$-module and an $\mathcal{R}$-module scheme.

Theorem 1.12. The category of quasi-coherent modules over $\mathcal{R}$ is equivalent to the category of $\mathcal{R}$-modules. The category of quasi-coherent modules over $\mathcal{R}$ is anti-equivalent to the category of $\mathcal{R}$-module schemes (the correspondence is established by taking the dual functor).

In [G, Exposé VII, 1.2.3], the anti-equivalence between the category of flat $\mathcal{R}$-modules and the category of projective pseudocompact $\mathcal{R}$-modules is established, where $\mathcal{R}$ is a (commutative) pseudocompact ring.

Proposition 1.13. The $\mathcal{R}$-linear morphism $\mathcal{M} \to \mathcal{M}'$ is surjective, in the category of $\mathcal{R}$-module functors, if and only if the morphism $\mathcal{M}'^\ast \to \mathcal{M}^\ast$ is injective, in the category of $\mathcal{R}$-module functors.

Proof. It follows immediately that if the morphism $\mathcal{M} \to \mathcal{M}'$ is surjective, then the morphism $\mathcal{M}'^\ast \to \mathcal{M}^\ast$ is injective. Inversely, let us suppose the morphism $\mathcal{M}'^\ast \to \mathcal{M}^\ast$ is injective. If $V$ is the cokernel of the morphism $\mathcal{M} \to \mathcal{M}'$, we obtain $V^\ast = 0$. Hence $V = V^\ast = 0$ and the morphism $\mathcal{M} \to \mathcal{M}'$ is surjective. □

If a morphism $\mathcal{M}'^\ast \to \mathcal{M}^\ast$ is surjective then the associated morphism $\mathcal{M} \to \mathcal{M}'$ is injective and it has a retraction. Let us consider the $\mathcal{R}$-algebra $S := R \oplus M$, where $e_1 \cdot e_2 = 0$ for all $e_1, e_2 \in M$. Let $w \in \mathcal{M}'(S) = \text{Hom}_\mathcal{R}(M, S)$ be defined by $w(e) := e$. Then, there exists a $w' \in \text{Hom}_\mathcal{R}(M', S)$ such that $w'(e) = e$ for all $e \in M$. If $\pi : S \to M$ is the natural projection, then $\pi \circ w'$ is a retraction of the morphism $\mathcal{M} \to \mathcal{M}'$.

Let us recall the Formula of adjoint functors.

Definition 1.14. Let us consider the inclusion of categories

$$\mathcal{C}_R = \{\text{commutative $\mathcal{R}$-algebras}\} \supset \mathcal{C}_S = \{\text{commutative $\mathcal{S}$-algebras}\}$$

where $S$ is an $\mathcal{R}$-algebra. Given a functor $\mathcal{N}$ on $\mathcal{C}_S$ we define $(i_* \mathcal{N})(R') := \mathcal{N}(S \otimes R)$ for each object $R'$ of $\mathcal{C}_R$. Given a functor $\mathcal{M}$ on $\mathcal{C}_R$ we define $(i^* \mathcal{M})(S') := \mathcal{M}(S')$ for each object $S'$ of $\mathcal{C}_S$.

Let us give a direct proof of the following theorem, although it can be obtained from [B, 8.4.8.5] after many precisions and complicated technical terms.

Theorem 1.15 (Formula of adjoint functors). Let $\mathcal{M}$ be an $\mathcal{R}$-module functor and let $\mathcal{N}$ be an $\mathcal{S}$-module functor. Then it holds that

$$\text{Hom}_\mathcal{S}(i^* \mathcal{M}, \mathcal{N}) = \text{Hom}_\mathcal{R}(\mathcal{M}, i_* \mathcal{N})$$

Proof. Given a $w \in \text{Hom}_\mathcal{S}(i^* \mathcal{M}, \mathcal{N})$, we have morphisms $w_{S \otimes R'} : \mathcal{M}(S \otimes R') \to \mathcal{N}(S \otimes R')$ for each $\mathcal{R}$-algebra $R'$. By composition with the morphisms $M(R') \to \mathcal{M}(S \otimes R')$, we have the morphisms $\phi_{R'} : M(R') \to \mathcal{N}(S \otimes R') = i_* \mathcal{N}(S')$, which in their turn define $\phi \in \text{Hom}_\mathcal{R}(\mathcal{M}, i_* \mathcal{N})$.

Given a $\phi \in \text{Hom}_\mathcal{R}(\mathcal{M}, i_* \mathcal{N})$, we have morphisms $\phi_{S'} : \mathcal{M}(S') \to i_* \mathcal{N}(S') = \mathcal{N}(S \otimes S')$ for each $\mathcal{S}$-algebra $S'$. By composition with the morphisms $\mathcal{N}(S \otimes S') \to \mathcal{N}(S \otimes S')$
Proof. It holds that instead of and the whole diagram is commutative.

\[ \phi \text{ defines } \text{Hom} \]

\[ \text{Proposition 1.16.} \]

For simplicity of notation, given a functor \( M \) we will sometimes write \( w \in M \) instead of \( w \in \text{M}(S) \).

**Proposition 1.16.** Let \( \text{M}_i \) be \( K \)-vector space functors and let \( M \) be a \( K \)-vector space. It holds that

\[ \text{Hom}_K(\prod_i \text{M}_i, \mathcal{M}) = \oplus_i \text{Hom}_K(\text{M}_i, \mathcal{M}) \]

**Proof.** From the injective morphism \( \oplus_i \text{M}_i \rightarrow \prod_i \text{M}_i \) one obtains the morphism

\[ j^*: \text{Hom}_K(\prod_i \text{M}_i, \mathcal{M}) \rightarrow \text{Hom}_K(\oplus_i \text{M}_i, \mathcal{M}) = \prod_i \text{Hom}_K(\text{M}_i, \mathcal{M}) \]

The aim is to prove that this morphism is injective and its image is \( \oplus \text{Hom}_K(\text{M}_i, \mathcal{M}) \).

For the first question, let \( w \in \text{Hom}_K(\prod_i \text{M}_i, \mathcal{M}) \) be a linear form such that \( w \neq 0 \) but \( w|_{\oplus_i \text{M}_i} = 0 \). Then there exists a \( K \)-algebra \( S \) and elements \( f_i \in \text{M}_i(S) \) such that \( w((f_i)_i) \neq 0 \), and composing with the morphisms \( \phi : \prod_i S \rightarrow \prod_i \text{M}_i, \phi((s_i)_i) = (s_if_i)_i \) we get a linear form \( w \circ \phi \in \text{Hom}_S(\prod_i S, \mathcal{M} \otimes S) \) that is not null but is null on \( \oplus_i S \), which is impossible since \( \text{Hom}_S(\prod_i S, \mathcal{M} \otimes S) = \text{Hom}_S((\oplus_i S)^*, \mathcal{M} \otimes S) \cong (\oplus_i S) \otimes_S (\mathcal{M} \otimes S) = \oplus_i \mathcal{M} \otimes S \).
To prove that $\text{Im} j^* = \bigoplus_i \text{Hom}_K(M_i, M)$ it is enough to prove that $\text{Hom}_K(\prod_i M_i, M) = \bigoplus_i \text{Hom}_K(M_i, M)$, because in that case we will have

$$\text{Hom}_S(\prod_i M_{i,S}, M_{i|S}) \cong \text{Hom}_K(\prod_i M_i, M \otimes S) = \bigoplus_i \text{Hom}_K(M_i, M \otimes S)$$

Given a linear form $w \in \text{Hom}_K(\prod_i M_i, M)$ we have to prove that there exists at most a finite subset of indices $i$ such that $w|_{M_i} \neq 0$. Let us suppose that this is not true, i.e., that there exists a set of indices $i_n$, where $n \in \mathbb{N}$, and $K$-algebras $S_n$ such that $w(m_{i_n}) \neq 0$ for some $m_{i_n} \in M_{i_n}(S_n)$. Let $S = \otimes_n S_n$ and denote by $h_r : S_r \hookrightarrow \otimes_n S_n$ the natural injections and by $\bar{m}_r$ the image of $m_{i_n}$ by the induced morphism $M_{i_n}(h_r) : M_{i_n}(S_r) \to M_{i_n}(\otimes_n S_n)$. It is easy to see that $w(\bar{m}_r) = h_r(w(m_{i_n})) \neq 0$. Therefore, we get a linear form $\bar{w} : \prod_n S \to M \otimes S$, $\bar{w}((s_n)_n) := w((s_n \bar{m}_n)_n)$, which is not null on any factor $S \subset \prod_n S$. Again this contradicts the fact that $\text{Hom}_S(\prod_n S, M \otimes S) = \bigoplus_n M \otimes S$. \hfill $\Box$

2. Characterizations of vector space schemes.

Let $R$ be a commutative ring with unit and let $K$ be a commutative field.

**Definition 2.1.** An $R$-module functor $\mathcal{M}$ is said to be reflexive if $\mathcal{M} = \mathcal{M}^{**}$.

**Theorem 2.2.** If $\mathcal{M}$ is a reflexive functor of $K$-vector spaces such that $\text{Hom}_K(\mathcal{M}, -)$ commutes with direct sums, i.e.,

$$\text{Hom}_K(\mathcal{M}, \bigoplus_i M_i) = \bigoplus_i \text{Hom}_K(\mathcal{M}, M_i)$$

for all $K$-vector space functors $M_i$, then $\mathcal{M}$ is a $K$-vector space scheme.

**Proof.** From the adjoint functor formula, given a commutative $K$-algebra $S$, we have that

$$\text{Hom}_K(\mathcal{M}, S) = \text{Hom}_K(\mathcal{M}, i_* i^* K) = \text{Hom}_S(\mathcal{M}_{i|S}, S) = \mathcal{M}^*(S)$$

However, $S = \bigoplus_i K$ and the property that $\mathcal{M}$ satisfies by hypothesis means that $\text{Hom}_K(\mathcal{M}, S) = \mathcal{M}^*(K) \otimes S$. Hence, $\mathcal{M}^*(S) = \mathcal{M}^*(K) \otimes_K S$ and $\mathcal{M}^* = \mathcal{M}$, where $\mathcal{M} = \mathcal{M}^*(K)$, and therefore $\mathcal{M} = \mathcal{M}^{**} = \mathcal{M}^*$. \hfill $\Box$

We can now rephrase this result in terms of direct limits. The definition that we work with is taken from [E, Appendix 6].

**Theorem 2.3.** Let $\mathcal{M}$ be a reflexive $K$-vector space functor. The functor on the category of quasi-coherent $K$-vector spaces, $\text{Hom}_K(\mathcal{M}, -)$, commutes with direct limits if and only if $\mathcal{M}$ is a $K$-vector space scheme.

**Proof.** The necessary condition is a consequence of the previous theorem, since it was only necessary that $\text{Hom}_K(\mathcal{M}, -)$ commuted with direct sums of quasi-coherent vector spaces for $\mathcal{M}$ to be a $K$-vector space scheme.
The sufficient condition is obtained as an immediate consequence of Proposition 1.8, since the functor \( \lim_{i \in I} M_i \) is again a quasi-coherent \( K \)-vector space and

\[
\text{Hom}_K(M, \lim_{i \in I} M_i) \overset{1.8}{=} M^* \otimes (\lim_{i \in I} M_i) = \lim_{i \in I} (M^* \otimes M_i) \overset{1.8}{=} \lim_{i \in I} \text{Hom}_K(M, M_i)
\]

\( \square \)

**Definition 2.4.** Given an \( R \)-module functor \( M \), we shall say that \( \bar{M} \) is the \( R \)-module scheme closure of \( M \) if \( \bar{M} \) is an \( R \)-module scheme and

\[
\text{Hom}_R(M, M^*) = \text{Hom}_R(\bar{M}, M^*)
\]

for every \( R \)-module \( M \).

As \( \bar{M} \) is defined to be the representant on the category of \( R \)-module schemes of the functor \( \text{Hom}_R(\bar{M}, -) \) it is unique up to isomorphisms, and there exists a canonical morphism \( \bar{M} \to \bar{M} \) corresponding to the identity morphism \( \bar{M} \to \bar{M} \).

**Notation 2.5.** We mean by \( M^*(R) \) the quasi-coherent \( R \)-module corresponding to the \( R \)-module \( M^*(R) \), i.e., \( M^*(R)(S) = M^*(R) \otimes_R S \).

**Lemma 2.6.** Let \( M, N \) be functors of \( R \)-modules. Then

\[
\text{Hom}_R(M, N^*) = \text{Hom}_R(N, M^*)
\]

**Proof.**

\[
\text{Hom}_R(M, N^*) = \text{Hom}_R(M \otimes_R N, R) = \text{Hom}_R(N, M^*)
\]

\( \square \)

**Proposition 2.7.** Let \( M \) be an \( R \)-module functor. It holds that \( \bar{M} = M^*(R)^* \).

**Proof.**

\[
\text{Hom}_R(M, M^*) \overset{2.6}{=} \text{Hom}_R(M, M^*) \overset{1.3}{=} \text{Hom}_R(M, M^*(R)) \overset{1.3}{=} \text{Hom}_R(M, M^*(R)) \overset{2.6}{=} \text{Hom}_R(M^*(R)^*, M^*)
\]

\( \square \)

Unfortunately, the \( R \)-module scheme closure of an \( R \)-module functor \( M \) is not stable under base change.

**Proposition 2.8.** Let \( M \) be an \( R \)-module functor. If \( M^* \) is a quasi-coherent \( R \)-module then \( \bar{M} = M^{**} \) and \( \bar{M} = M^* \).

