FUNCTORIAL CARTIER DUALITY

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Introduction

In this paper we obtain the Cartier duality for k-schemes of commutative monoids functorially without providing the vector spaces of functions with a topology (as in [DGr, Exposé VII_B by P. Gabriel, 2.2.1]), generalizing a result for finite commutative algebraic groups by M. Demazure & P. Gabriel ([DG, II, §1, 2.10]).

All functors we consider are functors defined over the category of commutative k-algebras. Given a k-module E, we denote by $\mathbf{E}$ the functor of k-modules $\mathbf{E}(B) := E \otimes_k B$. Given two functors of k-modules F and H, $\mathbf{Hom}_k(F, H)$ will denote the functor of k-modules

$$\mathbf{Hom}_k(F, H)(B) = \mathrm{Hom}_B(F_{|B}, H_{|B})$$

where $F_{|B}$ is the functor F restricted to the category of commutative B-algebras. We denote $F^* := \mathbf{Hom}_k(F, \mathbf{k})$. It holds that $\mathbf{E}^{**} = \mathbf{E}$ ([A1, 1.10]). Given $\mathbf{E}$ there exists a k-module V such that $\mathbf{E} = \mathbf{V}^*$ if and only if E is a projective k-module of finite type.

Given a functor of commutative k-algebras $\mathcal{A}$, we define $\mathrm{Spec}\,\mathcal{A}$ to be the functor $\mathrm{Spec}\,\mathcal{A}(B) := \mathrm{Hom}_{k-alg}(\mathcal{A}, \mathbf{B})$. Let $X = \mathrm{Spec}\, A$ be a k-scheme and let $X^{\cdot}$ be its functor of points ($X^{\cdot}(B) := \mathrm{Hom}_{k-alg}(A, B)$). It is easy to see that $X^{\cdot} = \mathrm{Spec}\,\mathbf{A}$. If $\mathbf{C}^*$ is a functor of commutative algebras, we prove (Theorem 1.7) that $\mathrm{Spec}\,\mathbf{C}^*$ is a direct limit (of functor of points) of finite k-schemes.

If $\mathcal{A} = \mathbf{A}$, in [A1, 3.4] we proved that the linear enveloping of $\mathrm{Spec}\,\mathcal{A}$ is $\mathcal{A}^*$, that is,

$$\mathrm{Hom}_{functors}(\mathrm{Spec}\,\mathcal{A}, H) = \mathrm{Hom}_k(\mathcal{A}^*, H)$$

for all dual functors of k-modules H (i.e., $H = F^*$). If in addition $G = \mathrm{Spec}\,\mathcal{A}$ is an affine k-group, in [A1, 5.4] we proved that the algebraic enveloping of $\mathrm{Spec}\,\mathcal{A}$ is $\mathcal{A}^*$ again, that is,

$$\mathbf{Hom}_{monoids}(\mathrm{Spec}\,\mathcal{A}, \mathcal{B}) = \mathbf{Hom}_{k-alg}(\mathcal{A}^*, \mathcal{B})$$

for all dual functors of k-algebras $\mathcal{B}$. As a consequence, we obtained that the category of G-modules is equivalent to the category of $\mathcal{A}^*$-modules.

In this paper we prove simultaneously the same results for $\mathcal{A} = \mathbf{A}, \mathbf{C}^*$ (Corollary 2.3, Proposition 2.6, Theorem 2.7). In particular,

$$\mathbf{Hom}_{functors}(\mathrm{Spec}\,\mathcal{A}, \mathbf{k}) = \mathbf{Hom}_k(\mathcal{A}^*, \mathbf{k}) = \mathcal{A}.$$

Moreover, if $\mathrm{Spec}\,\mathcal{A}$ is a functor of abelian monoids, then we get

$$\mathbf{Hom}_{monoids}(\mathrm{Spec}\,\mathcal{A}, \mathbf{k}) = \mathbf{Hom}_{k-alg}(\mathcal{A}^*, \mathbf{k}) = \mathrm{Spec}\,\mathcal{A}^*$$

where $\mathbf{k}$ is regarded as a functor of monoids with the multiplication operation. We also prove that

$$\mathrm{Hom}_{monoids}(\mathrm{Spec}\,\mathcal{A}, \mathcal{B}) = \mathrm{Hom}_{k-alg}(\mathcal{A}^*, \mathcal{B})$$

for every dual functor of k-algebras $\mathcal{B}$, and we prove that the category of $\mathrm{Spec}\,\mathcal{A}$-modules is equivalent to the category of $\mathcal{A}^*$-modules (2.6, 2.7).

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Given a functor of monoids H, we say that $H^* = \mathbf{Hom}_{monoids}(H, \mathbf{k})$ is the dual functor of monoids of H and we prove the following theorem.

Theorem 1. *The category of abelian affine monoids $G = \operatorname{Spec} A$ is anti-equivalent to the category of functors $\operatorname{Spec} \mathbf{C}^*$ of abelian monoids. The functors giving the anti-equivalence are the ones that assign to each functor of monoids its dual functor of monoids.*

In particular, $G^{**} = G$.

Finally, we get three immediate applications of the functorial Cartier duality:

(1) The tangent space to G^* at the identity element is isomorphic to the vector space V of linear functions of G.
(2) If $\operatorname{car} k = 0$ and G is a unipotent abelian k-group, then it is isomorphic to the dual of V.
(3) If G is semisimple, then the transposed Fourier transform of G^* is the inverse of the Fourier transform of G.

1. Spectrum of a functor of algebras

The basic references for reading this paper are [A1] and [S].

Let k be a commutative ring with unit. If E is a k-module we will say that $\mathbf{E}$ is a cuasi-coherent k-module. The category of k-modules is equivalent to the category of cuasi-coherent k-modules ([A1, 1.12]). In particular, $\operatorname{Hom}_k(\mathbf{E}, \mathbf{E}') = \operatorname{Hom}_k(E, E')$. $\mathbf{E}^* = \mathbf{Hom}_k(\mathbf{E}, \mathbf{k})$ is the functor of points of the scheme $\operatorname{Spec} S^\cdot_k E$ and we say that it is a k-module scheme. As it holds that $\mathbf{E}^{**} = \mathbf{E}$, the category of cuasi-coherent modules is anti-equivalent to the category of module schemes ([A1, 1.12]) .

Notation 1.1. *Throughout this paper we assume that $\mathbf{A}^*$ is a commutative k-algebra scheme (equivalently, A is a cocommutative k-coalgebra,* [A1, 4.2]*). From Proposition* 1.5 *on, we will always assume that A is a projective k-module.*

Definition 1.2. *Given a functor of k-algebras $\mathcal{A}$, the functor $\operatorname{Spec} \mathcal{A}$, "spectrum of $\mathcal{A}$", is defined to be*

$$(\operatorname{Spec} \mathcal{A})(B) := \operatorname{Hom}_{k-alg}(\mathcal{A}, \mathbf{B})$$

for each commutative k-algebra B.

Example 1.3. *If C is a commutative k-algebra, then*

$$\operatorname{Spec} \mathbf{C} = (\operatorname{Spec} C)^\cdot.$$

Example 1.4. *Let X be a set. Let us consider the discrete topology on X. Let $\mathcal{X}$ be the functor, which we will call the constant functor X, defined by*

$$\mathcal{X}(B) := \operatorname{Aplic}_{cont.}(\operatorname{Spec} B, X)$$

for each commutative k-algebra B.

