CHARACTERIZATION OF QUASI-COHERENT MODULES THAT ARE MODULE SCHEMES

Introduction

Let R be a commutative ring with unit. All functors we consider are functors over the category of commutative R-algebras. Given an R-module E, we denote by $\mathbf{E}$ the functor of R-modules $\mathbf{E}(B) := E \otimes_R B$. We will say that $\mathbf{E}$ is a quasi-coherent R-module. If E is an R-module of finite type we will say that $\mathbf{E}$ is a coherent R-module. Given F, H functors of R-modules, $\mathbf{Hom}_R(F, H)$ will denote the functor of R-modules

$$\mathbf{Hom}_R(F, H)(B) = \mathrm{Hom}_B(F_{|B}, H_{|B})$$

where $F_{|B}$ is the functor F restricted to the category of commutative B-algebras. It holds that $\mathrm{Hom}_R(\mathbf{E}, \mathbf{E}') = \mathrm{Hom}_R(E, E')$. The category of R-modules is equivalent to the category of quasi-coherent R-modules ([A, 1.12]).

We denote $F^* := \mathbf{Hom}_R(F, \mathbf{R})$. We will say that $\mathbf{E}^*$ is an R-module scheme.

The R-module functors that are essential for the development of the theory of the linear representations of an affine R-group are the quasi-coherent R-modules and the R-module schemes ([A]). The aim of this paper is to study when a quasi-coherent R-module is an R-module scheme. We will prove that it is equivalent to giving a characterization of projective R-modules of finite type.

The main result we are going to use is the following proposition.

Proposition 0.1. [A, 1.8] *Let E, E' be R-modules. Then:*

$$\mathbf{Hom}_R(\mathbf{E}^*, \mathbf{E}') = \mathbf{E} \otimes_R \mathbf{E}'.$$

Two immediate consequences are:

(a) $\mathbf{E}^{**} = \mathbf{E}$ ([A, 1.10]).
(b) *If E is a projective module, every morphism $\mathbf{E}^* \to \mathbf{V}$ factorizes via an epimorphism onto a coherent submodule of* $\mathbf{V}$ *(*[A, 4.5]*).*

1. Characterization of quasi-coherent modules that are module schemes

Theorem 1.1. *Let E be an R-module. Then, $\mathbf{E}$ is an R-module scheme if and only if E is a projective module of finite type.*

Proof. $\Rightarrow$) Assume $\mathbf{E} = \mathbf{V}^*$. For every R-module E',

$$\mathrm{Hom}_R(E, E') = \mathrm{Hom}_R(\mathbf{E}, \mathbf{E}') = \mathrm{Hom}_R(\mathbf{V}^*, \mathbf{E}') \overset{0.1}{=} V \otimes_R E'.$$

E is a projective module because $\mathrm{Hom}_R(E, -) = V \otimes_R -$, which is exact on the right.

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Since $\mathbf{V} \overset{0.1(a)}{=} \mathbf{V}^{**} = \mathbf{E}^*$, we have that V is a projective module. The image of the isomorphism $\mathbf{V}^* = \mathbf{E}$ is coherent (0.1 (b)). Then, E is an R-module of finite type.

$\Leftarrow$) Let us consider an epimorphism from a finitely generated free module L to E, $L \to E$. Taking dual we have an injective morphism $\mathbf{E}^* \hookrightarrow \mathbf{L}^*$. $\mathbf{L}^*$ is isomorphic to a coherent module. The image of the morphism $\mathbf{E}^* \hookrightarrow \mathbf{L}^*$, which is $\mathbf{E}^*$, is coherent (0.1(b)). Then, $\mathbf{E} = \mathbf{E}^{**}$ is a module scheme. □

Corollary 1.2. *An R-module E is projective of finite type if and only if*

$$\mathrm{Hom}_R(E, B) = \mathrm{Hom}_R(E, R) \otimes_R B$$

for every commutative R-algebra.

Proof. $\mathrm{Hom}_R(E, B) = \mathrm{Hom}_R(E, R) \otimes_R B$ for every commutative R-algebra if and only if $\mathbf{E}^*$ is a quasi-coherent R-module. That is to say, if and only if $\mathbf{E}$ is a module scheme, from the previous proposition, if and only if E is a projective module of finite type. □

Corollary 1.3. *An R-module E is projective of finite type if and only if*

$$\mathrm{Hom}_R(E, E') = \mathrm{Hom}_R(E, R) \otimes_R E'$$

for every R-module E'.

In [B, Ch. II, §4.2, Prop. 2] it can be found the necessary condition of this corollary. For the sufficient one, we will only say that if $\mathrm{Id} = \sum_i w_i \otimes e_i \in \mathrm{Hom}_R(E, E) = E^* \otimes_R E$, then $E = < e_i >$ is a module of finite type.

Corollary 1.4. *The quasi-coherent R-module corresponding to the R-module E^* is $\mathbf{E}^*$ if and only if E is a projective module of finite type.*

References

[A] Álvarez, A., Sancho, C., Sancho,P., *Algebra schemes and their representations*, J. Algebra **296/1** (2006) 110-144.

[B] Bourbaki, N., Éléments de Mathématique. Algèbre I, Hermann, Paris, 1970.