**Proof.** If \( M^* \) is a quasi-coherent \( R \)-module then

\[
\text{Hom}_R(M, M^*) = \text{Hom}_R(M, M^*) = \text{Hom}_R(M^{**}, M^*)
\]

Therefore \( \bar{M} = M^{**} \). Moreover, \( \bar{M} = (M^{**})^* = (M^*)^{**} = M^* \). \( \square \)

**Proposition 2.9.** The \( R \)-module scheme closure of an \( R \)-module functor \( M \) is stable under base change if and only if \( M^* \) is a quasi-coherent \( R \)-module.

**Proof.** If \( \bar{M}_{|S} = \bar{M}_{|S} \), then taking \( \text{Hom}_S(-, S) \) we obtain that \( \bar{M}^*(S) = M^*(R) \otimes_R S \). Inversely, if \( M^* \) is quasi-coherent then \( \bar{M}_{|S} = M^{**}_{|S} = M_{|S} = \bar{M}_{|S} \). \( \square \)
If the morphism \( \text{Hom}_R(M_1 \otimes \ldots \otimes M_n, R) = \text{Hom}_R(M_1 \otimes \ldots \otimes M_{n-1}, M_n^*) \) is injective, then \( M \text{-module schemes} \) become \( \text{lim} \) of complete reflexive functors. First, we need some technical results before Definition 2.14.

Lastly, we will show another characterization of \( K \)-vector space schemes by means of complete reflexive functors. First, we need some technical results before Definition 2.14.

**Example 2.10.** If \( M_1, \ldots, M_n \) are \( R \)-module functors whose duals are quasi-coherent \( R \)-modules, then \( (M_1 \otimes \ldots \otimes M_n)^* = M_1^* \otimes \ldots \otimes M_n^* \), which in particular is a quasi-coherent \( R \)-module:

\[
\text{Hom}_R(M_1 \otimes \ldots \otimes M_n, R) = \text{Hom}_R(M_1 \otimes \ldots \otimes M_{n-1}, M_n^*) = \text{Hom}_R(M_1 \otimes \ldots \otimes M_{n-2}, \text{Hom}_R(M_{n-1}, M_n^*)) = \text{Hom}_R(M_1 \otimes \ldots \otimes M_{n-2}, \text{Hom}_R(M_{n-1}, M_n^*)) = \ldots = M_1^* \otimes \ldots \otimes M_n^*.
\]

Hence, \( M_1 \otimes \ldots \otimes M_n = (M_1 \otimes \ldots \otimes M_n)^* \) and \( M_1 \otimes \ldots \otimes M_n = M_1^* \otimes \ldots \otimes M_n^* \).

If we denote by \( \otimes \) the tensorial product in the category of \( R \)-module schemes then \( M_1^* \otimes M_2^* = M_1^* \otimes M_2^* = (M_1 \otimes M_2)^* \). Moreover, \( \otimes \) commutes with inverse limits:

\[
\lim_i (M_i^*) \otimes M^* = \left( \lim_i M_i \right)^* \otimes M^* = ((\lim_i M_i) \otimes M)^* = (\lim_i (M_i \otimes M))^* = \lim_i (M_i \otimes M)^* = \lim_i (M_i^* \otimes M^*)
\]

Henceforward, we shall only work with functors of \( K \)-vector spaces.

**Proposition 2.11.** The morphism \( M \rightarrow M^{**} \) is injective if and only if the morphism \( M \rightarrow M \) is injective.

**Proof.** Let us prove the necessary condition. Given \( s \in M(S) \) such that \( s = 0 \) in \( M(S) = M^*(S) \), then \( s(w) := w(s) = 0 \) for all \( w \in M^*(K) \). Given a commutative \( S \)-algebra \( T \), if one writes \( T = \bigoplus K \cdot e_i \), one notices that

\[
M^*(T) = \text{Hom}_T(M, T) = \text{Hom}_K(M, T) = \text{Hom}_K(M, \bigoplus K) \subset \prod \text{Hom}_K(M, K)
\]

which assigns to every \( w_T \in M^*(T) \) a \( (w_i) \in \prod M^*(K) \). Explicitly, given \( t \in M(T) \), then \( w_T(t) = \sum_i w_i(t) \cdot e_i \). Therefore \( w_T(s) = 0 \) for all \( w_T \in M^*(T) \). Since the morphism \( M \rightarrow M^{**} \) is injective, this means that \( s = 0 \), i.e., the morphism \( M \rightarrow M \) is injective.

For the sufficient condition, we consider the morphism \( M^*(K) \rightarrow M^* \), which, by taking duals, becomes \( M^{**} \rightarrow M^*(K)^* \). Since the composite morphism \( M \rightarrow M^{**} \rightarrow M = M^*(K)^* \) is injective, so is the morphism \( M \rightarrow M^{**} \).

Lastly, we will show another characterization of \( K \)-vector space schemes by means of complete reflexive functors. First, we need some technical results before Definition 2.14.

**Proposition 2.12.**

1. The morphism \( M^*(K) \rightarrow M(K)^* \) is injective if and only if the morphism \( M^* \rightarrow M(K)^* \) is injective.

2. The morphism \( M^*(K) \rightarrow M(K)^* \) is injective if and only if for every quasi-coherent vector space \( V \) the image of any \( K \)-linear morphism \( M \rightarrow V \) is a quasi-coherent subspace of \( V \).
Proof. 

(1) If the morphism $M^* \to \mathcal{M}(K)^*$ is injective then taking sections on $K$ the morphism $M^*(K) \to \mathcal{M}(K)^*$ is injective. Inversely, from the commutative diagram

$$
\begin{array}{ccc}
\text{Hom}_K(M, \oplus K) & \to & \text{Hom}_K(M(K), \oplus K) \\
\cap & & \cap \\
\cap \text{Hom}_K(M, K) & \leftarrow & \cap \text{Hom}_K(M(K), K)
\end{array}
$$

one has that $\text{Hom}_K(M, \oplus K) \subset \text{Hom}_K(M(K), \oplus K)$. Since $S = \oplus K$, then

$$
M^*(S) = \text{Hom}_S(M(S), S) \supset \text{Hom}_K(M, \oplus K) \subset \text{Hom}_K(M(K), \oplus K)
$$

$$
\supset \text{Hom}_K(M(K) \otimes K S, S) \supset \text{Hom}_S(M(K), K) = M(K)^*(S)
$$

i.e., the morphism $M^* \to \mathcal{M}(K)^*$ is injective.

(2) Let us suppose that the image of any morphism $\mathcal{M} \to V$ is a quasi-coherent subspace of $V$. Given $w \in M^*(K)$, i.e., a morphism $w : M \to K$, $\text{Im} w$ is equal to the quasi-coherent vector space associated to $w(M(K))$. Hence if $w(M(K)) = 0$ then $w = 0$.

Inversely, let $V'$ be the image of the morphism $\mathcal{M}(K) \to V$ and consider $\mathcal{M} \to W := V/V'$. The morphism $W^* \to \mathcal{M}(K)^*$ is null. Hence the morphism $W^* \to M^*$ is null, and the composite morphism $\mathcal{M} \to M^* \to W^* = W$ is null. Therefore, the image of the morphism $\mathcal{M} \to V$ is $V'$.

\[\square\]

Corollary 2.13. Let $\mathcal{M}$ be a reflexive functor and let $V$ be a $K$-vector space. Then the image of any $K$-linear morphism $\mathcal{M} \to V$ is a quasi-coherent subspace of $V$.

Proof. If $\mathcal{M} = M^{**}$, by Proposition 2.11 the morphism $M^* \to \mathcal{M} = \mathcal{M}(K)^*$ is injective. Then by Proposition 2.12 the proof is complete. \[\square\]

Definition 2.14. Given a $K$-vector space functor $\mathcal{M}$ such that the image of any $K$-linear morphism $\mathcal{M} \to V$ is a quasi-coherent subspace of $V$, let us consider the $K$-vector space subfunctors $\mathcal{M}_i \subset \mathcal{M}$ such that $\mathcal{M}/\mathcal{M}_i$ are coherent $K$-vector spaces. Then we define $\mathcal{M} := \text{lim}_{\leftarrow i} \mathcal{M}/\mathcal{M}_i$.

The direct limit of quasi-coherent vector spaces, in the category of $K$-vector space functors, is a quasi-coherent vector space. Therefore, the inverse limit of $K$-vector space schemes is a $K$-vector space scheme. Hence $\mathcal{M}$ is a $K$-vector space scheme, namely, $\mathcal{M} := (\text{lim}_{\leftarrow i} (\mathcal{M}/\mathcal{M}_i))^*$.

Proposition 2.15. Let $V$ be a $K$-vector space. Then, $\mathcal{V}^*$ is complete and separate, i.e., $\mathcal{V}^* = \mathcal{V}^*$.

Proof. By the reflexivity theorem, the coherent cokernels of $\mathcal{V}^*$ correspond to the subspaces $V' \subset V$ of finite dimension. Hence,

$$
\mathcal{V}^* = \lim_{\text{dim}_K V' < \infty} (\mathcal{V}')^* = (\lim_{\text{dim}_K V' < \infty} \mathcal{V}')^* = \mathcal{V}^*
$$

\[\square\]
Proposition 2.16. Let $\mathcal{M}$ be a $K$-vector space functor such that the image of any $K$-linear morphism $\mathcal{M} \rightarrow \mathcal{V}$ is a quasi-coherent subspace of $\mathcal{V}$. Then the vector space closure of $\mathcal{M}$ is equal to the completion of $\mathcal{M}$, i.e., $\bar{\mathcal{M}} = \mathcal{M}$.

In particular, $\bar{\mathcal{M}} = \mathcal{V}^*$, where $\mathcal{V} = \mathcal{M}^*(K)$, and $\bar{\mathcal{M}}$ is complete, separate, and reflexive.

Proof. First, let us suppose that $\mathcal{V}$ is a finite-dimensional space. Observe that the dual of an inverse limit of $K$-vector space schemes is equal to the direct limit of the quasi-coherent dual vector spaces, $(\varinjlim \mathcal{V}_i)^* = (\varprojlim \mathcal{V}_i)^* = \lim \mathcal{V}_i$, then

$$\hom_K(\bar{\mathcal{M}}, \mathcal{V}^*) = \hom_K(\varinjlim \mathcal{M}/\mathcal{M}_i, \mathcal{V}^*) = \varprojlim \hom_K(\mathcal{M}/\mathcal{M}_i, \mathcal{V}^*) = \hom_K(\mathcal{M}, \mathcal{V}^*)$$

In general, $\mathcal{V}^* = \varinjlim \mathcal{V}_i^*$, where $\dim \mathcal{V}_i < \infty$. Then

$$\hom_K(\bar{\mathcal{M}}, \mathcal{V}^*) = \hom_K(\varinjlim \mathcal{M}/\mathcal{M}_i, \mathcal{V}_i^*) = \varprojlim \hom_K(\mathcal{M}/\mathcal{M}_i, \mathcal{V}_i^*) = \lim \hom_K(\mathcal{M}, \mathcal{V}_i^*)$$

Therefore, $\bar{\mathcal{M}} = \mathcal{M}$. \hfill \Box

Theorem 2.17. Let $\bar{\mathcal{M}}$ be a reflexive $K$-vector space functor. Then $\mathcal{M}$ is a $K$-vector space scheme if and only if $\bar{\mathcal{M}}$ is complete and separate.


Definition 3.1. Let $X = \spec A$ be an affine $\mathcal{R}$-scheme and let us denote by $X^*$ the functor of points of $X$, i.e., $X(\mathcal{S}) = \hom_{\mathcal{R}_{\text{alg}}}(A, \mathcal{S})$. Let $RX$ be the $\mathcal{R}$-module functor defined by $RX(\mathcal{S}) := \varinjlim \mathcal{S} = \{ \text{the formal finite } \mathcal{S}\text{-linear combinations of points of } X \text{ in } \mathcal{S} \}$.

It is clear that for every $\mathcal{R}$-module functor $\mathcal{M}$ it holds that

$$\hom_{\mathcal{R}}(RX, \mathcal{M}) = \hom_{\text{func}}(X^*, \mathcal{M})$$

Since every morphism of $\mathcal{R}$-algebras $A \rightarrow \mathcal{S}$ is in particular $\mathcal{R}$-linear, we have a morphism of functors $\phi : X^* \rightarrow A^*$, where the morphism between schemes is given by the natural epimorphism of $\mathcal{R}$-algebras $S_{R^*}A \rightarrow A$. Then we have a morphism $RX^* \rightarrow A^*$.

Notation 3.2. It is usual the notation $X_S = \spec A \times_R \spec S = \spec(A \otimes_R \mathcal{S})$ and $A_S = A \otimes_S \mathcal{S}$.

Theorem 3.3. Let $X = \spec A$ be an affine $\mathcal{R}$-scheme. It holds that

1. $RX^* = A$
2. $\overline{RX}^* = RX^{**} = A^*$

Proof.

$$RX^*(\mathcal{R}) = \hom_{\mathcal{R}}(RX^*, \mathcal{R}) = \hom_{\text{func}}(X^*, \mathcal{R}) = A$$

and likewise

$$RX^*(\mathcal{S}) = \hom_{\mathcal{S}}(RX^*_\mathcal{S}, \mathcal{S}) = \hom_{\mathcal{S}}(SX^*_\mathcal{S}, \mathcal{S}) = A_S = A(\mathcal{S})$$

Hence, $RX^* = A$ and taking duals $A^* = RX^{**} = \overline{RX}^*$. \hfill \Box
Theorem 3.4. If $X = \text{Spec} A$ is an $R$-scheme and $M$ is a reflexive $R$-module functor, then the morphism

$$\text{Hom}_R(A^*, M) \to \text{Hom}_\text{functors}(X^*, M)$$

$$A^* \to M \mapsto X^* \mapsto A^* \to M$$

is an isomorphism.