Let $\mathcal{A}_X$ be the functor of algebras defined by

$$\mathcal{A}_X(B) := \operatorname{Aplic}(X, B) = \prod_X B$$

for each commutative k-algebra B. Let us observe that $\mathcal{A}_X = \prod_X \mathbf{k} = (\bigoplus_X \mathbf{k})^*$ is a commutative algebra scheme.

Let us prove that $\operatorname{Spec} \mathcal{A}_X = \mathcal{X}$.

The morphisms of functors of k-algebras are, in particular, morphisms of functors of k-modules

$$(\operatorname{Spec} \mathcal{A}_X)(B) = \operatorname{Hom}_{k-alg}(\mathcal{A}_X, \mathbf{B}) \subset \operatorname{Hom}_k(\mathcal{A}_X, \mathbf{B}) = \operatorname{Hom}_k(\prod_{x \in X} \mathbf{k}, \mathbf{B})$$

and from [A1, 4.5] *we know that morphisms of functors of k-modules from $\prod_X \mathbf{k}$ to $\mathbf{B}$ factorize through the projection onto a finite number of factors. Therefore, given a morphism $\prod_X \mathbf{k} \to \mathbf{B}$ there exists a subset $Y \subset X$ of finite order, such that we get the factorization*

$$\prod_X \mathbf{k} \to \prod_Y \mathbf{k} \to \mathbf{B}.$$

Every continuous morphism $\operatorname{Spec} B \to X$ *has finite image, because* $\operatorname{Spec} B$ *is compact and X is discrete. We can assume that X is a finite set.*

Thus, we get

$$\begin{aligned}\operatorname{Hom}_{k-alg}(\prod_X \mathbf{k}, \mathbf{B}) &= \operatorname{Hom}_{k-alg}(\prod_X k, B)\\ &= \text{Cont.Aplic.}_{\operatorname{Spec} k}(\operatorname{Spec} B, \operatorname{Spec} \prod_X k = \coprod_X \operatorname{Spec} k)\\ &= \operatorname{Aplic}_{cont.}(\operatorname{Spec} B, X).\end{aligned}$$

Later we will see that the ring of functions of $\operatorname{Spec} \mathcal{A}_X$ *coincides with* $\mathcal{A}_X$*, that is,*

$$\mathbf{Hom}_{functors}(\operatorname{Spec} \mathcal{A}_X, \mathbf{k}) = \mathcal{A}_X.$$

Obviously, $\operatorname{Spec}(\varinjlim_{i\in I} \mathcal{A}_i) = \varprojlim_{i\in I}(\operatorname{Spec} \mathcal{A}_i)$.

Proposition 1.5. *Let $\mathcal{A}$ a functor of commutative algebras. Then,* $\operatorname{Spec} \mathcal{A} = \mathbf{Hom}_{k-alg}(\mathcal{A}, \mathbf{k})$.

Proof. By the adjoint functor formula ([A1, 1.15]) (restricted to the morphisms of algebras) it holds that

$$\begin{aligned}\mathbf{Hom}_{k-alg}(\mathcal{A}, \mathbf{k})(B) &= \operatorname{Hom}_{B-alg}(\mathcal{A}_{|B}, \mathbf{B}) = \operatorname{Hom}_{k-alg}(\mathcal{A}, \mathbf{B})\\ &= (\operatorname{Spec} \mathcal{A})(B).\end{aligned}$$

□

Therefore, $\operatorname{Spec} \mathbf{A}^* = \mathbf{Hom}_{k-alg}(\mathbf{A}^*, \mathbf{k}) \subset \mathbf{Hom}_k(\mathbf{A}^*, \mathbf{k}) = \mathbf{A}$.

Theorem 1.6. *Let $\{\mathbf{A_i^*}, f_{ij}\}_{i,j\in I}$ be a projective system of morphisms of functors of algebras. Then*

$$\operatorname{Spec} \varprojlim_{i\in I} \mathbf{A_i^*} = \varinjlim_{i\in I} \operatorname{Spec} \mathbf{A_i^*}$$

Proof. We have to prove that $\operatorname{Hom}_{k-alg}(\varprojlim_{i\in I} \mathbf{A_i^*}, \mathbf{B}) = \varinjlim_{i\in I} \operatorname{Hom}_{k-alg}(\mathbf{A_i^*}, \mathbf{B})$, for every commutative k-algebra B. Observe that

$$\begin{aligned}\operatorname{Hom}_k(\varprojlim_{i\in I} \mathbf{A_i^*}, \mathbf{B}) &= \operatorname{Hom}_k((\varinjlim_{i\in I} \mathbf{A_i})^*, \mathbf{B}) \overset{[A1,\ 1.8]}{=} (\varinjlim_{i\in I} A_i) \otimes B\\ &= \varinjlim_{i\in I}(A_i \otimes B) = \varinjlim_{i\in I} \operatorname{Hom}_k(\mathbf{A_i^*}, \mathbf{B})\end{aligned}$$

Similarly we have that $\operatorname{Hom}_k(\varprojlim_{i\in I} \mathbf{A_i^*} \otimes \varprojlim_{i\in I} \mathbf{A_i^*}, \mathbf{B}) = \varinjlim_{i\in I} \operatorname{Hom}_k(\mathbf{A_i^*} \otimes \mathbf{A_i^*}, \mathbf{B})$. Given a scheme of algebras $\mathbf{C}^*$ let us denote $\mu\colon \mathbf{C}^* \otimes \mathbf{C}^* \to \mathbf{C}^*$ the multiplication morphism. The commutative diagram

$$\begin{array}{ccc} \operatorname{Hom}_k(\varprojlim_{i\in I} \mathbf{A_i^*}, \mathbf{B}) & = & \varinjlim_{i\in I} \operatorname{Hom}_k(\mathbf{A_i}^*, \mathbf{B}) \\ \downarrow \mu^* & & \downarrow \mu^* \\ \operatorname{Hom}_k(\varprojlim_{i\in I} \mathbf{A_i^*} \otimes \varprojlim_{i\in I} \mathbf{A_i^*}, \mathbf{B}) & = & \varinjlim_{i\in I} \operatorname{Hom}_k(\mathbf{A_i}^* \otimes \mathbf{A_i^*}, \mathbf{B}) \end{array}$$

shows that $\operatorname{Hom}_{k-alg}(\varprojlim_{i\in I} \mathbf{A_i^*}, \mathbf{B}) = \varinjlim_{i\in I} \operatorname{Hom}_{k-alg}(\mathbf{A_i^*}, \mathbf{B})$.

□

If E is a finitely-generated k-module, we will say that $\mathbf{E}$ is coherent.

