A DIRECT PROOF OF THE THEOREM ON FORMAL FUNCTIONS

FERNANDO SANCHO DE SALAS AND PEDRO SANCHO DE SALAS

Abstract. We give a direct and elementary proof of the theorem on formal functions by studying the behaviour of the Godement resolution of a sheaf of modules under completion.

Introduction

Let \( \pi : X \to \text{Spec } A \) be a proper scheme over a ring \( A \). Let \( \mathcal{M} \) be a coherent \( \mathcal{O}_X \)-module and \( Y \subset \text{Spec } A \) a closed subscheme. Let us denote by \( ^\wedge \) the completion along \( Y \) (respectively, along \( \pi^{-1}(Y) \)). The theorem on formal functions states that

\[
H^i(X, \mathcal{M})^\wedge = H^i(X, \hat{\mathcal{M}})
\]

Two important corollaries of this theorem are Stein’s factorization theorem and Zariski’s Main Theorem ([H] III, 11.4, 11.5).

Hartshorne [H] gives a proof of the theorem on formal functions for projective schemes (over a ring). Grothendieck [G] proves it for proper schemes. He first gives sufficient conditions for the commutation of the cohomology of complexes of \( A \)-modules with inverse limits (0, 13.2.3 [G]); secondly, he gives a general theorem on the commutation of the cohomology of sheaves with inverse limits (0, 13.3.1 [G]); finally, he laboriously checks that the theorem on formal functions is under the hypothesis of this general one (4.1.5 [G]).

In this paper we give the “obvious direct proof” of the theorem on formal functions. Very briefly, we prove that the completion of the Godement resolution of a coherent sheaf is a flasque resolution of the completion of the coherent sheaf and that taking sections in the Godement complex commutes with completion.

1. Theorem on formal functions

Definition 1. Let \( X \) be a scheme, \( p \subset \mathcal{O}_X \) a sheaf of ideals and \( \mathcal{M} \) an \( \mathcal{O}_X \)-module. The \( p \)-adic completion of \( \mathcal{M} \), denoted by \( \hat{\mathcal{M}} \), is

\[
\hat{\mathcal{M}} := \varprojlim_n \mathcal{M}/p^n\mathcal{M}
\]

If \( U = \text{Spec } A \) is an affine open subset and \( I = p(U) \), one has a natural morphism

\[
\Gamma(U, \mathcal{M}) \otimes_A A/I^n \to \Gamma(U, \mathcal{M}/p^n\mathcal{M})
\]
We say that

\[ C \]

and then a morphism

\[ \Gamma(U, \mathcal{M})^\wedge \to \Gamma(U, \hat{\mathcal{M}}) \]

where \( \Gamma(U, \mathcal{M})^\wedge \) is the I-adic completion of \( \Gamma(U, \mathcal{M}) \).

**Definition 2.** We say that \( \mathcal{M} \) is affinely \( p \)-acyclic if for any affine open subset \( U \) and any natural number \( n \), the sheaves \( \mathcal{M} \) and \( \mathcal{M}/p^n\mathcal{M} \) are acyclic on \( U \) and the morphism \( \Gamma(U, \mathcal{M}) \otimes_A A/I^n \to \Gamma(U, \mathcal{M}/p^n\mathcal{M}) \) is an isomorphism. In particular, \( \Gamma(U, \mathcal{M})^\wedge \to \Gamma(U, \hat{\mathcal{M}}) \) is an isomorphism.

Every quasi-coherent module is affinely \( p \)-acyclic.

**Notations:** For any sheaf \( F \), let us denote

\[ 0 \to F \to C^0F \to C^1F \to \cdots \to C^nF \to \cdots \]

its Godement resolution. We shall denote \( C^iF = \bigoplus_{i \geq 0} C^iF \) and \( F_i = \text{Ker}(C^iF \to C^{i+1}F) \). One has that \( C^0F_i = C^0F \).

**Lemma 3.** Let \( X \) be a scheme, \( \mathfrak{p} \) a coherent ideal and \( \mathcal{M} \) an \( \mathcal{O}_X \)-module. Denote \( I = \Gamma(X, \mathfrak{p}) \) and assume that \( \mathfrak{p} \) is generated by a finite number of global sections (this holds for example when \( X \) is affine). For any open subset \( V \subseteq X \) one has

\[ \Gamma(V, C^0(\mathfrak{p}, \mathcal{M})) = I \cdot \Gamma(V, C^0\mathcal{M}) \]

In particular, the natural morphism \( pC^0\mathcal{M} \to C^0(\mathfrak{p}, \mathcal{M}) \) is an isomorphism.