Moreover, if $A$ is a free $R$-module such linear applications of functors are determined each by its value on global sections, i.e.,

$$\text{Hom}_R(A^*, M) \subset \text{Hom}_R(A^*, M(R))$$

Proof. Firstly, we have

$$\text{Hom}_R(A^*, M) \overset{2.6}{=} \text{Hom}_R(M^*, A) \overset{3.3}{=} \text{Hom}_R(M^*, RX^*)$$

$$\overset{2.6}{=} \text{Hom}_R(RX^*, M) = \text{Hom}_\text{func}(X^*, M)$$

which is the isomorphism to compose with $\phi$.

Secondly, since $A = \oplus R \subseteq \prod R$ we get

$$\text{Hom}_R(A^*, M) \overset{2.6}{=} \text{Hom}_R(M^*, A) \subseteq \text{Hom}_R(M^*, \prod R)$$

$$\overset{2.6}{=} \text{Hom}_R(\oplus R, M) \overset{1.3}{=} \text{Hom}_R(\oplus R, M(R))$$

Since the injective morphism $\text{Hom}_R(A^*, M) \hookrightarrow \text{Hom}_R(\oplus R, M(R))$ factors through $\text{Hom}_R(A^*, M(R))$, the morphism $\text{Hom}_R(A^*, M) \to \text{Hom}_R(A^*, M(R))$ is injective.

Theorem 3.5. Let us suppose that the only function $a \in A$ of the $K$-scheme $X = \text{Spec} A$ that is null on every $K$-rational point is the zero function $a = 0$. Then, it holds that $\hat{K}X^* = A^*$.

Proof. By hypothesis the morphism $KX^*(K) = A \hookrightarrow (KX^*)(K)^*$ is injective, hence we are under the hypothesis of Definition 2.14 and Proposition 2.12. Therefore, by Proposition 2.16 $\hat{K}X^* = \hat{KX^*} \overset{\text{(1)}}{=} A^*$.

Maybe it is more natural the definition

$$\mathcal{R}X^*(S) := \langle \text{Hom}_{\text{alg}}(A, S) \rangle_S \subset \text{Hom}_R(A, S)$$

i.e., $\mathcal{R}X^*$ is the image of $\mathcal{R}X^*$ in $A^*$.

Proposition 3.6. It holds

1. $\text{Hom}_R(\mathcal{R}X^*, M) = \text{Hom}_\text{func}(X^*, M)$ for every reflexive functor $M$.
2. $\mathcal{R}X^* = A$.
3. $\mathcal{R}X' = A^*$.
4. The minimum reflexive subfunctor of $A^*$ that contains $\mathcal{R}X^*$ is $A^*$.

Proof.

(1) It is a consequence of the equalities

$$\text{Hom}_R(A^*, M) \overset{3.4}{=} \text{Hom}_\text{func}(X^*, M) = \text{Hom}_R(\mathcal{R}X^*, M)$$

(2),(3) They are consequences of (1).
Let us suppose we have morphisms \( R^X \leftrightarrow M \hookrightarrow A^* \), where \( M \) is a reflexive functor. Taking double duals, we obtain that the composite morphism \( A^* \rightarrow M \rightarrow A^* \) is the identity morphism. Therefore, the morphism \( \bar{M} \rightarrow A^* \) is surjective and (4) follows.

\[ \square \]

4. **Algebra schemes.**

**Definition 4.1.** We call an \( R \)-module scheme \( A^* \) an \( R \)-algebra scheme if it is also an \( R \)-algebra functor (i.e., \( A^*(S) \) is a \( S \)-algebra and the morphisms \( A^*(S) \rightarrow A^*(S') \) are morphisms of \( S \)-algebras for every morphism \( S \rightarrow S' \) of \( R \)-algebras).

**Proposition 4.2.** The category of coalgebras with counit, \( C_{\text{coalg}} \), is anti-equivalent to the category of algebra schemes, \( C_{\text{alg}} \). The functors which give the equivalence are \( C_{\text{coalg}} \rightarrow C_{\text{alg}} \), \( B \rightsquigarrow B^* \) and \( C_{\text{alg}} \rightarrow C_{\text{coalg}} \), \( A^* \rightsquigarrow A \).

**Proof.** Observe that \( \text{Hom}_R(M^*_1 \otimes \ldots \otimes M^*_n, N^*) \cong \text{Hom}_R(N, (M^*_1 \otimes \ldots \otimes M^*_n)^*) \cong \text{Hom}_R(N, M^*_1 \otimes \ldots \otimes M^*_n) \).

Giving an \( R \)-algebra functor structure on a scheme \( A^* \) is equivalent to giving the morphism of multiplication \( A^* \otimes A^* \rightarrow A^* \) and the unit \( \mathbb{R} \rightarrow A^* \), so that the diagrams that state distributive, associative and the like properties are commutative. This is equivalent to giving morphisms \( A \rightarrow A \otimes A, A \rightarrow \mathbb{R} \) which endow \( A \) with a coalgebra structure with counit. \( \square \)

**Notation 4.3.** From now on, in this and next sections, \( A^* \) denotes an \( R \)-algebra scheme.

**Definition 4.4.** Let \( M \) be an \( R \)-module functor and let \( \mathbb{A} \) be an \( R \)-algebra functor. We say that \( M \) is an \( \mathbb{A} \)-module if there exists a morphism of \( R \)-algebra functors \( \mathbb{A} \rightarrow \text{End}_R(M) \).

We will say that an \( R \)-module \( M \) is an \( \mathbb{A} \)-module if \( M \) is an \( \mathbb{A} \)-module.

Giving a structure of \( A^\ast \)-module on \( M \) is equivalent to the existence of a morphism \( A^\ast \otimes M \rightarrow M \) verifying the obvious properties, which is equivalent to the existence of a morphism \( M \rightarrow A \otimes M \) verifying the obvious properties, since

\[
\text{Hom}_R(A^*, \text{Hom}_R(M, M)) = \text{Hom}_R(A^* \otimes M, M) = \text{Hom}_R(M, \text{Hom}_R(A^*, M)) = \text{Hom}_R(M, A \otimes M)
\]

By these equivalences, if we have the morphism \( M \rightarrow A \otimes M, m \mapsto \sum_i a_i \otimes m_i \), then \( w \cdot m = \sum_i w(a_i) m_i \) given \( w \in A^\ast \). If \( w \) is the general linear form, i.e., \( w = \text{Id} \in A^\ast(A) = \text{Hom}_R(A, A) \), then \( w \cdot m = \sum_i a_i \otimes m_i \).

If \( A^\ast \) is an algebra scheme, then \( A \) is in a natural way a right and left \( A^\ast \)-module as it follows:

\[
(w \cdot a)(w') := a(w' \cdot w) \\
(a \cdot w)(w') := a(w \cdot w')
\]

where \( a \in A \), \( w, w' \in A^\ast \). We shall say that \( A \) is the regular \( A^\ast \)-module.

Given an \( R \)-submodule \( M' \subset M \) we will say that \( M' \rightarrow M \) is a quasicohherent submodule.
Lemma 4.5. Let $M_1, \ldots, M_n$ be projective $R$-modules and let $M_0$ be an $R$-module. The $R$-linear morphism $T : M_1^* \otimes_R \cdots \otimes_R M_n^* \to M_0$ factors via an epimorphism onto a coherent submodule of $M_0$.

Proof. As $M_1, \ldots, M_n$ are projective $R$-modules, they are direct summands of free modules $L_1, \ldots, L_n$. Then, $M_i^*$ is a direct summand of $L_i^*$ and we can assume that $M_i = L_i$ are free modules.

By Proposition 1.8, $\text{Hom}_R(M_1^* \otimes_R \cdots \otimes_R M_n^*, M_0) = M_1 \otimes_R \cdots \otimes_R M_n \otimes_R M_0$. Let $\{e_{ij}\}$ be a basis for $M_j$, for every $j$. Then for every $T \in \text{Hom}_R(M_1^* \otimes_R \cdots \otimes_R M_n^*, M_0)$ we can write

$$T = \sum_{i_1, \ldots, i_n} e_{i_1,1} \otimes \cdots \otimes e_{i_n,n} \otimes e_{i_1,\ldots,i_n}$$

where only a finite number of the elements $e_{i_1,\ldots,i_n} \in M_0$ are not null. It is easy to check that $T$ factors via an epimorphism onto the image of the coherent $R$-module associated to $M = \langle e_{i_1,\ldots,i_n} >_{i_1,\ldots,i_n}$.

Proposition 4.6. Let $\mathcal{A}^*$ be an $R$-algebra scheme, let $M$ be an $\mathcal{A}^*$-module and let $M' \subset M$ be an $R$-submodule. Let us suppose that $A$ is a projective $R$-module. Then, $M'$ is an $\mathcal{A}^*$-submodule of $M$ if and only if $M'$ is an $\mathcal{A}^*$-submodule of $M$.

Proof. Obviously, if $M'$ is an $\mathcal{A}^*$-submodule of $M$ then $M'$ is an $\mathcal{A}^*$-submodule of $M$. Inversely, let us suppose $M'$ is an $\mathcal{A}^*$-submodule of $M$ and let us consider the natural morphism of multiplication $\mathcal{A}^* \otimes M' \to M$. By the previous lemma the morphisms $\mathcal{A}^* \to M$, $w \mapsto w \cdot m'$, for each $m' \in M'$, factors via $M'$, then $\mathcal{A}^* \otimes M' = \mathcal{A}^* \otimes M' \to M$ factors via $M'$, which proves that $M'$ is an $\mathcal{A}^*$-submodule of $M$.

Proposition 4.7. Let $\mathcal{A}^*$ be an $R$-algebra scheme and let $M$ be an $\mathcal{A}^*$-module (respectively a right and left $\mathcal{A}^*$-module). Let us suppose $A$ is a projective $R$-module. Every finitely generated $R$-submodule of $M$ is included in an $\mathcal{A}^*$-submodule of $M$ (respectively a right and left $\mathcal{A}^*$-module) that is a finitely generated $R$-submodule.

Proof. Given a finitely generated $R$-module $M' \subset M$ then $A^* \cdot M'$ (respectively $A^* \cdot M' \cdot A^*$), the obvious image of the morphism $\mathcal{A}^* \otimes M' \to M$ (respectively $\mathcal{A}^* \otimes M' \otimes A^* \to M$), is an $\mathcal{A}^*$-submodule (respectively a right and left $\mathcal{A}^*$-submodule) of $M$ that is a finitely generated $R$-module.

Remark 4.8. In particular, an $\mathcal{A}^*$-module $M$ is a $K$-vector space of finite dimension if and only if is a finitely-generated $\mathcal{A}^*$-module, i.e., there exists an epimorphism of $\mathcal{A}^*$-modules $\mathcal{A}^* \otimes \mathcal{A}^* \to M$.

Definition 4.9. Let $\mathcal{A}^*$ be an $R$-algebra scheme. We will say that a submodule scheme $T^* \subseteq \mathcal{A}^*$ is an ideal scheme if it is an ideal subfunctor. We will say that $T^* \subseteq \mathcal{A}^*$ is a bilateral ideal scheme if it is a bilateral ideal subfunctor.

The kernel of a morphism of algebra schemes is a bilateral ideal scheme.

Definition 4.10. Given a finite $R$-algebra $B$, we will say that $B$ is a coherent $R$-algebra.

Remark 4.11. Owing to the categorial equivalence between the category of $R$-modules and the category of quasi-coherent $R$-modules, there is an obvious categorial equivalence between finite $R$-algebras and coherent $R$-algebras.
Proposition 4.12. Let $A^*$ be an $R$-algebra scheme. Let us suppose $A$ is a projective $R$-module. Then $A^*$ is an inverse limit of quotients $B_i$, which are coherent $R$-algebras.

Proof. $A$ is a direct limit of its finitely generated $R$-submodules $M_i \subset A$. Then by Proposition 4.7 it is a direct limit of its right and left $A^*$-submodules $N_i$ that are finitely generated $R$-modules.

The kernels of the morphisms $A^* \rightarrow N_i^*$ are bilateral ideal schemes $I_i^*$ of $A^*$. Let $R^* \rightarrow N_i$ an epimorphism of $R$-modules. The composite morphism $A^* \rightarrow N_i^* \leftarrow (R^*)^* = R$ factors via the epimorphism $A^* \rightarrow B_i$, where $B_i = A^*/I_i^*$ and it is a finite $R$-algebra, by Lemma 4.5. Dually, we obtain the morphisms $N_i^* \rightarrow B_i^* \hookrightarrow A$. Taking direct limit we obtain the sequence $A \rightarrow \lim_i B_i^* \hookrightarrow A$.

Hence, $\lim_i B_i^* = A$. Dually, $\lim_i B_i = A^*$.

5. Closure of an algebra functor.

Definition 5.1. Let $M$ be an $R$-algebra functor. We define $\hat{M}$ to be the representant on the category of $R$-algebra schemes, if it exists, of the functor $\text{Hom}_{R-\text{alg}}(M, -)$. I.e.,

$$\text{Hom}_{R-\text{alg}}(M, A^*) = \text{Hom}_{R-\text{alg}}(\hat{M}, A^*)$$

Notation 5.2. We will denote by $A^* \hat{\otimes} B^*$ the representant, on the category of $R$-algebra schemes, of the functor $\text{Hom}_{R-\text{alg}}(A^* \otimes B^*, -)$.