Theorem 1.7. *Let $\mathbf{A}^*$ be a commutative algebra scheme and let us assume that A is a projective k-module. By* [A1, 4.12]*, $\mathbf{A}^* = \varprojlim_i \mathbf{A}_i$, where $\mathbf{A}_i$ are the coherent algebras that are quotients of $\mathbf{A}^*$. It holds that*

$$\operatorname{Spec} \mathbf{A}^* = \varinjlim_i \operatorname{Spec} \mathbf{A}_i.$$

Proof. By [A1, 4.5], the image of every k-linear morphism $\mathbf{A}^* \to \mathbf{B}$ is a coherent k-module; therefore, every morphism of functors of k-algebras $\mathbf{A}^* \to \mathbf{B}$ factorizes through a coherent algebra $\mathbf{A}_i$ that is a quotient of $\mathbf{A}^*$. Then,

$$\begin{aligned} (\operatorname{Spec} \mathbf{A}^*)(B) = \operatorname{Hom}_{k-alg}(\mathbf{A}^*, \mathbf{B}) &= \varinjlim_i \operatorname{Hom}_{k-alg}(\mathbf{A}_i, \mathbf{B}) \\ &= \varinjlim_i (\operatorname{Spec} \mathbf{A}_i)(B). \end{aligned}$$

□

Proposition 1.8. *Let $\mathcal{B}$ be a functor of commutative k-algebras. Then,*

(1) $\operatorname{Hom}_{functors}(\operatorname{Spec} \mathbf{A}, \operatorname{Spec} \mathcal{B}) = \operatorname{Hom}_{k-alg}(\mathcal{B}, \mathbf{A})$.
(2) $\operatorname{Hom}_{functors}(\operatorname{Spec} \mathbf{A}^*, \operatorname{Spec} \mathcal{B}) = \operatorname{Hom}_{k-alg}(\mathcal{B}, \mathbf{A}^*)$.

Proof. It holds that $\operatorname{Hom}_{functors}((\operatorname{Spec} A)^{\cdot}, F) = F(A)$ for every functor F, by Yoneda's lemma.

(1)

$$\begin{aligned} \operatorname{Hom}_{functors}(\operatorname{Spec} \mathbf{A}, \operatorname{Spec} \mathcal{B}) &= \operatorname{Hom}_{functors}((\operatorname{Spec} A)^{\cdot}, \operatorname{Spec} \mathcal{B}) \\ &= \operatorname{Hom}_{k-alg}(\mathcal{B}, \mathbf{A}). \end{aligned}$$

(2)

$$\begin{aligned} \operatorname{Hom}_{functors}(\operatorname{Spec} \mathbf{A}^*, \operatorname{Spec} \mathcal{B}) &= \operatorname{Hom}_{functors}(\varinjlim_i \operatorname{Spec} \mathbf{A}_i, \operatorname{Spec} \mathcal{B}) \\ &= \varprojlim_i \operatorname{Hom}_{functors}(\operatorname{Spec} \mathbf{A}_i, \operatorname{Spec} \mathcal{B}) \\ &= \varprojlim_i \operatorname{Hom}_{k-alg}(\mathcal{B}, \mathbf{A}_i) \\ &= \operatorname{Hom}_{k-alg}(\mathcal{B}, \mathbf{A}^*). \end{aligned}$$

□

Proposition 1.9. *For $\mathcal{A} = \mathbf{A}, \mathbf{A}^*$, it holds that*

$$\mathbf{Hom}_{functors}(\operatorname{Spec}\mathcal{A}, \mathbf{k}) = \mathcal{A}.$$

Proof.

$$\begin{aligned}\mathbf{Hom}_{functors}(\operatorname{Spec}\mathcal{A}, \mathbf{k}) &= \mathbf{Hom}_{functors}(\operatorname{Spec}\mathcal{A}, \operatorname{Spec}\mathbf{k}[\mathbf{x}]) \\ &= \mathbf{Hom}_{k-alg}(\mathbf{k}[\mathbf{x}],\mathcal{A}) = \mathcal{A}.\end{aligned}$$

□

Explicitly, $\mathcal{A} = \mathbf{Hom}_{functors}(\operatorname{Spec}\mathcal{A}, \mathbf{k})$, $w \mapsto \tilde{w}$, where $\tilde{w}(\phi) = \phi(w)$, for every $\phi \in \operatorname{Spec}\mathcal{A} = \mathbf{Hom}_{k-alg}(\mathcal{A}, \mathbf{k})$.

2. Closure of dual modules and algebras

Definition 2.1. *We will say that a functor of k-modules F is dual if there exists a functor of k-modules H such that $F \simeq H^*$.*

Proposition 2.2. *Let F be a functor of k-modules such that F^* is a reflexive functor. The closure of dual functors of k-modules of F is F^{**}, that is, it holds the functorial equality*

$$\operatorname{Hom}_k(F, G) = \operatorname{Hom}_k(F^{**}, G)$$

for every dual functor of k-modules G.

Proof. Let us write $G = H^*$, then

$$\begin{aligned}\operatorname{Hom}_k(F, G) &= \operatorname{Hom}_k(F, H^*) = \operatorname{Hom}_k(H, F^*) = \operatorname{Hom}_k(H, F^{***}) \\ &= \operatorname{Hom}_k(F^{**}, H^*) = \operatorname{Hom}_k(F^{**}, G).\end{aligned}$$

□

Corollary 2.3. *Let $\mathcal{A}$ be a functor of commutative k-algebras. Let us assume that $\mathbf{Hom}_{functors}(\operatorname{Spec}\mathcal{A}, \mathbf{k}) = \mathcal{A}$ and that $\mathcal{A}$ is a reflexive functor of k-modules.Then, the closure of dual functors of k-modules of $k[\operatorname{Spec}\mathcal{A}]$ is $\mathcal{A}^*$, that is,*

$$\operatorname{Hom}_{functors}(\operatorname{Spec}\mathcal{A}, F) = \operatorname{Hom}_k(k[\operatorname{Spec}\mathcal{A}], F) = \operatorname{Hom}_k(\mathcal{A}^*, F)$$

for every dual functor of k-modules F.

Proof. As $k[\operatorname{Spec}\mathcal{A}]^* = \mathbf{Hom}_{functors}(\operatorname{Spec}\mathcal{A}, \mathbf{k}) = \mathcal{A}$, the closure of dual functors of k-modules of $k[\operatorname{Spec}\mathcal{A}]$ is $k[\operatorname{Spec}\mathcal{A}]^{**} = \mathcal{A}^*$. □

Proposition 2.4. *Let F be a functor of k-algebras such that F^* is a reflexive functor of k-modules. The closure of dual functors of k-algebras of F is F^{**}, that is, it holds the functorial equality*

$$\operatorname{Hom}_{k-alg}(F, G) = \operatorname{Hom}_{k-alg}(F^{**}, G)$$

for every dual functor of k-algebras G. In particular, $\operatorname{Spec} F = \operatorname{Spec} F^{**}$.