Then, we have that

$$\text{Hom}_{R-\text{alg}}(A^* \hat{\otimes} B^*, C^*) = \text{Hom}_{R-\text{alg}}(A^* \otimes B^*, C^*) = \text{Hom}_{R-\text{alg}}(A^* \hat{\otimes} B^*, C^*)$$

Therefore, $A^* \hat{\otimes} B^* = A^* \hat{\otimes} B^*$.

Proposition 5.3. If $M$ is an $R$-algebra functor such that $M^*$ is a quasi-coherent $R$-module, then $\hat{M} = \hat{M} \hat{\otimes} M^*$. Moreover, if $E$ is an $R$-module functor such that $N := E^*$ is an $R$-algebra functor; then

$$\text{Hom}_{R-\text{alg}}(M, N) = \text{Hom}_{R-\text{alg}}(\hat{M}, N)$$

Proof. By Lemma 2.6, Example 2.10 and Proposition 2.8 it holds for every $R$-module functor $N_1 := N_2$ that

$$\text{Hom}_R(M \otimes \ldots \otimes M, N_1) = \text{Hom}_R(N_2, M^* \otimes \ldots \otimes M^*) = \text{Hom}_R(\hat{M} \otimes \ldots \otimes \hat{M}, N_1)$$

If we consider $N_1 = M$, it follows easily that the structure of algebra of $M$ define a structure of algebra on $\hat{M}$. Finally, if we consider $N_1 = N$, we see at once that $\text{Hom}_{R-\text{alg}}(M, N) = \text{Hom}_{R-\text{alg}}(\hat{M}, N)$.

Remark 5.4. In particular,

1. If $G = \text{Spec} A$ is an $R$-group, then $\hat{R}G = \hat{R}G$.

2. If $A^*$ and $B^*$ are $R$-algebra schemes, then $A^* \hat{\otimes} B^* = A^* \hat{\otimes} B^*$.

Theorem 5.5. Let $G = \text{Spec} A$ be an $R$-group scheme. The category of $G$-modules is equal to the category of $A^*$-modules.
Proof. Let $M$ be an $R$-module. Let us observe that $\text{End}_R(M) = (M^* \otimes M)^*$. Therefore, by Proposition 5.3 and Theorem 3.3, (2), $\text{Hom}_{R_{\text{alg}}}(RG, \text{End}_R(M)) = \text{Hom}_{R_{\text{alg}}}(A^*, \text{End}_R(M))$. In conclusion, endowing $M$ with a structure of $G$-module is equivalent to endowing $M$ with structure of $A^*$-module.

Defining a morphism $RG \otimes M \to M$ is equivalent to defining a morphism $A^* \otimes M \to M$, because $\text{Hom}_R(RG \otimes M, M) = \text{Hom}_R(A^* \otimes M, M)$ by Lemma 2.6, since $(RG \otimes M)^* = (A^* \otimes M)^*$. Now it is easy to check that $\text{Hom}_{G_{\text{mod}}}(M, M') = \text{Hom}_{A^*_{\text{alg}}}(M, M')$.

\begin{proposition}
Let $G = \text{Spec } A$ be an $R$-group and let $G_m = \text{Hom}_R(R, R) \subset \text{End}_R(R)$. It holds that
\begin{align*}
\text{Hom}_{R_{\text{grp}}}(G, G_m) = \text{Hom}_{R_{\text{alg}}}(A^*, R)
\end{align*}
\end{proposition}

\begin{proposition}
[\text{W}, 3.3] Let $V$ be a $G$-module. Every vector subspace of $V$ of finite dimension is included in a $G$-submodule of $V$ of finite dimension.
\end{proposition}

Proof. It is a consequence of Proposition 4.7 \hfill \Box

\begin{proposition}
[\text{W}, 3.4] If $G = \text{Spec } A$ is an algebraic group then it is a subgroup of a linear group $G_{\text{lin}}$.
\end{proposition}

Proof. Let us consider the natural inclusion $G \hookrightarrow A^*$. By Proposition 4.12 we know that $A^* = \lim_{\leftarrow i} A_i^*$ is an inverse limit of finite quotient $K$-algebras. By the noetherianity of $G$, there exists an index $i$ such that the morphism $G \to A_i^*$ is injective. However, we have the natural injection $A_i^* \hookrightarrow \text{End}_K(A_i)$, then an injection $G \hookrightarrow \text{End}_K(A_i)$. \hfill \Box

In this section, from now on, $A$ will be an algebraic function such that the image of any $K$-linear morphism $A \to V$ is a quasi-coherent subspace of $V$, for example, if $A$ is a reflexive $K$-vector space functor.

\begin{theorem}
$\hat{A} = \lim_{\leftarrow i} A/\mathbb{I}_i$, where $\{\mathbb{I}_i\}_i$ is the set of bilateral ideal subfunctors of $A$ such that $A/\mathbb{I}_i$ is a coherent $K$-vector space.
\end{theorem}

Proof. Let us denote $\hat{A}' = \lim_{\leftarrow i} A/\mathbb{I}_i$. We must proof the functorial expression
\begin{align*}
\text{Hom}_{K_{\text{alg}}}(A, B^*) = \text{Hom}_{K_{\text{alg}}}(\hat{A}', B^*)
\end{align*}

First let us suppose that $B^*$ is a finite $K$-algebra scheme. Every morphism of $K$-algebra functors $A \to B^*$ has as kernel an $\mathbb{I}_i$, then it factors through $A/\mathbb{I}_i$, then through $A'$. Inversely, let us see that every morphism $A' \to B^*$ factors through $A/\mathbb{I}_i$:
\begin{align*}
\text{Hom}_K(A', B^*) = \text{Hom}_K(\lim_{\leftarrow i} A/\mathbb{I}_i, B^*) = \text{Hom}_K(B, \lim_{\leftarrow i} (A/\mathbb{I}_i)^*) \\
\cong \lim_{\leftarrow i} \text{Hom}_K(B, (A/\mathbb{I}_i)^*) = \lim_{\leftarrow i} \text{Hom}_K(A/\mathbb{I}_i, B^*)
\end{align*}

where $\cong$ holds because $B$ is a finite-dimensional $K$-vector space.
In the general case,
\[
\text{Hom}_{K-\text{alg}}(A, B^*) \cong \text{Hom}_{K-\text{alg}}(A, \lim_{i} B_i^*) = \lim_{i} \text{Hom}_{K-\text{alg}}(A, B_i^*) = \text{Hom}_{K-\text{alg}}(A', B^*)
\]

Proposition 5.10. Let $A$ be a $K$-algebra functor. Then,

1. The category of $K$-coherent $A$-modules is the same as the category of $K$-coherent $\tilde{A}$-modules.
2. The natural morphism $\bar{A} \to \tilde{A}$ is surjective.

Proof. (1) If $I_i \hookrightarrow A$ is a bilateral ideal functor such that $A/I_i$ is a coherent $K$-vector space, then the epimorphism $A \to A/I_i$ factors through $\tilde{A}$ and hence the morphism $\tilde{A} \to A/I_i$ is surjective.

If $\tilde{I}_i \hookrightarrow \tilde{A}$ is a bilateral ideal functor such that $\tilde{A}/\tilde{I}_i$ is a coherent $K$-vector space, then the image of the morphism $A \to \tilde{A}/\tilde{I}_i$ is an algebra scheme, therefore the induced morphism $\tilde{A} \to \tilde{A}/\tilde{I}_i$ values on that image. In conclusion, the morphism $A \to \tilde{A}/\tilde{I}_i$ is surjective.

Now (1) follows easily.

(2) By the last argument, the composite morphism $\bar{A} \to \tilde{A} \to \tilde{A}/\tilde{I}_i$ is surjective. The inverse limit of such surjections is surjective, because dually the direct limit of injections of quasi-coherent vector spaces is an injection. Then the morphism $\bar{A} \to \tilde{A}$ is surjective.

□

Theorem 5.11. Let $A$ be a $K$-algebra functor such that $\bar{A}$ is a $K$-algebra functor and $\bar{A} \to \tilde{A}$ is a morphism of $K$-algebra functors. Then $\bar{A} = \tilde{A}$.

Proof. The morphism of $K$-algebra functors $A \to \bar{A}$ factors through a morphism $i : \bar{A} \to A$. The morphism of $K$-algebra functors $A \to \tilde{A}$ is a $K$-linear morphism, then it factors through a morphism $j : \tilde{A} \to \bar{A}$. As $i \circ j : \tilde{A} \to \tilde{A}$ is the identity morphism on $A$, $i \circ j = \text{Id}$. Then the morphism $j$ is injective and, since it is surjective by the previous proposition, this proves that $\bar{A} = \tilde{A}$.

□

Definition 5.12. Let $A$ be a $K$-algebra functor. We call the $K$-vector space of distributions of finite support of $A$, and we denote it by $D$, the vector subspace $D \subseteq A^*(K)$ consisting of linear $1$-forms of $A$ that are null on some bilateral ideal of $A$ whose cokernel is a coherent $K$-vector space.

By Theorem 5.9, $\tilde{A}^* = D$, then $\tilde{A} = D^*$. It holds that
\[
\text{Hom}_{\text{coalg}}(B, D) = \text{Hom}_{K-\text{alg}}(D^*, B^*) = \text{Hom}_{K-\text{alg}}(A, B^*)
\]
for every coalgebra $B$.

Given a commutative $K$-algebra $A$ and a closed point $x \in \text{Spec} A$, if we consider it as an ideal of $A$ we will write $m_x$ for it.

Proposition 5.13. Let $A$ be a commutative $K$-algebra of finite type. It holds that
(1) $\hat{A} = \prod_{x \in \text{Spec}_{\text{max}} A} \hat{A}_x$, where $\hat{A}_x := \lim_{n} A/m_x^n$.

(2) The natural morphism $\hat{D} \to \hat{A}^*$ is surjective, where $D$ is the $K$-vector space of the distributions of finite support of $A$.

**Proof.**

(1) If $A/I$ is a finite $K$-algebra, then $\text{Spec}(A/I)$ correspond to a finite number of closed points of $\text{Spec} A$, $\{x_1, \ldots, x_n\}$, and there exists an $m \in \mathbb{N}$ such that $(m_{x_1} \cdot \ldots \cdot m_{x_n})^m \subset I$. Therefore,

$$\begin{align*}
\hat{A} &= \lim_{x_1, \ldots, x_n, m} A/(m_{x_1} \cdot \ldots \cdot m_{x_n})^m \\
&= \lim_{x_1, \ldots, x_n, m} A/m_{x_1}^m \times \ldots \times A/m_{x_n}^m = \prod_{x \in \text{Spec}_{\text{max}} A} \hat{A}_x
\end{align*}$$

(2) The morphism $\hat{D} \to \hat{A}^*$ is surjective if and only if the morphism $A \to D^*(K) = D^*$ is injective. By (1) this morphism is obviously injective. \hfill \Box

**Lemma 5.14.** Let $\phi : M_1 \to M_2$ be a morphism of vector space functors and let $\tilde{\phi} : \hat{M}_1 \to \hat{M}_2$ be the induced morphism on the vector space scheme closure. It holds that $\text{Coker} \tilde{\phi} = \text{Coker} \phi$ and $\phi(M_1)$ is the vector space scheme closure of the image of $M_1$ in $M_2$.

**Proof.** Obviously, $\tilde{\phi}(\hat{M}_1)$ is the same as the minimum vector space subscheme in $\hat{M}_2$ that contains the image of $\hat{M}_1$. It follows immediately from the functorial definition of Coker and the vector space scheme closure that $\text{Coker} \tilde{\phi} = \text{Coker} \phi$. \hfill \Box

**Notation 5.15.** In the next proposition, given $M \subset M^*$, we will denote by $M'$ the module scheme closure of $M$ in $M^*$.

**Proposition 5.16.** Let $I_1^*, \ldots, I_n^* \subseteq A^*$ be bilateral ideal schemes and let $M$ be an $A^*$-module. It holds that

(1) $I_1^* \cdot M$ is a quasi-coherent submodule of $M$.

(2) $I_1^* \cdot I_2^* \cdot M = (I_1^* \cdot I_2^*)' \cdot M$.

(3) $\{e \in M : I_1^* \cdot e = 0\}$ is a quasi-coherent submodule of $M$.

(4) $(M^* \cdot I_1^*, \ldots, I_n^*)'$ is an $A^*$-submodule of $M^*$ and to take the module scheme closure is stable under base change, i.e., given a morphism of rings $K \to B$, then $(M^* \cdot I_1^* \cdot \ldots \cdot I_n^*)'_B = (M^*|_B \cdot I_1^*|_B \cdot \ldots \cdot I_n^*|_B)'$. Therefore, $(M^*/(M^* \cdot I_1^* \cdot \ldots \cdot I_n^*))' = (M^*/(M^*|_B \cdot I_1^*|_B \cdot \ldots \cdot I_n^*))'$.

(5) $(M^* \cdot I_1^* \cdot \ldots \cdot I_n^*)' \cdot (I_{r+1}^* \cdot \ldots \cdot I_n^*)' = (M^* \cdot I_1^* \cdot \ldots \cdot I_n^*)'$.

**Proof.**

(1) The image of the morphism of $A^*$-modules $I_1^* \otimes_K M \to M$ is a quasi-coherent $A^*$-submodule and it coincides with $I_1^* \cdot M$.