Proof. Let us observe that

$$\begin{aligned}(F \otimes \overset{n}{\cdots} \otimes F)^* &= \operatorname{Hom}_k(F \otimes \overset{n}{\cdots} \otimes F, \mathbf{k}) = \operatorname{Hom}_k(F, (F \otimes \overset{n-1}{\cdots} \otimes F)^*) \\ &\underset{Induction}{=} \operatorname{Hom}_k(F, (F^{**} \otimes \overset{n-1}{\cdots} \otimes F^{**})^*) = \operatorname{Hom}_k(F^{**}, (F^{**} \otimes \overset{n-1}{\cdots} \otimes F^{**})^*) \\ &= (F^{**} \otimes \overset{n}{\cdots} \otimes F^{**})^*.\end{aligned}$$

Therefore, given a dual functor of k-modules S^*,

$$\begin{aligned}\operatorname{Hom}_k(F \otimes \ldots \otimes F, S^*) &= \operatorname{Hom}_k(S, (F \otimes \ldots \otimes F)^*) \\ &= \operatorname{Hom}_k(S, (F^{**} \otimes \ldots \otimes F^{**})^*) \\ &= \operatorname{Hom}_k(F^{**} \otimes \ldots \otimes F^{**}, S^*).\end{aligned}$$

If we consider $S^* = F^{**}$, it follows easily that the structure of algebra of F defines a structure of algebra on F^{**}. Finally, if we consider $S^* = G$, we obtain that $\mathrm{Hom}_{k-alg}(F, G) = \mathrm{Hom}_{k-alg}(F^{**}, G)$. □

Remark 2.5. *Let us observe that if F is also a functor of commutative algebras, then so is F^{**}: since $\mathrm{Hom}_k(F \otimes F, F^{**}) = \mathrm{Hom}_k(F^{**} \otimes F^{**}, F^{**})$, the morphism $F \otimes F \to F$, $f \otimes f' \mapsto ff' - f'f = 0$ extends to a unique morphism $F^{**} \otimes F^{**} \to F^{**}$ (which is $f \otimes f' \mapsto ff' - f'f = 0$).*

Let $\mathbf{A}^*$ and $\mathbf{B}^*$ be commutative k-algebra schemes. By [A1, 1.8], we have that $(\mathbf{A}^* \otimes_k \mathbf{B}^*)^* = \mathbf{Hom}_k(\mathbf{A}^* \otimes_k \mathbf{B}^*, \mathbf{k}) = \mathbf{Hom}_k(\mathbf{A}^*, \mathbf{B}) = \mathbf{A} \otimes_k \mathbf{B}$. Then, $\mathrm{Spec}\, \mathbf{A}^* \times \mathrm{Spec}\, \mathbf{B}^* = \mathrm{Spec}\, (\mathbf{A}^* \otimes_k \mathbf{B}^*) \overset{2.4}{=} \mathrm{Spec}\, (\mathbf{A} \otimes_k \mathbf{B})^*$.

Let $\mathcal{A}$ be a functor of k-algebras and let us assume that $G = \mathrm{Spec}\, \mathcal{A}$ is a functor of monoids. The functor of k-modules $k[G]$ is obviously a functor of k-algebras. Given a functor of k-algebras $\mathcal{B}$, it is easy to check the equality

$$\mathrm{Hom}_{monoids}(G, \mathcal{B}) = \mathrm{Hom}_{k-alg}(k[G], \mathcal{B}).$$

Proposition 2.6. *Let $G = \mathrm{Spec}\, \mathcal{A}$ be a functor of monoids. Let us suppose that $\mathbf{Hom}_{functors}(G, \mathbf{k}) = \mathcal{A}$ and that $\mathcal{A}$ is a reflexive functor of k-modules. Then the closure of dual functors of algebras of $k[G]$ coincides with $\mathcal{A}^*$, that is,*

$$\mathrm{Hom}_{monoids}(G, \mathcal{B}) = \mathrm{Hom}_{k-alg}(k[G], \mathcal{B}) = \mathrm{Hom}_{k-alg}(\mathcal{A}^*, \mathcal{B})$$

for every dual functor of k-algebras $\mathcal{B}$.

Proof. $k[G]^* = \mathcal{A}$ is reflexive, then the closure of dual functors of algebras of $k[G]$ is $\mathcal{A}^*$. □

Theorem 2.7. *Let $G = \mathrm{Spec}\, \mathcal{A}$ be a functor of monoids. Let us assume that $\mathbf{Hom}_{functors}(\mathrm{Spec}\, \mathcal{A}, \mathbf{k}) = \mathcal{A}$ and that $\mathcal{A}$ is a reflexive functor of k-modules. The category of G-modules is equivalent to the category of $\mathcal{A}^*$-modules. Likewise, the category of dual functors of G-modules is equivalent to the category of dual functors of $\mathcal{A}^*$-modules.*

Proof. Let E be a k-module. Let us observe that $\mathbf{End}_k(\mathbf{E}) = (\mathbf{E}^* \otimes \mathbf{E})^*$ is a dual functor. Therefore,

$$\mathrm{Hom}_{k-alg}(k[G], \mathbf{End}_k(\mathbf{E})) = \mathrm{Hom}_{k-alg}(\mathcal{A}^*, \mathbf{End}_k(\mathbf{E})).$$

In conclusion, endowing E with a structure of G-module is equivalent to endowing E with a structure of $\mathcal{A}^*$-module.

Let us observe that $\mathrm{Hom}_k(k[G], \mathbf{E}) = \mathrm{Hom}_k(\mathbf{E}^*, \mathcal{A}) = \mathrm{Hom}_k(\mathcal{A}^*, \mathbf{E})$. Hence, given any two G-modules (or $\mathcal{A}^*$-modules) E, E', a linear morphism $f : E \to E'$ and $e \in E$, we will have that the morphism $f_1 : k[G] \to \mathbf{E}'$, $f_1(g) := f(ge) - gf(e)$ is null if and only if the morphism $f_2 : \mathcal{A}^* \to \mathbf{E}'$, $f_2(a) := f(ae) - af(e)$ is null. In conclusion, $\mathrm{Hom}_{G-mod}(\mathbf{E}, \mathbf{E}') = \mathrm{Hom}_{\mathcal{A}^*}(\mathbf{E}, \mathbf{E}')$. □

Let us highlight that the structure of functor of algebras of $\mathcal{A}^*$ is the one that makes the embedding $\mathrm{Spec}\, \mathcal{A} \hookrightarrow \mathcal{A}^*$ be a morphism of monoids.

Example 2.8. *The $\mathbb{C}$-linear representations of $\mathbb{Z}$ are equivalent to the $\mathbb{C}[\mathbb{Z}]$-modules. $\mathbb{C}[\mathbb{Z}] = \mathbb{C}[x, 1/x]$, $n \mapsto x^n$, is a principal ideal domain. Thus, if E is a finite $\mathbb{C}$-linear representation of $\mathbb{Z}$, then*

$$E = \underset{\alpha \neq 0, n_\alpha > 0}{\oplus} \mathbb{C}[x]/(x - \alpha)^{n_\alpha}$$

such that $n \cdot (\bar{p}_{n_\alpha})_{n_\alpha} = (\overline{x^n \cdot p_{n_\alpha}})_{n_\alpha}$.

Proposition 2.9. *Let $G = \operatorname{Spec}\mathcal{A}$ be a functor of monoids. Let us assume that $\mathbf{Hom}_{functors}(\operatorname{Spec}\mathcal{A}, \mathbf{k}) = \mathcal{A}$ and that $\mathcal{A}$ is a reflexive functor of k-modules. Let $\mathcal{V}$ be a dual functor of G-modules (on the left and on the right). Then,*

$$Der_k(\mathcal{A}^*, \mathcal{V}) = Der(G, \mathcal{V})$$

where $Der(G, \mathcal{V}) = \{D \in \operatorname{Hom}_{functors}(G, \mathcal{V}) : D(g \cdot g') = D(g) \cdot g' + g \cdot D(g')\}$.

Proof.

$$\begin{aligned} Der_k(\mathcal{A}^*, \mathcal{V}) &= \{f \in \operatorname{Hom}_{k-alg}(\mathcal{A}^*, \mathcal{A}^* \oplus \mathcal{V}\epsilon), \epsilon^2 = 0, f(a) = a \operatorname{mod}(\epsilon)\} \\ &= \{f \in \operatorname{Hom}_{monoids}(G, \mathcal{A}^* \oplus \mathcal{V}\epsilon), \epsilon^2 = 0, f(g) = g \operatorname{mod}(\epsilon)\} \\ &= Der(G, \mathcal{V}). \end{aligned}$$

□

3. Functorial Cartier Duality

Definition 3.1. *Given a functor of abelian monoids G, where we regard $\mathbf{k}$ as a monoid with the multiplication operation,*

$$G^* := \mathbf{Hom}_{monoids}(G, \mathbf{k})$$

will be referred to as the dual monoid of G.

If G is a functor of groups, then $G^* = \mathbf{Hom}_{groups}(G, G_m^{\cdot})$.

Theorem 3.2. *Assume that $\operatorname{Spec}\mathcal{A}$ is a functor of abelian monoids, $\mathcal{A}$ is a reflexive functor of k-modules and $\mathbf{Hom}_{functors}(\operatorname{Spec}\mathcal{A}, \mathbf{k}) = \mathcal{A}$. Then*

$$\operatorname{Spec}\mathcal{A}^* = (\operatorname{Spec}\mathcal{A})^*$$

(in particular, this equality shows that $\operatorname{Spec}\mathcal{A}^*$ is a functor of abelian monoids).

Proof.

$$\begin{aligned} \operatorname{Spec}\mathcal{A}^* &= \mathbf{Hom}_{k-alg}(\mathcal{A}^*, \mathbf{k}) = \mathbf{Hom}_{monoids}(\operatorname{Spec}\mathcal{A}, \mathbf{k}) \\ &= (\operatorname{Spec}\mathcal{A})^*. \end{aligned}$$

□

Remark 3.3. *Explicitly,* $\operatorname{Spec}\mathcal{A}^* = \mathbf{Hom}_{monoids}(\operatorname{Spec}\mathcal{A}, G_m^{\cdot})$, $\phi \mapsto \tilde{\phi}$, *where* $\tilde{\phi}(x) = \phi(x)$, *for every* $\phi \in \operatorname{Spec}\mathcal{A}^* = \mathbf{Hom}_{k-alg}(\mathcal{A}^*, \mathbf{k})$ *and* $x \in \operatorname{Spec}\mathcal{A} \subset \mathcal{A}^*$.

Let $G = \operatorname{Spec}\mathcal{A}$. G^* is a functor of abelian monoids ($(f \cdot f')(g) := f(g) \cdot f'(g)$, for every $f, f' \in G^*$ and $g \in G$), the inclusion $G^* = \mathbf{Hom}_{monoids}(G, \mathbf{k}) \subset \mathbf{Hom}_{functors}(G, \mathbf{k}) = \mathcal{A}$ is a morphism of monoids and the diagram

$$\begin{array}{ccc} (\operatorname{Spec}\mathcal{A})^* & \hookrightarrow & \mathbf{Hom}_{functors}(\operatorname{Spec}\mathcal{A}, \mathbf{k}) = \mathcal{A} \\ \| & & \| \\ \operatorname{Spec}\mathcal{A}^* & \hookrightarrow & \mathbf{Hom}_k(\mathcal{A}^*, \mathbf{k}) = \mathcal{A}^{**} \end{array}$$

is commutative.

Theorem 3.4. *Assume that $G = \operatorname{Spec}\mathcal{A}$ is a functor of abelian monoids, $\mathcal{A}$ is a reflexive functor of k-modules and $\mathbf{Hom}_{functors}(\operatorname{Spec}\mathcal{A}, \mathbf{k}) = \mathcal{A}$, $\mathbf{Hom}_{functors}(\operatorname{Spec}\mathcal{A}^*, \mathbf{k}) = \mathcal{A}^*$. Then:*

1. *The morphism $G \overset{**}{\to} G^{**}$, $g \mapsto g^{**}$, where $g^{**}(f) := f(g)$ for every $f \in G^*$, is an isomorphism.*
2. $\operatorname{Hom}_{monoids}(G_1, G_2) = \operatorname{Hom}_{monoids}(G_2^*, G_1^*)$.

Proof.

(1) It is easy to check that the diagram

$$\begin{array}{ccccc} \operatorname{Spec}\mathcal{A}^{**} & \xrightarrow[3.2]{\sim} & (\operatorname{Spec}\mathcal{A}^{*})^{*} & \xleftarrow[3.2]{\sim} & (\operatorname{Spec}\mathcal{A})^{**} \\ & \diagdown\!\diagdown & & \nearrow^{**} & \\ & & \operatorname{Spec}\mathcal{A} & & \end{array}$$

is commutative. Hence the morphism $**$ is an isomorphism.

(2) Every morphism of monoids $G_1 \to G_2$, taking $\mathbf{Hom}_{monoids}(-,\mathbf{k})$, defines a morphism $G_2^* \to G_1^*$. Taking $\mathbf{Hom}_{monoids}(-,\mathbf{k})$ we get the original morphism $G_1 \to G_2$, as it is easy to check. Likewise, every morphism of monoids $G_2^* \to G_1^*$, taking $\mathbf{Hom}_{monoids}(-,\mathbf{k})$, defines a morphism $G_1 \to G_2$. Taking $\mathbf{Hom}_{monoids}(-,\mathbf{k})$ we get the original morphism $G_2^* \to G_1^*$.

□

Theorem 3.5. *The category of abelian affine k-monoids $G = \operatorname{Spec} A$ is anti-equivalent to the category of functors $\operatorname{Spec}\mathbf{A}^*$ of abelian monoids (we assume the k-modules A are projective).*

In particular, we get the Cartier duality for finite commutative algebraic groups ([S, §9.9]). Dieudonné ([D, Ch. I, §2, 13]) proves the equivalence between the category of k-coalgebras in groups and the dual category of linearly compact k-algebras in cogroups (where k is a field).

Proposition 3.6. *Let G_1 and G_2 be a pair of functors of abelian monoids. It holds that*

$$(G_1 \times G_2)^* = G_1^* \times G_2^*.$$

Proof.

$$\begin{aligned}(G_1 \times G_2)^* &= \mathbf{Hom}_{monoids}(G_1 \times G_2, \mathbf{k}) \\ &= \mathbf{Hom}_{monoids}(G_1, \mathbf{k}) \times \mathbf{Hom}_{monoids}(G_2, \mathbf{k}) = G_1^* \times G_2^*.\end{aligned}$$

□

Let us show some examples.