(2) It is enough to prove $I_1^* \cdot I_2^* \cdot e = (I_1^* \cdot I_2^*)' \cdot e$. Let us consider the commutative diagram

$$\begin{array}{ccc}
I_1^* \cdot I_2^* & \xrightarrow{e \cdot} & M \\
\downarrow & & \downarrow \\
A^* & \xrightarrow{e} & M
\end{array}$$
Let us consider the exact sequence
\[ \mathcal{M} \rightarrow \mathcal{I}_1 \otimes \mathcal{M} \]
We say a \( M \) is a simple right \( A \)-module, therefore of finite dimension, then \( V^* \) is a simple right \( A^* \)-module. Hence, for every \( w \in V^* \) not null, \( V^* = w \cdot A^* \). I.e., \( V^* \) is a quotient of \( A^* \), as a right \( A^* \)-module. Therefore, \( V \) is a submodule of \( A \), as left modules. Let us suppose now that \( V \) is not simple. The morphism of multiplication \( V^* \otimes A^* \rightarrow V^* \) is obviously surjective and it is of right \( A^* \)-modules, where \( A^* \) acts on \( V^* \otimes A^* \) by the second factor (on the right). Taking duals we have the desired injection \( \mathcal{V} \rightarrow A \otimes V = \oplus A \).

**Notation 5.17.** From now on, when we are in the context of algebra schemes and vector spaces, given a bilateral ideal scheme \( I^* \subset A^* \) and a right \( A^* \)-module \( \mathcal{M} \) we will understand by \( \mathcal{M}^* \cdot I^* \) its module scheme closure \( (\mathcal{M}^* \cdot I^*)' \) in the vector space scheme \( \mathcal{M}^* \).

**6. MAXIMAL QUOTIENT SEMISIMPLE ALGEBRA SCHEME.**

**Definition 6.1.** We say a \( K \)-algebra scheme \( A^* \) is simple if it does not contain any proper bilateral ideal. We say that an \( A^* \)-module \( V \neq 0 \) is simple if it does not contain any proper \( A^* \)-submodule. We say that an \( A^* \)-module \( V \) is semisimple if it is a sum of simple \( A^* \)-modules.

An \( A^* \)-module \( V \) is semisimple if and only if it is a direct sum of simple \( A^* \)-modules.

By Proposition 4.7, the simple \( A^* \)-modules are \( K \)-vector spaces of finite dimension.

**Theorem 6.2.** \( A^* \) is simple if and only if it is isomorphic to the endomorphism ring of a finite-dimensional vector space over a non-commutative field of finite degree.

**Proof.** If \( A^* \) is simple, by Proposition 4.12 \( A^* \) is a finite \( K \)-algebra scheme. Now, this theorem is a consequence of Wedderburn Theorem ([P, 3.5]).

**Theorem 6.3.** Every simple \( A^* \)-module is an \( A^* \)-submodule of the regular module. Every \( A^* \)-module is a submodule of a direct sum of regular modules.

**Proof.** If \( V \) is a simple left \( A^* \)-module, therefore of finite dimension, then \( V^* \) is a simple right \( A^* \)-module. Hence, for every \( w \in V^* \) not null, \( V^* = w \cdot A^* \). I.e., \( V^* \) is a quotient of \( A^* \), as a right \( A^* \)-module. Therefore, \( V \) is a submodule of \( A \), as left modules. Let us suppose now that \( V \) is not simple. The morphism of multiplication \( V^* \otimes A^* \rightarrow V^* \) is obviously surjective and it is of right \( A^* \)-modules, where \( A^* \) acts on \( V^* \otimes A^* \) by the second factor (on the right). Taking duals we have the desired injection \( V \rightarrow A \otimes V = \oplus A \).

**Corollary 6.4.** ([W, 3.5]) Every simple \( G \)-module is a \( G \)-submodule of the regular \( G \)-module. Every \( G \)-module is a \( G \)-submodule of a direct sum of regular modules.
Definition 6.5. We say that a $K$-algebra scheme $A^*$ is a semisimple $K$-algebra scheme if every quasi-coherent $A^*$-module is semisimple.

Proposition 6.6. $A^*$ is a semisimple algebra scheme if and only if $A$ is a semisimple $A^*$-module.

Proof. If $A^*$ is a semisimple algebra scheme then in particular $A$ is a semisimple $A^*$-module. Inversely, if $A$ is a semisimple $A^*$-module, as by Proposition 6.3 every $A^*$-module $V$ is a submodule of a direct sum of $A$'s, that is semisimple, we have that $V$ is semisimple. Then $A^*$ is a semisimple algebra scheme. □

Definition 6.7. A bilateral ideal scheme $I^* \subseteq A^*$ is said to be a maximal bilateral ideal scheme if $A^*/I^*$ is simple. We shall call maximal spectrum of $A^*$ the set of its maximal bilateral ideal schemes, which we will denote by $\text{Spec}_{\text{max}} A^*$.

If $A^* = A^*_1 \times A^*_2$, then
\[
\text{Spec}_{\text{max}} A^* = \text{Spec}_{\text{max}} A^*_1 \cup \text{Spec}_{\text{max}} A^*_2
\]

because every bilateral ideal scheme $I^* \subseteq A^*$ is $I^* = I^*_1 \times I^*_2$, where $I^*_i$ is a bilateral ideal scheme of $A^*_i$. Therefore, every epimorphism from a product of two $K$-algebra schemes to a simple $K$-algebra scheme factors through the projection on one of the two factors. If $A^*_i$, $B^*_i$ are simple $K$-algebras and $\phi : A^*_1 \times \ldots \times A^*_r \to B^*_1 \times \ldots \times B^*_s$ is an epimorphism, then there exist isomorphisms $\phi_j : A^*_i \to B^*_j$ ($i_j \neq i_k$, if $j \neq k$) such that $\phi(a_1, \ldots, a_r) = (\phi_1(a_{i_1}), \ldots, \phi_s(a_{i_s}))$.

Theorem 6.8. $A^*$ is a semisimple $K$-algebra scheme if and only if it is a direct product of simple $K$-algebras.

Proof. Let us suppose that $A^*$ is a semisimple algebra scheme. We know that $A^*$ is an inverse limit of quotients $A^*_i$ which are finite $K$-algebras. Obviously, the $A^*_i$-modules are $A^*$-modules, then $A^*_i$ is a semisimple algebra. By the theory of semisimple rings, $A^*_i$ is a direct product of simple finite $K$-algebras, therefore $A^*$ is a direct product of simple finite $K$-algebras. □

Proposition 6.9. Every $A^*$-module $V \neq 0$ contains an only maximal semisimple $A^*$-submodule not null.

Proof. The maximal semisimple submodule is the sum of every semisimple submodule. As well there exist simple submodules, since given $0 \neq e \in V$, this $e$ is contained in a finite-dimensional $A^*$-module, which contains simple $A^*$-submodules. □

Proposition 6.10. The dual of the maximal semisimple submodule of $A$ is the maximal semisimple quotient algebra scheme of $A^*$, i.e., any other semisimple quotient $K$-algebra scheme of $A^*$ is a quotient of this one.

Proof. Let $A_M \subseteq A$ be the maximal semisimple submodule. Let us see that it is bilateral. We must prove that it is a right $A^*$-module. Given $w \in A^*$, it is clear that $A_M \cdot w$ is a left $A^*$-submodule of $A$. Then it is a left $A^*$-submodule of $A$. It is also clear that it is semisimple, then $A_M \cdot w \subseteq A_M$. Hence $A_M$ is a right $A^*$-submodule of $A$, then it is a right $A^*$-submodule of $A$.

Moreover, the counit $w : A \to K$ (i.e., the unit of $A^*$) is not null on the whole $A_M$: if $m(w) = w(m) = 0$ for every $m \in A_M$, then $0 = (m \cdot w')(w) = m(w' \cdot w) = m(w')$ for every $w' \in A^*$ and $m = 0$, then $A_M = 0$, a contradiction.
Therefore, \( A^*_M \) is a \( \mathcal{K} \)-algebra scheme. \( A_M \) as an \( A^*_M \)-module is semisimple because as an \( A^* \)-module it is semisimple. Hence, by Proposition 6.6, \( A^*_M \) is a semisimple \( \mathcal{K} \)-algebra scheme. If \( B^* \) is a semisimple quotient of \( A^* \) then \( B \) is a \( B^* \)-semisimple module, then it is a semisimple \( A^* \)-submodule of \( A \). Therefore \( B \subseteq A_M \) and \( B^* \) is a quotient of \( A^*_M \).

**Notation 6.11.** We will denote by \( A^*_M \) the maximal semisimple quotient algebra scheme of \( A^* \).

If \( V \) is a simple \( A^* \)-module then the image of the natural morphism \( A^* \twoheadrightarrow \text{End}_K(V) \) is a simple \( \mathcal{K} \)-algebra scheme, then it is a quotient of \( A^*_M \). Then \( V \) is an \( A^*_M \)-module. If \( V \) is a semisimple \( A^* \)-module then it is a semisimple \( A^*_M \)-module. Obviously,

\[
\text{Spec}_{\text{max}} A^* = \text{Spec}_{\text{max}} A^*_M = \{ \text{Set of isomorphism classes of simple } A^*-\text{modules} \}
\]

**Definition 6.12.** We call the radical (ideal) of a \( \mathcal{K} \)-algebra scheme the kernel of the quotient morphism from the algebra scheme to its maximal semisimple quotient algebra scheme.

Let \( V \) be an \( A^* \)-module and let \( \mathcal{I}^* \) be the radical of \( A^* \). \( V \) is semisimple if and only if it is an \( A^*_M \)-module, i.e., if it is cancelled by \( \mathcal{I}^* \). If \( 0 \neq V_1 \subseteq V \) is the maximal semisimple \( A^* \)-submodule of \( V \), then

\[
V_1 = \{ e \in V : \mathcal{I}^* \cdot e = 0 \}
\]

or equivalently, \( V_1 = \{ e \in V : \mathcal{I}^*(\mathcal{K}) \cdot e = 0 \} \).

**Proposition 6.13.** Let \( V \) be an \( A^* \)-module and let \( \mathcal{I}^* \) be the radical of \( A^* \). Let \( V_1 \) be the maximal semisimple submodule of \( V \), then

\[
V_1 = (V^* \otimes_{A^*} A^*_M)^* = (V^*/V^* \cdot \mathcal{I}^*)^*
\]

**Proof.** By base change it is enough to prove that \( V_1 = \text{Hom}_{\mathcal{K}}(V^*/V^* \cdot \mathcal{I}^*, \mathcal{K}) \).

However, \( \text{Hom}_{\mathcal{K}}(V^*/V^* \cdot \mathcal{I}^*, \mathcal{K}) \) identifies with the vectors \( e \in \text{Hom}_{\mathcal{K}}(V^*, \mathcal{K}) = V \) such that \( e(V^* \cdot \mathcal{I}^*) = 0 \). As \( e(w \cdot i) = w(i \cdot e) \) for every \( w \in V^* \) and \( i \in \mathcal{I}^* \), it follows that \( e \in V \) holds that \( e(V^* \cdot \mathcal{I}^*) = 0 \) if and only if \( e \in V_1 \). \( \square \)

The functor \( F(V) := V_1 \) from the category of \( A^* \)-modules to the category of \( A^*_M \)-modules is a left exact functor represented by \( A^*_M \), because

\[
F(V) = V_1 = \text{Hom}_{A^*}(A^*_M, V)
\]

Let us consider the quotient \( V' = V/V_1 \) and \( V'_1 \) the maximal semisimple \( A^* \)-submodule of \( V' \). Let \( V_2 := \pi^{-1}(V'_1) \), where \( \pi : V \to V' \) is the quotient morphism. Then \( V_1 \subseteq V_2 \) and \( V_2/V_1 = V'_1 \). So on we construct a canonical chain \( V_1 \subseteq V_2 \subseteq V_3 \subseteq \ldots \), such that every quotient \( V_i/V_{i+1} \) is a semisimple \( A^* \)-module and \( V_i/V_{i+1} = \{ \bar{e} \in V/V_{i+1} : \mathcal{I}^* \cdot \bar{e} = 0 \} \). Inductively we deduce that

\[
V_i = \{ e \in V : \mathcal{I}^{*i} \cdot e = 0 \}
\]

Again, as in Proposition 6.13, we obtain that

\[
V_i = (V^* \otimes_{A^*} A^*/\mathcal{I}^{*i})^* = (V^*/V^* \cdot \mathcal{I}^{*i})^*
\]

**Notation 6.14.** Given an \( A^* \)-module \( V \), we will denote by \( V_1 \subseteq V_2 \subseteq \ldots \) the canonical chain of \( A^* \)-submodules of \( V \) we have just constructed. We will denote

\[
G(V) := \bigoplus_{i=1}^{\infty} V_i/V_{i-1}, \text{ where } V_0 = 0 \text{ and } G_{\mathcal{I}^*}V^* := \prod_{i=1}^{\infty} (V^*/V^* \cdot \mathcal{I}^{*i})^*.
\]
Proposition 6.15. Let $V$ be an $A^*$-module. Then
\[(G_V^*)^* = G(V)\]

In case of the regular $A^*$-module $A$, the canonical chain of semisimple factors is $A_1 \subset A_2 \subset \ldots \subset A$ where $A_i = (A^*/I^*)^*$. 

Lemma 6.16. Let $V$ be a finitely-generated $A^*$-module and let $I^*$ be the radical of $A^*$. There exists an $n > 0$ such that $I^* \cdot V^n = 0$.