Example 3.7. *Let k be a field. Let $\mathbb{Z} := \operatorname{Spec}\prod_{\mathbb{Z}}\mathbf{k}$. Obviously $\mathbb{Z}^* = G_m$, therefore $G_m^* = \mathbb{Z}$. In other words, $\mathbf{Hom}_{grupos}(G_m, G_m) = \mathbb{Z}$: given $\tau : G_m \to G_m$, there exists $n \in \mathbb{Z}$ such that $\tau(\alpha) = \alpha^n$.*

Given a quasi-coherent $\prod_X \mathbf{k}$-module $\mathbf{E}$, then $\mathbf{E} = \oplus_{x\in X}\mathbf{E}_x$ in the obvious way. Given a morphism of algebra schemes $\prod_X \mathbf{k} \to \mathbf{A}^*$, then $\mathbf{A}^* = \prod_{x\in X}\mathbf{A}_x^*$. Moreover, if the morphism $\operatorname{Spec}\mathbf{A}^* \to \operatorname{Spec}\prod_X \mathbf{k}$ is injective, then $\mathbf{A}_x^* = \mathbf{k}$ or $\mathbf{A}_x^* = 0$ for each $x \in X$. Therefore, $\operatorname{Spec}\mathbf{A}^* \subset \operatorname{Spec}\prod_{\mathbb{Z}}\mathbf{k} = \mathbb{Z}$ is a subgroup if and only if $\operatorname{Spec}\mathbf{A}^* = n\cdot\mathbb{Z}$. Dually, the quotient algebraic groups of G_m are the epimorphisms $G_m \to G_m$, $t \mapsto t^n$. Hence, the subgroups of G_m are the $\mu_n = \{\alpha \in G_m : \alpha^n = 1\} = \operatorname{Spec} k[x]/(x^n - 1) = (\mathbb{Z}/(n))^*$.

Likewise, $(G_m \times \overset{n}{\dots} \times G_m)^* = \mathbb{Z} \times \overset{n}{\dots} \times \mathbb{Z}$. By the theory of abelian groups (or $\mathbb{Z}$-modules), the subgroups of $\mathbb{Z} \times \overset{n}{\dots} \times \mathbb{Z}$ are isomorphic to $\mathbb{Z}^r$ through an injective morphism ϕ (let us say that its matrix is (n_{ij})),

$$\mathbb{Z} \times \overset{r}{\dots} \times \mathbb{Z} \overset{\phi}{\to} \mathbb{Z} \times \overset{n}{\dots} \times \mathbb{Z}$$

whose cokernel is isomorphic to $\mathbb{Z}^{n-r} \times \mathbb{Z}/(n_1) \times \ldots \times \mathbb{Z}/(n_r)$. Dually, the subgroups of $G_m \times \overset{n}{\dots} \times G_m$ are isomorphic to $G_m^{n-r} \times \mu_{n_1} \times \ldots \times \mu_{n_r}$, that is isomorphic to the kernel of the epimorphism

$$G_m^n \to G_m^r,\ (t_1, \ldots, t_n) \mapsto (t_1^{n_{11}} \cdot \ldots \cdot t_n^{n_{n1}}, \ldots, t_1^{n_{1r}} \cdot \ldots \cdot t_n^{n_{nr}}).$$

Proposition 3.8. [B, Ch. III, 8.12] *The category of diagonalizable algebraic groups is anti-equivalent to the category of $\mathbb{Z}$-modules of finite type (or finitely-generated abelian groups).*

Example 3.9 (Affine toric varieties). *Let M be a set with structure of abelian (multiplicative) monoid. The constant functor $\mathcal{M} = \operatorname{Spec} \prod_M \mathbf{k}$ is a functor of abelian monoids. The dual functor is the abelian k-monoid $\mathcal{M}^* = \operatorname{Spec} \oplus_M \mathbf{k} = \operatorname{Spec} k[M]$.*

We will say that an abelian monoid M is classic if it is finitely generated and its associated group G is torsion-free and the natural morphism $M \to G$ is injective. It is easy to prove that M is classic if and only if $k[M]$ is an integral finitely generated k-algebra.

Theorem 3.10. *The category of abelian monoids (respectively finitely generated, classic) is anti-equivalent to the category of affine k-schemes of semisimple abelian monoids (respectively algebraic, integral algebraic).*

As $G = \mathbb{Z}^n$, the morphism $M \to G$ induces a morphism $G_m^n \to \mathcal{M}^*$. In particular, G_m^n operates on $\mathcal{M}^*$. Furthermore, as $k[G]$ is the localization of $k[M]$ by the algebraically closed system M, the morphism $G_m^n \to \mathcal{M}^*$ is an open injection. We will say that an integral affine algebraic variety on which the torus operates with a dense orbit is an affine toric variety. It is easy to prove that there exists a one-to-one correspondence between affine toric varieties with a fixed point whose orbit is transitive and dense, and classic monoids.

Example 3.11. *Let $G_a = \operatorname{Spec} k[x]$ be the additive group. Let us compute G_a^*. Assume $\operatorname{car} k = 0$. Let $\mathbf{k}[[\mathbf{z}]]$ be the k-algebra scheme defined by $\mathbf{k}[[\mathbf{z}]](B) := B[[z]]$. Let $\phi\colon \mathbf{k}[\mathbf{x}]^* \to \mathbf{k}[[\mathbf{z}]]$ be the linear isomorphism defined by $\phi^{-1}(z^i)(x^n) = i!\cdot\delta_{in}$. The composition of the natural morphism $G_a \to \mathbf{k}[\mathbf{x}]^*$ with ϕ is equal to the morphism $G_a \to \mathbf{k}[[\mathbf{z}]]$, $\alpha \mapsto e^{\alpha z}$, which is a morphism of monoids. Therefore, ϕ is an isomorphism of functors of algebras.*

$$\begin{aligned} G_a^*(B) &= \operatorname{Hom}_{k-alg}(\mathbf{k}[\mathbf{x}]^*, \mathbf{B}) = \operatorname{Hom}_{k-alg}(\mathbf{k}[[\mathbf{z}]], \mathbf{B}) \\ &= \varinjlim_n \operatorname{Hom}_{k-alg}(k[z]/(z^n), B) = \operatorname{rad} B. \end{aligned}$$

Let us follow the notation $G_a^* = \operatorname{rad}\mathbf{k}$. Explicitly, we assign to $n \in \operatorname{rad}\mathbf{k}$ the morphism $G_a \to G_m$, $\alpha \mapsto e^{\alpha\cdot n}$. Finally, $(\operatorname{rad}\mathbf{k})^* = G_a$.

Proposition 3.12. *Let $G = \operatorname{Spec} A$ be an abelian k-monoid. Let $p \in G^*(k)$ and let us regard k as a G-module via p. Then*

$$T_p G^* := Der_k(\mathbf{A}^*, \mathbf{k}) = Der(G, \mathbf{k}).$$

Let $q \in G(k)$ and let us regard k as a G^*-module via q. Then

$$T_q G = Der_k(\mathbf{A}, \mathbf{k}) = Der(G^*, \mathbf{k}).$$

Proof. It is a consequence from Proposition 2.9, where $\mathcal{V} = \mathbf{k}$. □

Let $\mathbf{I}_p^* := \ker p$, then $\operatorname{Hom}_k(\mathbf{I}_p^*/\mathbf{I}_p^{*2}, \mathbf{k}) = Der_k(\mathbf{A}^*, \mathbf{k}) = Der(G, \mathbf{k})$.