Proof. In the natural chain $V_1 \subseteq V_2 \subseteq \ldots$ of $V$, an inclusion $V_n \subseteq V_{n+1}$ is an equality when $V_n = V$ by Proposition 6.9. Because $V$ is of finite dimension the equality $V = V_n$ must be true for some $n \in \mathbb{N}$. Therefore $I^* \cdot V^n = 0$. □

Theorem 6.17. Let $V$ be an $A^*$-module and let $I^*$ be the radical of $A^*$. It holds that
\[
(1) \quad V = \lim_{\rightarrow} V_i \\
(2) \quad V^* = \lim_{\leftarrow} V^*/V^* \cdot I^* \cdot V^n = 0.
\]

Proof.

(1) Every $e \in V$ is included in a finite-dimensional $A^*$-submodule $V'$ of $V$. Therefore, there exists an $n \in \mathbb{N}$ such that $I^* \cdot V^n = 0$. Then $V = \lim_{\rightarrow} V_i$.

(2) As $V = \lim_{\rightarrow} V_i$, taking duals and remembering that $V_i = (V^*/V^* \cdot I^*)^*$, it follows that $V^* = \lim_{\leftarrow} V^*/V^* \cdot I^*$. □

Proposition 6.18. (Nakayama) Let $V, V'$ be $A^*$-modules and let $A^*_M$ be the maximal semisimple quotient algebra scheme of $A^*$.

(1) $V^* = 0 \iff V^* \otimes A^*. A^*_M = 0$.
(2) A morphism of $A^*$-modules $V \rightarrow V'$ is surjective $\iff$ the morphism $V^* \otimes A^*. A^*_M \rightarrow V'^* \otimes A^*. A^*_M$ is surjective.
(3) The morphism $V^* \rightarrow V'^*$ is an isomorphism $\iff$ the morphism $G_V^* \rightarrow G_V'^*$ is an isomorphism.

Proof.

(1) If $V^* \otimes A^*. A^*_M = 0$ then $V_i = (V^* \otimes A^*. A^*_M)^* = 0$, then $V = 0$ and $V^* = 0$.
(2) (1) must be applied to the cokernels.
(3) If the morphism $G_V^* \rightarrow G_V'^*$ is an isomorphism, then so is the morphism between the completions, which coincide with the vector space schemes themselves by Theorem 6.17 (2). □

If $V$ is an $A^*$-module of finite dimension we define the character associated to $V$, $\chi_V : A^* \rightarrow K$, to be
\[\chi_V(w) := \text{tr} h_w\]
where $h_w$ is the homotethy on $V$ of factor $w \in A^*$ and $\text{tr} h_w$ is the trace of such linear endomorphism. But the trace of $h_w : V \rightarrow V$ is the same as the trace of the
induced endomorphism  \( h_v : G(V) := \bigoplus_i V_i/V_i-1 \to \bigoplus_i V_i/V_i-1 =: G(V) \). So we have the commutative diagram

\[
\begin{array}{ccc}
\mathcal{A}^* & \xrightarrow{\chi_V} & K \\
\downarrow{\chi_{\oplus V_i/V_i-1}} \quad & & \quad \downarrow{\chi_{\oplus \bar{V}_i/V_i-1}} \\
\mathcal{A}_M^* \\
\end{array}
\]

Proposition 6.19. Let us suppose that \( \text{car} \ K = 0 \). Then \( \chi_V = \chi_{\bar{V}} \) if and only if \( G(V) \) and \( G(V') \) are isomorphic \( \mathcal{A}_M^* \)-modules.

Proposition 6.20. Let \( K \) be an algebraically closed field. The characters associated to the simple \( \mathcal{A}^* \)-modules are linearly independent.

Notation 6.21. Let \( V^* \) be an \( \mathcal{R} \)-module scheme (respectively \( \mathcal{R} \)-algebra scheme) and let \( R \to S \) be an extension of commutative rings. We will denote by \( V^*_S \) the \( S \)-module scheme (respectively \( S \)-algebra scheme) \( V^*_{|S} = (V \otimes_R S)^* \) associated to the \( S \)-module \( V \otimes_R S \). We will say that \( V^* \) is \( V^*_S \) under the base change \( R \to S \).

Proposition 6.22. Let \( \mathcal{A}^* \) be an algebra scheme, let \( \mathcal{A}_M^* \) be its maximal semisimple quotient algebra scheme and let \( K \to K' \) be an extension of commutative fields. The maximal semisimple quotient \( K' \)-algebra scheme of \( \mathcal{A}^*_K \) is a quotient of \( (\mathcal{A}^*_M)_{K'} \). If \( K \) is algebraically closed, then the maximal semisimple quotient \( K' \)-algebra scheme of \( \mathcal{A}^*_K \) is the same as \( (\mathcal{A}^*_M)_{K'} \).

Proof. Let \( 0 \subset A_1 \subset A_2 \subset \ldots \) be the canonical filtration of \( \mathcal{A}^* \)-modules of \( A \). Let us consider the filtration \( A_1 \otimes_K K' \subset A_2 \otimes_K K' \subset \ldots \) of \( A \otimes_K K' \). If \( V \) is a simple \( A_{\mathcal{A}^*_M} \)-module, then it injects into \( \mathcal{A} \otimes_K K' \) and for some \( i \) there exists an injection \( V \hookrightarrow (A_i \otimes_K K')/(A_{i-1} \otimes_K K') \) of \( \mathcal{A}^*_M \)-modules. However, \( (A_i \otimes_K K')/(A_{i-1} \otimes_K K') \) is an \( (\mathcal{A}^*_M)_{K'} \)-module, then \( V \) is an \( (\mathcal{A}^*_M)_{K'} \)-module. In conclusion, every morphism from \( \mathcal{A}^*_K \) to a simple algebra factors through \( (\mathcal{A}^*_M)_{K'} \). Therefore, the maximal semisimple quotient \( K' \)-algebra scheme of \( \mathcal{A}^*_K \) is a quotient of \( (\mathcal{A}^*_M)_{K'} \).

If \( K \) is algebraically closed then \( \mathcal{A}^*_M = \prod_i M_n(K) \), then \( (\mathcal{A}^*_M)_{K'} = \prod_i M_n(K') \) is semisimple and the maximal semisimple quotient \( K' \)-algebra scheme of \( \mathcal{A}^*_K \) is isomorphic to \( (\mathcal{A}^*_M)_{K'} \).

\( \square \)

Proposition 6.23. Let \( \mathcal{A}_M^* \) be the maximal semisimple quotient algebra scheme of \( \mathcal{A}^* \) and let \( \mathcal{B}_M^* \) be the maximal semisimple quotient algebra scheme of \( \mathcal{B}^* \). Then the maximal semisimple quotient algebra scheme of \( \mathcal{A}^* \otimes \mathcal{B}^* \) is a quotient of \( \mathcal{A}_M^* \otimes \mathcal{B}_M^* \).

If \( k \) is algebraically closed, then \( \mathcal{A}_M^* \otimes \mathcal{B}_M^* \) is the maximal semisimple quotient algebra scheme of \( \mathcal{A}^* \otimes \mathcal{B}^* \) and an \( \mathcal{A}^* \otimes \mathcal{B}^* \)-module is simple if and only if it is a tensorial product of a simple \( \mathcal{A}^* \)-module and a simple \( \mathcal{B}^* \)-module.

Proof. Let \( V \) be a simple \( \mathcal{A}^* \otimes \mathcal{B}^* \)-module. In particular \( \text{dim}_k V < \infty \). Let us consider the canonical chain \( V_1 \subset V_2 \subset \ldots \subset V_n = V \) of \( \mathcal{A}^* \)-modules. As \( \mathcal{B}^* \) commutes with \( \mathcal{A}^* \) in \( \mathcal{A}^* \otimes \mathcal{B}^* \), then it has to leave stable the chain. Since \( V \) is simple, then \( V = V_1 \), i.e., \( V \) is a semisimple \( \mathcal{A}^* \)-module. Likewise, \( V \) is a semisimple \( \mathcal{B}^* \)-module. In conclusion, \( V \) is an \( \mathcal{A}_M^* \otimes \mathcal{B}_M^* \)-module, then it is an \( \mathcal{A}_M^* \otimes \mathcal{B}_M^* \)-module. Therefore, every morphism from \( \mathcal{A}^* \otimes \mathcal{B}^* \) to a simple algebra scheme factors through \( \mathcal{A}_M^* \otimes \mathcal{B}_M^* \), then the maximal semisimple quotient algebra scheme of \( \mathcal{A}^* \otimes \mathcal{B}^* \) is a quotient of \( \mathcal{A}_M^* \otimes \mathcal{B}_M^* \).
Let $K$ be an algebraically closed field. As $\text{End}_K(V) \otimes_K \text{End}_K(V') = \text{End}_K(V \otimes_K V')$ $(\dim_K V, V' < \infty)$, $A^*_M = \prod_i \text{End}_{\mathcal{K}}(V_i)$ and $B^*_M = \prod_j \text{End}_K(V'_j)$, then

$$A^*_M \otimes B^*_M = \prod_{i,j} \text{End}_K(V_i \otimes V'_j)$$

\[\square\]

**Corollary 6.24.** Let $K$ be an algebraically closed field. $A^*$, $B^*$ are semisimple $\mathcal{K}$-algebra schemes if and only if $A^* \otimes B^*$ is semisimple.

Let us give some final examples of the application of the representation theory of algebra schemes to the representation theory of algebraic groups.

Let $G = \text{Spec } A$ be an algebraic group. It is easy to prove that $G$ is unipotent if and only if $A^*$ is local (i.e., it only contains one bilateral ideal scheme) and that $G$ is triangulable if and only if $A^*$ is basic (i.e., $A^*_M = \prod K$). It is also easy to prove that subschemes and quotients of a basic algebra scheme (respectively local and basic) are basic (respectively local and basic).

**Corollary 6.25.** Subgroups, quotients, direct products of triangulable (respectively unipotent) groups are triangulable (respectively unipotent) groups.

We shall say that $X \subset A^*$ is a dense subset in $A^*$ if the minimum vector space subscheme of $A^*$ that contains $X$ is $A^*$. Dually, $X$ is dense in $A^*$ if the only $a \in A$ such that $a(x) = 0$ for all $x \in X$ is $a = 0$.

**Proposition 6.26.** Let $K$ be an algebraically closed field, let $\chi : A^* \rightarrow \mathcal{K}$ be a morphism of functors of $\mathcal{K}$-algebras and let $X \subset A^*$ be a dense subset in $A^*$. $A^*$ is local if and only if any of the following conditions holds:

1. For every $x \in X$, $x - \chi(x)$ belongs to the radical of $A^*$.
2. For every $x \in X$ and every morphism of $\mathcal{K}$-algebra functors $\phi : A^* \rightarrow \text{End}_K(V)$, where $\dim_K V < \infty$, $\phi(x - \chi(x))$ is nilpotent.

**Proof.**

1. If $A^*$ is local then $x - \chi(x)$ belongs to the kernel of $\chi$, which is the radical of $A^*$. Let us see the inverse. Let $V$ be a simple $A^*$-module and let $\phi : A^* \rightarrow \text{End}_K(V)$ be the natural epimorphism. Given an $x \in X$, as $x - \chi(x)$ belongs to the radical of $A^*$ it holds that $\phi(x - \chi(x))$ is nilpotent, then $\chi_V(x - \chi(x)) = 0$. Therefore $\chi_V(x) = n\chi(x)$, where $n = \dim_K V$. Then $\chi_V = n\chi$ because $X$ is dense in $A^*$. From here it follows that $V = K$ and $\chi_V = \chi$. In conclusion, $A^*$ is local.
2. The proof above works here.

\[\square\]

Let $\chi : A^* \rightarrow \mathcal{K}$ be a morphism of $\mathcal{K}$-algebra schemes. An $A^*$-module $V$ is called $\chi$-unipotent if there exists a filtration of $A^*$-modules $0 \subset V'_1 \subset V'_2 \subset \cdots \subset V$ such that $A^*$ operates on $V'_i/V'_{i-1}$ via $\chi$ for all $i$ (and $V = \bigcup_i V'_i$). Then $A^*$ is local if and only if $A$ is $\chi$-unipotent. If $V' \rightarrow V$ is an epimorphism of $A^*$-modules and $V'$ is $\chi$-unipotent then $V$ is $\chi$-unipotent. If $V$ is $\chi$-unipotent then $V \otimes_K \mathcal{K} \otimes_K V$ is $\chi$-unipotent.
Corollary 6.27. Let \( K \) be an algebraically closed field and let \( G \subseteq \text{GL}_n \) be an integral algebraic group. \( G \) is unipotent if and only if every closed point \( g \in G \) is a unipotent matrix, i.e. \( g - \text{id} \) is nilpotent.

7. Separable algebra schemes.


Definition 7.1. We call a \( K \)-algebra scheme \( \mathcal{A}^* \) separable if and only if under every base change \( K \hookrightarrow K' \), \( \mathcal{A}^*_{K'} := (\mathcal{A} \otimes_K K')^* \) is a semisimple \( K' \)-algebra scheme.

Definition 7.2. Let \( \mathcal{A} \) be a \( K \)-algebra functor. We will call centre of \( \mathcal{A} \) the \( K \)-algebra subfunctor of \( \mathcal{A} \), that we denote by \( Z(\mathcal{A}) \), defined by

\[
Z(\mathcal{A})(S) := \{ a \in \mathcal{A}(S) \mid a = a \}
\]

where \( a : \mathcal{A}|_S \rightarrow \mathcal{A}|_S, \ b \mapsto a \cdot b \) and \( a : \mathcal{A}|_S \rightarrow \mathcal{A}|_S, \ b \mapsto b \cdot a \).