4. Structure of commutative k-groups in characteristic zero

For simplicity we assume that k is an algebraically closed field. Let $\mathbf{A}^*$ be a k-scheme of commutative algebras and $\mathbf{M}^* = \prod_J \mathbf{k}$ the maximal semisimple quotient of $\mathbf{A}^*$. The maximal spectrum of a scheme of algebras is defined to be the set of its maximal bilateral ideal schemes and it is the same as the set of isomorphism classes of simple $\mathbf{A}^*$-modules (see [A1, 6.7]). Then, $\operatorname{Spec}_{max}\mathbf{A}^* = \operatorname{Spec}_{max}\mathbf{M}^* = J$. $\mathbf{A}^*$ is the inverse limit of its cokernels of finite dimension. As every finite commutative

k-algebra is a direct product of local k-algebras, it is easy to prove that $\mathbf{A}^* = \prod_{j\in J} \mathbf{A}_j^*$, where the maximal semisimple quotient of $\mathbf{A}_i^*$ is $\mathbf{k}$. Then, there exists an only section of algebra k-schemes $s : \mathbf{M}^* \to \mathbf{A}^*$ of the epimorphism $\mathbf{A}^* \to \mathbf{M}^*$.

The maximal semisimple quotient of $\mathbf{A}^*\bar{\otimes}\mathbf{A}^* = (\mathbf{A}\otimes\mathbf{A})^*$ is $\mathbf{M}^*\bar{\otimes}\mathbf{M}^* = (\mathbf{M}\otimes\mathbf{M})^*$ ([A1, 6.23]). If $\operatorname{Spec}\mathbf{A}^*$ is also a functor of abelian groups, then $\operatorname{Spec}\mathbf{M}^*$, which is the constant functor J, is a functor of abelian subgroups. Furthermore, the morphism $s^* : \operatorname{Spec}\mathbf{A}^* \to \operatorname{Spec}\mathbf{M}^*$ induced by s is a morphism of functors of groups. Therefore, if $1 \in J$ is the identity element, we have the following decomposition of group functors

$$\operatorname{Spec}\mathbf{A}^* = \operatorname{Spec}\mathbf{M}^* \times \operatorname{Spec}\mathbf{A}_1^*.$$

In particular, $G = \operatorname{Spec} A$, which is the dual of $G^* = \operatorname{Spec}\mathbf{A}^*$, is a direct product of a diagonalizable group (every linear representation is a direct sum of simple modules of rank 1) and a unipotent group (every linear representation has got a filtration of invariant factors).

Given a k-vector space scheme $\mathbf{E}^*$, let us denote by $\bar{S}^n\mathbf{E}^*$ the representant of the functor

$$F(\mathbf{V}^*) = \operatorname{Sim}_k(\mathbf{E}^*\times \overset{n}{\cdots} \times\mathbf{E}^*, \mathbf{V}^*).$$

$\bar{S}^n\mathbf{E}^*$ is the quotient of $\mathbf{E}^*\bar{\otimes} \overset{n}{\cdots} \bar{\otimes}\mathbf{E}^* = (\mathbf{E}\otimes \overset{n}{\cdots} \otimes\mathbf{E})^*$ by the minimum k-vector space subscheme that contains elements of the following type

$$\ldots\otimes w\otimes\ldots\otimes w'\otimes\ldots - \ldots\otimes w'\otimes\ldots\otimes w\otimes\ldots .$$

$\bar{S}^n\mathbf{E}^*$ coincides with the closure of k-vector space schemes of $S^n\mathbf{E}^*$ and it coincides with $((\mathbf{E}\otimes \overset{n}{\cdots} \otimes\mathbf{E})^{S_n})^*$. In characteristic zero, $(E\otimes \overset{n}{\cdots} \otimes E)^{S_n} = S^nE$, then $\bar{S}^n\mathbf{E}^* = (S^n\mathbf{E})^*$.

If $E = \oplus_X k$, then $\mathbf{E}^* = \prod_X \mathbf{k}$ and $\bar{S}^n\mathbf{E}^* = \prod_{S^nX}\mathbf{k}$.

We will denote $k[[\mathbf{E}^*]] := \prod_{n\in\mathbb{N}} \bar{S}^n\mathbf{E}^*$. If $\operatorname{car} k = 0$ and $G = \mathbf{V}^*$, then $G^* = \operatorname{Spec} k[[\mathbf{V}^*]]$. Specifically, the natural morphism $\mathbf{V}^* \to k[[\mathbf{V}^*]]$ assigns $w \in \mathbf{V}^*$ to e^w.

Definition 4.1. *The kernel of the morphism $\mathbf{A}^* \to \mathbf{M}^*$ (the morphism of a scheme of algebras onto its maximal semisimple quotient) is said to be the radical of $\mathbf{A}^*$. We will say that $\mathbf{A}^*$ is local if $\mathbf{M}^* = \mathbf{k}$.*

Lemma 4.2. *Let $f : \mathbf{A}^* \to \mathbf{B}^*$ be a morphism between local k-algebra schemes and let $\mathbf{I}_A^*$ and $\mathbf{I}_B^*$ be the radical ideals of $\mathbf{A}^*$ and $\mathbf{B}^*$, respectively. If the induced morphism $\mathbf{I}_A^*/\mathbf{I}_A^{*2} \to \mathbf{I}_B^*/\mathbf{I}_B^{*2}$ is surjective, then f is a surjective morphism. If $\mathbf{B}^* = k[[\mathbf{E}^*]]$ and the morphism $\mathbf{I}_A^*/\mathbf{I}_A^{*2} \to \mathbf{I}_B^*/\mathbf{I}_B^{*2}$ is injective, then f is an injective morphism.*

Proof. By abuse of notation, we have regarded $\mathbf{I}^{*n}$ as the closure of vector space schemes of $\mathbf{I}^{*n}$ in $\mathbf{A}^*$ (as we did in [A1]). It is convenient to denote by $(\mathbf{I}^{*n})'$ this closure in this paragraph. The natural morphism $S^n\mathbf{I}^*/\mathbf{I}^{*2} \to \mathbf{I}^{*n}/\mathbf{I}^{*n+1}$ is surjective and the closure of k-vector space schemes of the image of $\mathbf{I}^{*n}/\mathbf{I}^{*n+1}$ in $\mathbf{A}^*/(\mathbf{I}^{*n+1})'$ is $(\mathbf{I}^{*n})'/(\mathbf{I}^{*n+1})'$. Therefore, the morphism $\bar{S}^n(\mathbf{I}^*/\mathbf{I}^{*2}) \to (\mathbf{I}^{*n})'/(\mathbf{I}^{*n+1})'$ is surjective.

If $\mathbf{I}_A^*/\mathbf{I}_A^{*2} \to \mathbf{I}_B^*/\mathbf{I}_B^{*2}$ is a surjective morphism, then the morphisms $\bar{S}^n(\mathbf{I}_A^*/\mathbf{I}_A^{*2}) \to \bar{S}^n(\mathbf{I}_B^*/\mathbf{I}_B^{*2})$ are surjective and from the commutative diagram

$$\begin{array}{ccc} \bar{S}^n(\mathbf{I}_A^*/\mathbf{I}_A^{*2}) & \xrightarrow{epi} & \bar{S}^n(\mathbf{I}_B^*/\mathbf{I}_B^{*2}) \\ \downarrow{\scriptstyle epi} & & \downarrow{\scriptstyle epi} \\ \mathbf{I}_A^{*n}/\mathbf{I}_A^{*n+1} & \longrightarrow & \mathbf{I}_B^{*n}/\mathbf{I}_B^{*n+1} \end{array}$$

it is deduced that the morphisms $\mathbf{I}_A^{*n}/\mathbf{I}_A^{*n+1} \to \mathbf{I}_B^{*n}/\mathbf{I}_B^{*n+1}$ are surjective. Then the morphisms $\mathbf{A}^*/\mathbf{I}_A^{*n+1} \to \mathbf{B}^*/\mathbf{I}_B^{*n+1}$ are surjective. Taking inverse limits (which is the dual of a direct limit of injections) we get the epimorphism $f : \mathbf{A}^* \to \mathbf{B}^*$ ([A1, 6.17 (2)]).

If $\mathbf{I}_A^*/\mathbf{I}_A^{*2} \to \mathbf{I}_B^*/\mathbf{I}_B^{*2}$ is an injective morphism, then the morphisms $\bar{S}^n(\mathbf{I}_A^*/\mathbf{I}_A^{*2}) \to \bar{S}^n(\mathbf{I}_B^*/\mathbf{I}_B^{*2})$ are injective and from the commutative diagram

$$\begin{array}{ccc} \bar{S}^n(\mathbf{I}_A^*/\mathbf{I}_A^{*2}) & \hookrightarrow & \bar{S}^n(\mathbf{I}_B^*/\mathbf{I}_B^{*2}) \\ \downarrow{\scriptstyle epi} & & \| \\ \mathbf{I}_A^{*n}/\mathbf{I}_A^{*n+1} & \longrightarrow & \mathbf{I}_B^{*n}/\mathbf{I}_B^{*n+1} \end{array}$$

it is deduced that the morphisms $\mathbf{I}_A^{*n}/\mathbf{I}_A^{*n+1} \to \mathbf{I}_B^{*n}/\mathbf{I}_B^{*n+1}$ are injective. Let $0 \neq a \in \mathbf{A}^*$ and let n be such that $a \in \mathbf{I}_A^{*n}$ and $a \neq \mathbf{I}_A^{*n+1}$ (see [A1, 6.17 (2)]). Then $f(a) \neq 0$, since $0 \neq \bar{a} \in \mathbf{I}_A^{*n}/\mathbf{I}_A^{*n+1}$ and $0 \neq \overline{f(a)} \in \mathbf{I}_B^{*n}/\mathbf{I}_B^{*n+1}$. □

Theorem 4.3. *Let $G = \operatorname{Spec} A$ be an abelian k-group and let k be an algebraically closed field of characterisic zero. Then, there exists an isomorphism of k-groups*

$$G = (\prod_X G_a) \times D$$

where D is a diagonalizable group.

Proof. We must prove that if G is unipotent, then it is isomorphic to a direct product of additive groups.

Thus, we have that $\mathbf{A}^*$ is local. Let $\mathbf{I}^*$ be its radical. The morphism $i : G \to 1 + \mathbf{I}^*/\mathbf{I}^{*2}$, $i(g) = 1 + \overline{g-1}$ is a morphism of k-groups. Let us prove that it is an isomorphism. Consider $\mathbf{k}$ as the trivial G-module. Then, $E := Der(G, \mathbf{k}) = \operatorname{Hom}_{lin}(G, \mathbf{k})$. The ring of functions of $1 + \mathbf{I}^*/\mathbf{I}^{*2} = \mathbf{I}^*/\mathbf{I}^{*2}$ is $\oplus_n S^n E$. The morphism between the rings induced by i is the natural morphism $\oplus_n S^n E \to A$. Dually, we have the morphism $\mathbf{A}^* \to k[[\mathbf{E}^*]] =: \mathbf{B}^*$. The induced morphism $\mathbf{I}^*/\mathbf{I}^{*2} \to \mathbf{I}_B^*/\mathbf{I}_B^{*2} = \mathbf{E}^*$ is the identity morphism. By the previous lemma, $\mathbf{A}^* = k[[\mathbf{E}^*]]$ and $G = \mathbf{I}^*/\mathbf{I}^{*2}$. □

5. Inversion formula

Let $\mathbf{B} \subseteq \prod_J \mathbf{k}$ be a quasi-coherent k-algebra (with or without unit). We define $\operatorname{Spec} \mathbf{B}$ to be the functor over the category of commutative algebras (with unit)

$$(\operatorname{Spec} \mathbf{B})(C) := \{f \in \operatorname{Hom}_{k-alg}(B, C) : (\operatorname{Im} f) = C\}.$$

If k is a field it is easily seen that $\operatorname{Spec} \mathbf{B} = \operatorname{Spec} \prod_J \mathbf{k}$ if and only if $\mathbf{B} = \oplus_J \mathbf{k}$.

Definition 5.1. *We will say that $\oplus_J k$ is the cuasi-coherent ring of functions of* $\operatorname{Spec} \prod_J \mathbf{k}$.

As well, $\oplus_J \mathbf{k}$ is characterized by being the maximal quasi-coherent $\prod_J \mathbf{k}$-submodule of $\prod_J \mathbf{k}$.

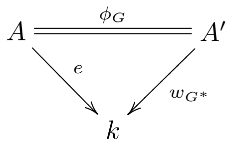
Let $G^* = \operatorname{Spec} \prod_J \mathbf{k}$ be a functor of discrete groups. G^* operates on G^* by translations on the left, then it operates on $\prod_J \mathbf{k}$, then on $\oplus_J k$. The mapping $w_{G^*} : \oplus_J k \to k$, $w_{G^*}((\lambda_i)) = \sum_i \lambda_i$ is the only G^*-invariant linear form up to a multiplicative factor. We will say that the linear mapping

$$\phi_{G^*} : \oplus_J \mathbf{k} \to (\prod_J \mathbf{k})^* \hookrightarrow (\oplus_J \mathbf{k})^*, a \mapsto w_{G^*}(a \cdot -)$$

is the transposed Fourier transform of G^*.

$G = \operatorname{Spec} A$ is a semisimple k-group if and only if there exists a (unique) linear mapping $w_G : A \to k$, G-invariant and such that $w_G(1) = 1$ (see [A2, 2.10]).

The morphism $\phi_G : A \to A^*$, $\phi_G(a) := w_G(a^* \cdot -)$ (where $* : A \to A$ is the morphism between rings induced by the morphism $G \to G, g \mapsto g^{-1}$) is called the Fourier transform of G (see [A3]). If G is also commutative, then $A^* = \prod_J k$ and $\operatorname{Im} \phi_G = \oplus_J k =: A'$. Let $e : A \to k$ the identity element of G (that is, the unit of A^*). The diagram



is commutative (see [A3], section 2).

Theorem 5.2. *The transposed Fourier transform of the dual group of a semisimple abelian k-group is the inverse of the Fourier transform of the semisimple abelian group.*

Proof. $\phi_{G^*}(\phi_G(a))(w) = w_{G^*}(\phi_G(a) \cdot w) = w_{G^*}(\phi_G(a \cdot w)) = (a \cdot w)(e) = a(w)$ for all $a \in A$ and $w \in A^*$. □

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