It holds that \( Z(\mathcal{A}^*)(S) = \{ w \in \mathcal{A}^*(S) \mid \mathcal{A}^*_S \overset{\cdot w}{\rightarrow} A^*_S \} \) coincides with the centre of the \( S \)-algebra \( (\mathcal{A} \otimes_K S)^* = \mathcal{A}^*(S) \).

\( Z(\mathcal{A}^*) \) is a \( K \)-algebra scheme: \( Z(\mathcal{A}^*) \) is the kernel of the morphism \( \phi : \mathcal{A}^* \rightarrow \text{End}_K(\mathcal{A}), \ w \mapsto w \cdot w \cdot w, \) and \( \text{End}_K(\mathcal{A}) \) is included in the \( K \)-vector space scheme \( \text{Hom}_K(\mathcal{A}, \mathcal{A}) \).

It holds that \( Z(\mathcal{A}^* \otimes B^*) = Z(\mathcal{A}^*) \otimes Z(B^*) \) is a \( K \)-vector space scheme isomorphic to \( \mathcal{A}^* \otimes B^* \) if \( \mathcal{A}^* \otimes B^* \) is a right and left \( \mathcal{A}^*-\text{module} \) isomorphic to \( \prod \mathcal{A}^* \). Now it is easily seen that \( Z(\mathcal{A}^*) \otimes B^* \subseteq Z(\mathcal{A}^*) \otimes Z(B^*). \) Hence, \( Z(\mathcal{A}^* \otimes B^*) \subseteq Z(\mathcal{A}^*) \otimes Z(B^*). \)

Notation 7.3. Given a ring \( (R, +, \cdot) \) we denote by \( (R^0, +, \cdot) \) the ring that is the same as \( R \) as a set, with the same addition and whose product \( \cdot \) is defined by \( a \cdot b := b \cdot a \).

Theorem 7.4. Let \( \mathcal{A}^* \) be a \( K \)-algebra scheme. The next conditions are equivalent:

1) \( \mathcal{A}^* \) is separable.
2) \( \mathcal{A}^*_{\bar{K}} \) is a direct product of algebras of matrices, where \( \bar{K} \) is an algebraically closed field.
3) \( \mathcal{A}^* \) is semisimple and its centre is a separable algebra scheme.
4) \( \mathcal{A}^* \otimes_K \mathcal{A}^{0^*} \) is semisimple.

Proof. 1) \( \Rightarrow \) 2) It is obvious.

2) \( \Rightarrow \) 3) \( Z(\mathcal{A}^*)_{\bar{K}} = Z(\mathcal{A}^*_{\bar{K}}) = \prod \bar{K}, \) then \( Z(\mathcal{A}^*) \) is separable. Obviously \( \mathcal{A}^* \) is semisimple.

3) \( \Rightarrow \) 2) \( \mathcal{A}^* \) is a direct product of simple algebras. As the centre of a direct product is the direct product of the centres, we can assume that \( \mathcal{A}^* \) is simple, that is a finite \( K \)-algebra. We can write \( \mathcal{A}^* = \mathcal{A}^* \). In this case \( Z(\mathcal{A}^*) \) is a field, because \( A^* = \text{End}_{\mathcal{A}^*}(V) \) and \( Z(\mathcal{A}^*) = Z(K^*) \). Therefore, \( \mathcal{A}^* \) is an Azumaya \( Z(\mathcal{A}^*) \)-algebra and \( \mathcal{A}^* \otimes_K \bar{K} = \mathcal{A}^* \otimes_{Z(\mathcal{A}^*)} Z(\mathcal{A}^*) \otimes_K \bar{K} = \mathcal{A}^* \otimes_{Z(\mathcal{A}^*)} \prod \bar{K} \) which is a direct product of algebras of matrices.

2) \( \Rightarrow \) 4) It is enough to prove that \( \mathcal{A}^* \otimes_K \mathcal{A}^{0^*} \) under base change to the algebraic closure of \( K \) is semisimple. As the tensorial product of algebras of matrices is an algebra of matrices, (4) is proved.
4) \implies 3) Because \(Z(A^* \otimes A^{**}) = Z(A^*) \otimes Z(A^*)\) and since \(A^* \otimes A^{**}\) is a direct product of algebras of matrices (over algebras of division of finite degree), it follows that \(Z(A^*) \otimes Z(A^*)\) is a direct product of commutative fields (of finite degree) and, hence, \(Z(A^*)\) is a direct product of separable finite extensions of commutative fields of \(K\), then it is separable. By Proposition 6.23 \(A^*\) is semisimple. \(\square\)

**Lemma 7.5.** If \(A^*\) is a semisimple \(K\)-algebra scheme, every \(A^*\)-module scheme is injective and projective (in the category of \(A^*\)-module schemes).

**Proof.** Dually, we must prove that every \(A^*\)-module \(V\) is projective and injective. However, because \(A^*\) is semisimple every exact sequence of \(A^*\)-modules is split, which implies that every \(A^*\)-module \(V\) is projective and injective. \(\square\)

**Definition 7.6.** Let \(\mathcal{A}\) be a \(K\)-algebra functor. We shall say \(D \in \text{Hom}_K(\mathcal{A}, \mathcal{M})\) is a derivation from \(\mathcal{A}\) to an \(A \otimes A^{**}\)-module \(\mathcal{M}\) if \(D(ab) = (Da)b + a(Db)\), for every \(a, b \in \mathcal{A}\). We will denote by \(\text{Der}_K(\mathcal{A}, \mathcal{M})\) the set of all derivations from \(\mathcal{A}\) to \(\mathcal{M}\).

**Lemma 7.7.** [P, 11.5] Let \(\Delta_{\mathcal{A}}\) be the kernel of the morphism \(\mathcal{A} \otimes_K \mathcal{A} \to \mathcal{A}, \ a \otimes b \mapsto ab\). It holds that

\[
\text{Der}_K(\mathcal{A}, \mathcal{M}) = \text{Hom}_{A \otimes A^{**}}(\Delta_{\mathcal{A}}, \mathcal{M})
\]

**Notation 7.8.** Given a \(K\)-algebra scheme \(A^*\) let us denote by \(\bar{\Delta}_{A^*}\) the kernel of the morphism \(\bar{\Delta}_{A^*}^*: \mathcal{A} \otimes_{K} A^* \to A^*, \ a \otimes a' \mapsto aa'\). Let us observe that \(\bar{\Delta}_{A^*}\) is the \(K\)-module scheme closure of \(\Delta_{A^*}\), since \(A^* \otimes_K A^* = A^* \otimes \bar{\Delta}_{A^*}\).

**Proposition 7.9.** Let \(\pi: B^* \to A^*\) be an epimorphism of \(K\)-algebra schemes with kernel \(I^*\). Then the “sequence of differentials”

\[
0 \to I^*/I^{*2} \xrightarrow{d^*} \bar{\Delta}_{B^*} \otimes_{B^*} (A^* \otimes A^{**}) \to \bar{\Delta}_{A^*} \to 0
\]

is exact, where \(d^* := i \otimes 1 - 1 \otimes i\) for all \(i \in I^*\).

**Proof.** If we apply \(\text{Hom}_{A^* \otimes A^{**}}(-, \mathcal{V}^*)\) to the sequence of differentials we obtain the exact sequence

\[
0 \to \text{Der}_K(A^*, \mathcal{V}^*) \to \text{Der}_K(B^*, \mathcal{V}^*) \to \text{Hom}_{A^* \otimes A^{**}}(I^*, \mathcal{V}^*)
\]

Therefore, there is only left to prove that \(d\) is injective. Let \(s: A^* \to B^*\) be a section of \(K\)-vector space schemes of the epimorphism \(\pi: B^* \to A^*\). The map

\[
\bar{\Delta}_{B^*} \otimes_{B^*} (A^* \otimes A^{**}) \to I^*/I^{*2}, \ \sum_i b_i \otimes b'_i \mapsto \sum_i (b_i - s(\pi(b_i))) \cdot b'_i
\]

is a retraction of \(d\). \(\square\)

**Theorem 7.10.** \(A^*\) is a separable \(K\)-algebra scheme if and only if \(A^*\) is a projective \(A^* \otimes A^{**}\)-module.

**Proof.** If \(A^*\) is a separable \(K\)-algebra then \(A^* \otimes A^*\) is a semisimple algebra and every module scheme is projective.

Inversely, let us suppose that \(A^*\) is a projective \(A^* \otimes A^{**}\)-module. Therefore, the sequence

\[
0 \to \bar{\Delta}_{A^*} \to A^* \otimes A^{**} \to A^* \to 0 \quad (1)
\]

is split. Let \(\bar{K}\) be the algebraic closure of \(K\). For simplicity of notation we write \(A^*\) instead of \(\bar{A}^*_K\). Let \(\bar{A}^*_M\) be the maximal semisimple quotient scheme of \(A^*\) and let \(I^*\) be the radical of \(A^*\). If we apply \(- \otimes_{A^* \otimes A^{**}} \bar{A}^*_M \otimes \bar{A}^*_M\) to (1) we obtain that

\[
\bar{\Delta}_{A^*} \otimes_{A^* \otimes A^{**}} \bar{A}^*_M \otimes \bar{A}^*_M = \bar{\Delta}_{A^*_M}.
\]
From the exact sequence $0 \rightarrow T^* \rightarrow A^* \rightarrow A^*_M \rightarrow 0$ and the exact sequence of differentials from 7.9

$$0 \rightarrow T^*/I^2 \xrightarrow{d} \Delta_A^* \otimes_A^* \Delta_A^* \otimes_A^* (A^*_M \otimes_A^* A^*_M) \rightarrow \Delta_A^*_M \rightarrow 0$$

we obtain that $T^*/I^2 = 0$. Therefore, $T^* = I^2$ and by Theorem 6.17 $A^* = \lim_{\rightarrow n} A^*/I^* = A^*_M$ and $A^*$ is separable. □

Remark 7.11. In the proof we have seen that if (and only if) the sequence of $A^* \otimes A^{*n}$-modules $0 \rightarrow \Delta_A^* \rightarrow A^* \otimes A^{*n} \rightarrow A^* \rightarrow 0$ is split, then $A^*$ is a separable $K$-algebra scheme.

8. Extensions of algebra schemes.

In this section the cohomological arguments and descent theory are concisely used to give a proof of the Principal Theorem of Wedderburn-Malcev (see [P, 11.6] or [M, X, 3.2]) in the context of algebra schemes.

Proposition 8.1. Let $\mathbb{A}$ be a $K$-algebra functor, let $C_{\mathbb{A}-Mod}$ be the category of $\mathbb{A}$-modules and let $C_{Vect}$ be the category of $K$-vector space functors. The functor “forget the structure of $\mathbb{A}$-module” $\phi : C_{\mathbb{A}-Mod} \rightarrow C_{Vect}$, $M \mapsto \hat{M}$ has got an adjoint functor, which is $\text{Ad}(\phi) : C_{Vect} \rightarrow C_{\mathbb{A}-Mod}$, $M \mapsto \text{Hom}_K(A, M)$. I.e., if $M$ is an $\mathbb{A}$-module and $N$ is a $K$-vector space functor it holds that

$$\text{Hom}_K(M, N) = \text{Hom}_\mathbb{A}(\hat{M}, \text{Hom}_K(A, N)) \quad (2)$$

Let us denote $R^0 := \text{Ad}(\phi) \circ \phi$, i.e., $R^0(M) = \text{Hom}_K(A, \hat{M})$. The morphism $\text{Id} : \hat{M} \rightarrow M$ defines a natural morphism $\hat{M} \rightarrow R^0(M) = \text{Hom}_K(A, \hat{M})$ by the equation (2). If we apply $R^0$ to this morphism then we obtain a new morphism $\hat{M} \rightarrow R^0(R^0(M)) = \text{Hom}_K(A \otimes A, \hat{M})$ besides the natural one, and so on we will obtain the sequence

$$\hat{M} \rightarrow R^0(M) \Rightarrow R^0(R^0(M)) \ldots$$

Let us denote by $M \rightarrow R(M)$ this complex, which is exact: The identity morphism $\text{Id} : \text{Ad}(\phi)(N) \rightarrow \text{Ad}(\phi)(N)$ defines by adjointness a canonical morphism $(\phi \circ \text{Ad}(\phi))(N) \rightarrow N$, then we have canonical morphisms $\phi(R(M)) \rightarrow \phi(R^{-1}(M))$ that turn out to be some operators of homotopy of the complex $\phi(M) \rightarrow \phi(R(M))$. Therefore, this complex is homotopic to zero and $M \rightarrow R(M)$ is exact.

If we now consider $A = A^*$, then $R^0(V) = A \otimes V$ and it turns out to be an injective quasi-coherent $A^*$-module, because $\text{Hom}_{A^*}(-, A \otimes V) = \text{Hom}_K(-, V)$ is exact on the category of quasi-coherent $A^*$-modules. Therefore $R(V)$ is a resolution of $V$ by injective quasi-coherent $A^*$-modules.

Let $E$ be an extension of quasi-coherent $A^*$-modules of the quasi-coherent $A^*$-module $V$ by the quasi-coherent $A^*$-module $W$ (see [H, III, 1] or [K, 2.6] for the definition of extension of modules). The automorphisms of extensions of $E$ identifies with $\text{Hom}_{A^*}(V, W)$. If $W$ is an injective $A^*$-module then $E = W \oplus V$. By the standard arguments from the descent theory we get the following

Theorem 8.2. The extensions of $A^*$-modules of $V$ by $W$, modulo isomorphisms of extensions, are classified by the group $\text{Ext}_{A^*}(V, W)$.
Given a morphism of $K$-algebra functors $\chi : A^* \to K$ and a quasi-coherent $A^*$-module $V$ let us denote by $V^\chi := \{ e \in V \mid g \cdot e = e \forall g \in A^* \}$. We will denote by $H(\chi, V)$ the derived functors of the functor $V \mapsto V^\chi$. Let us notice that

$$V^\chi = \text{Hom}_{A^*}(K, V)$$

hence $H(\chi, V) = \text{Ext}_{A^*}(K, V)$.

Let $G = \text{Spec } A$ be a $K$-group. Given a $G$-module $V$ let $V^G = \{ e \in V \mid g \cdot e = e \forall g \in G \}$. $H(G, V)$ is defined as the derived functors of the functor $V \mapsto V^G$. The morphism $G \to \{1\}, g \mapsto 1$ defines a morphism $\chi : A^* \to K$ and $V^G = V^\chi$. By Theorem 5.5 next proposition holds.

**Proposition 8.3.** $H(G, V) = H(\chi, V) = \text{Ext}_{A^*}(K, V)$

**Corollary 8.4.** [S, 8.6] Let $G = \text{Spec } A$ be an algebraic group. The extensions of $G$-modules of $K$ by $V$ are classified by $H^1(G, V)$.

**Proof.** The extensions of $G$-modules of $K$ by $V$ are classified by $\text{Ext}_{A^*}^1(K, V) = H^1(G, V)$.

Dually, let us consider the category of right $A^*$-modules (i.e., left $A^{op}$-modules). Given a right $A^*$-module $V^*$, we will have the resolution by projective $A^*$-module schemes of $V^*$

$$\ldots V^* \otimes A^* \otimes A^* \rightleftarrows V^* \otimes A^* \otimes \ldots \otimes a_i \mapsto a_0 \otimes a_1 \otimes \ldots \otimes a_n$$

(3)

where the morphisms are $V^* \otimes A^* \otimes \ldots \otimes A^* \to V^* \otimes A^* \otimes \ldots \otimes A^*$, $a_0 \otimes a_1 \otimes \ldots \otimes a_n \mapsto a_0 \otimes a_1 \otimes \ldots \otimes a_n$ for $0 \leq i \leq n$ and $a_0 \in V^*$. So we have that

$$\text{Ext}_{A^*}^i(W, V) = \text{Ext}_{A^{op}}^i(V^*, W^*)$$

Let us suppose $V^* = A^*$, which is also a left $A^*$-module, i.e., precisely it is an $A^* \otimes A^{op}$-module. Then the equation (3) is a resolution of $A^*$ by projective $A^* \otimes A^{op}$-module schemes, which is split as a sequence of left $A^*$-modules. Given a morphism of $K$-algebras $\chi : A^* \to K$, every left $A^*$-module $W^*$ can be seen as an $A^* \otimes A^{op}$-module, where $A^*$ operates on the right by $\chi$. It holds that

$$\text{Ext}_{A^*}^i(A^*, W^*) = \text{Ext}_{A^{op}}^i(K, W^*)$$

Let us suppose $V$ is a $G$-vector space of finite dimension, then

$$H^i(G, V) = \text{Ext}_{A^*}^i(K, V) = \text{Ext}_{A^{op}}^i(A^*, V)$$

Let $B$ be a (singular) extension of algebras of an algebra $A$ by an $A \otimes K A^*$-module $M$ (see [M, X, 3] for the definition of extension of algebras). Giving an isomorphism of extensions of algebras from $B$ to the trivial extension $T = A \otimes M$ ($m_i, m_j = 0 \forall m_i, m_j \in M$) is equivalent to giving a $K$-derivation $D : B \to M$ such that $D$ on $M$ is the identity morphism. Let us suppose $M = \text{Hom}_K(A \otimes_k A^*, N)$. If we apply $\text{Hom}_{A^{op}}(-, M) = \text{Hom}_K(-, N)$ to the exact sequence of differentials

$$0 \to M \to \Delta_B \otimes B \to A \otimes A^* \to \Delta_A \to 0$$

associated to the exact sequence $0 \to M \to B \to A \to 0$, we obtain that the morphism $\text{Der}_K(B, M) \to \text{Hom}_{A^{op}}(M, M)$ is a surjection. Therefore there exist a derivation $D : B \to M$ such that on $M$ is the identity. In conclusion, if $M = \text{Hom}_K(A \otimes_k A^*, N)$ then $B$ is isomorphic to the trivial extension.
The extensions of $K$-algebra schemes of a $K$-algebra scheme $A^*$ by an $A^* \otimes A^{**}$-module scheme $V^*$ are classified by the group 

$$\text{Ext}^2_{A^* \otimes A^{**}}(A^*, V^*) = \text{Ext}^2_{A^* \otimes A^{**}}(V, A)$$

**Proof.** Let us follow the standard notation from the descent theory (see [W, 17]).

The extensions of algebra functors of $A^*$ by $V^*$ are classified by the group 

$$H^1(K\frac{R^0(V^*)}{V^*}, \text{Aut}_{alg.ext.}(A^* \otimes V^*)) = H^1(K\frac{R^0(V^*)}{V^*}, \text{Der}_K(A^*, -)) = H^1(K\frac{\text{Hom}_{A^* \otimes A^{**}}(\Delta_{A^*}, -)}{V^*}) = H^1(K\frac{\text{Hom}_{A^* \otimes A^{**}}(\Delta_{A^*}, R (V^*))}{\text{Hom}_{A^* \otimes A^{**}}(A^*, R (V^*))})$$

where $\approx$ follows from applying $\text{Hom}_{A^* \otimes A^{**}}(-, R (V^*))$ to the $K$-split exact sequence $0 \to \Delta_{A^*} \to A^* \otimes A^* \to A^* \to 0$ and later taking cohomology.

Finally, we have 

$$H^2(\text{Hom}_{A^* \otimes A^{**}}(A^*, R (V^*))) = \ldots = H^2(\text{Hom}_{A^* \otimes A^{**}}(V, R (A))) = \text{Ext}^2_{A^* \otimes A^{**}}(V, A).$$

Let $G = \text{Spec} A$ be a $K$-group and let $V$ be a $G$-vector space where $\dim_K V < \infty$.

**Definition 8.6.** [S, 8.2] Let us denote by $1 + V$ the algebraic group $\text{Spec} S(V^*)$, whose functor of points is $V$. Given an exact sequence of affine $K$-groups 

$$1 \to 1 + V \to G' \xrightarrow{\pi} G \to 1$$

such that $g' \cdot (1 + v) \cdot g'^{-1} = 1 + g(v)$ for all $g' \in G'$ and $v \in V$ (where $g = \pi(g')$) we shall say $G'$ is an extension of groups of $G$ by $V$.

The morphism $G \to 1$ induces the morphism of $K$-algebra schemes $\chi : A^* \to K$.

**Theorem 8.7.** The set of extensions of groups of $G$ by $V$, modulo isomorphisms, is equal to the set of extensions of algebras of $A^*$ by the $A^* \otimes A^{**}$-module $V$, modulo isomorphisms (where $A^*$ operates on $V$ on the right by $\chi$).

**Proof.** By [H, VI, 10.3] or [S, 8.8], the set of extension of groups of $G$ by $V$, modulo isomorphisms, is equal to $H^2(G, V) = \text{Ext}^2_{A^*}(K, V) = \text{Ext}^2_{A^* \otimes A^{**}}(A^*, V)$, which coincides with the set of extensions of algebras of $A^*$ by the $A^* \otimes A^{**}$-module $V$, modulo isomorphisms. 

Let us give explicitly the correspondence between the extensions of groups of $G = \text{Spec} A$ by $V$ and the extensions of algebras of $A^*$ by $V$.

Let $B^*$ be a (singular) extension of algebras of $A^*$ by $V$, i.e., we have the exact sequence 

$$0 \to V \to B^* \xrightarrow{\pi} A^* \to 0$$

where $V^2 = 0$. If we consider the inclusion $G \subset A^*$, then $\pi^{-1}(G)$ is an extension of groups of $G$ by $V \overset{\sim}{=} \pi^{-1}(1)$, where $L(v) := 1 + v$.

Inversely, let be an extension of groups $1 \to N \to G' \xrightarrow{\pi} G \to 1$ where $N = 1 + V$. Let us consider the inclusion $V \to KG^*$, $v \mapsto (1 + v) - 1$, where $(1 + v) \in N$. Let us denote $v = (1 + v) - 1 \in KG^*$. Let $1 = \langle \lambda g' \cdot v - \pi(g')(\lambda v) \rangle_{g' \in G^*, \lambda \in K, v \in V}$ be the bilateral ideal functor of $KG^*$. Then it holds that the kernel of the natural
epimorphism $K\mathcal{G}^* / \mathbb{I} \to K\mathcal{G}$ is $\mathcal{V}$. Taking closure, if we denote $B^* = \overline{K\mathcal{G}^* / \mathbb{I}}$, we have an extension of algebras

$$0 \to \mathcal{V} \to B^* \to A^* \to 0$$

Both assignments are mutually inverse.

Now let us generalize the Principal Theorem of Wedderburn-Malcev to algebra schemes.

**Theorem 8.8.** Let $K$ be an algebraically closed field, let $A^*$ be a $K$-algebra scheme, let $A^*_M$ be its maximal semisimple quotient scheme and let $I^*$ be the radical of $A^*$. The morphism $A^* \to A^*_M$ has a section of algebra functors, which is the only one up to conjugations by elements of $1 + I^*$.

**Proof.** $A^*_M$ is a semisimple algebra scheme, then it is a product of algebras of matrices. Therefore, $A^*_M \otimes A^*_M$ is a product of algebras of matrices, then it is semisimple. By Lemma 7.5 every extension of algebra schemes of $A^*_M$ by any $A^*_M \otimes A^*_M$-module scheme is trivial. $A^*/I^2$ is an extension of algebra schemes of $A^*_M$ by $I^*/I^2$, therefore, the epimorphism $\pi_2 : A^*/I^2 \to A^*_M$ has a section $s_2$. Let $\pi : A^*/I^3 \to A^*/I^2$ be the natural epimorphism and let $B^* = \pi^{-1}(s_2(A^*_M)) \subset A^*/I^3$. $B^*$ is an algebra scheme extension of $A^*_M$ by $I^*/I^3$, therefore, there exists a section $s' : A^*_M \to B^*$. As $B^* \subset A^*/I^3$, we have a morphism $s_3 : A^*_M \to A^*/I^3$. Acting this way, we finally obtain a commutative diagram of arrows

$$
\cdots \longrightarrow A^*/I^m \longrightarrow \cdots \longrightarrow A^*/I^3 \longrightarrow A^*/I^2 \longrightarrow A^*/I^1 \\
\downarrow s_n \quad \downarrow s_2 \quad \downarrow s_3 \\
A^*_M
$$

that defines the section $A^*_M \to A^*$ we looked for, since $A^* = \lim_{\to} A^*/I^*n$ by Theorem 6.17.

Let $s_1, s_2$ be two sections of algebra schemes of the epimorphism $A^* \to A^*_M$. The induced morphisms $\bar{s}_1, \bar{s}_2 : A^*_M \to A^*/I^2$ differ on an element of $\text{Der}_K(A^*_M, \bar{I}^*) = \text{Hom}_{A^*_M \otimes K, A^*_M}(\bar{\Delta}_{A^*_M}, \bar{I}^*)$, where $\bar{I}^* = I^*/I^2$. Moreover, the natural morphism

$$\bar{I}^*(K) = \text{Hom}_{A^*_M \otimes K, A^*_M}(\bar{A}^*_M \otimes \bar{A}^*_M, \bar{I}^*) \to \text{Hom}_{A^*_M \otimes K, A^*_M}(\bar{\Delta}_{A^*_M}, \bar{I}^*)$$

is surjective because $\text{Ext}^1_{A^*_M \otimes A^*_M}(A^*_M, \bar{I}^*) = 0$ by Theorem 7.10. In conclusion, there exists an $i_1 \in \bar{I}^*(K)$ such that $s_2(m) = (1 + i_1) \cdot \bar{s}_1(m) \cdot (1 + i_1)^{-1}$. Let $s_2'$ be the composite of $s_1$ with the automorphism of $A^*$ which is to conjugate by $1 + i_1$. The induced morphisms $\bar{s}_2, \bar{s}_2' : A^*_M \to A^*/I^3$ differ on an element of $\text{Der}_K(A^*_M, I^{2*}) = \text{Hom}_{A^*_M \otimes K, A^*_M}(\bar{\Delta}_{A^*_M}, I^{2*})$, where $I^{2*} = I^2/I^3$. But the natural morphism $I^{2*}(K) = \text{Hom}_{A^*_M \otimes K, A^*_M}(\bar{A}^*_M \otimes \bar{A}^*_M, I^{2*}) \to \text{Hom}_{A^*_M \otimes K, A^*_M}(\bar{\Delta}_{A^*_M}, I^{2*})$ is surjective. Therefore there exists an $i_2 \in I^{2*}(K)$ such that $s_2'$ is the composite of $s_2'$ and the automorphism of conjugation by $1 + i_2$. Then, modulo $I^{3*}, s_2$ is equal to the composite of $s_1$ and the conjugation by $1 + i_1 + i_2$. Arguing this way we obtain that $s_2$ is equal to the composite of $s_1$ and the conjugation by an element of $1 + I^*$. □
References


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