**Proof.** If \( J \) is a finitely generated ideal of a ring \( A \) and \( M_i \) is a collection of \( A \)-modules, then \( J \cdot \prod M_i = \prod(J \cdot M_i) \). Now, by hypothesis \( \mathfrak{p} \) is generated by a finite number of global sections \( f_1, \ldots, f_r \). Let \( J = (f_1, \ldots, f_r) \). Then

\[ \Gamma(V, C^0(\mathfrak{p}, \mathcal{M})) = \prod_{x \in V} p_x \cdot \mathcal{M}_x = \prod_{x \in V} J \cdot \mathcal{M}_x = J \cdot \prod_{x \in V} \mathcal{M}_x = J \cdot \Gamma(V, C^0\mathcal{M}) \]

Since \( I \cdot \prod_{x \in V} \mathcal{M}_x \) is contained in \( \Gamma(V, C^0(\mathfrak{p}, \mathcal{M})) \) one concludes. In particular, if \( V \) is affine, then \( \Gamma(V, C^0(\mathfrak{p}, \mathcal{M})) = I_V \cdot \Gamma(V, C^0\mathcal{M}) \), with \( I_V = \Gamma(V, \mathfrak{p}) \). It follows that \( pC^0\mathcal{M} \to C^0(\mathfrak{p}, \mathcal{M}) \) is an isomorphism.

**Proposition 4.** Let \( X \) be a scheme and let \( \mathfrak{p} \) be a coherent ideal. For any \( \mathcal{O}_X \)-module \( \mathcal{M} \) one has:

1. \( pC^i\mathcal{M} = C^i(\mathfrak{p}, \mathcal{M}) \) and \( (C^i\mathcal{M})/p(C^i\mathcal{M}) = C^i(\mathcal{M}/p\mathcal{M}) \), for any \( i \).
2. \( C^0\mathcal{M} \) is affinely \( p \)-acyclic.
3. \( C^0\hat{\mathcal{M}} \) is flasque. Moreover, if \( \mathfrak{p} \) is generated by a finite number of global sections, then

\[ \Gamma(X, C^0\hat{\mathcal{M}}) = \Gamma(X, C^0\mathcal{M})^\wedge \]

**Proof.** 1. We may assume that \( X \) is affine. Hence \( pC^0\mathcal{M} = C^0(\mathfrak{p}, \mathcal{M}) \) by the previous lemma and \( (C^0\mathcal{M})/pC^0\mathcal{M} = C^0\mathcal{M}/C^0(\mathfrak{p}, \mathcal{M}) = C^0(\mathcal{M}/p\mathcal{M}) \). From the exact sequence

\[ \mathcal{M}/p\mathcal{M} \to C^0\mathcal{M}/pC^0\mathcal{M} \to M_1/p\mathcal{M} \to 0 \]

and the isomorphism \( C^0\mathcal{M}/pC^0\mathcal{M} = C^0(\mathcal{M}/p\mathcal{M}) \) it follows that \( M_1/pM_1 = (\mathcal{M}/p\mathcal{M})_1 \) and \( pM_1 = (p\mathcal{M})_1 \). Consequently \( pC^1\mathcal{M} = pC^0(M_1) = C^0(p\mathcal{M})_1 = C^0((p\mathcal{M})_1) = C^1(p\mathcal{M}) \), and analogously \( C^1\mathcal{M}/pC^1\mathcal{M} = C^1(\mathcal{M}/p\mathcal{M}) \). Repeating this argument one concludes 1.
2. Denote $\mathcal{N} = C^0\mathcal{M}$. By (1), $\mathcal{N}/p^n\mathcal{N}$ is acyclic on any open subset. From the long exact sequence of cohomology associated to $0 \to p^n\mathcal{N} \to \mathcal{N} \to \mathcal{N}/p^n\mathcal{N} \to 0$ and the acyclicity of $p^n\mathcal{N}$ (by (1)) one obtains that
\[ \Gamma(U, \mathcal{N}/p^n\mathcal{N}) = \Gamma(U, \mathcal{N})/\Gamma(U, p^n\mathcal{N}). \]
Moreover, if $U$ is affine $\Gamma(U, p^n\mathcal{N}) = p^n(U)\Gamma(U, \mathcal{N})$, by Lemma 3. We have concluded.

3. Let us prove that $\mathcal{N} = C^0\mathcal{M}$ is flasque. It suffices to prove that its restriction to any affine open subset is flasque, so we may assume that $X$ is affine. Let us denote $I = p(X)$. For any open subset $V$, one has as in the proof of (2)
\[ \Gamma(V, \hat{\mathcal{N}}) = \lim_{\to} \Gamma(V, \mathcal{N}/p^n\mathcal{N}) = \lim_{\to} \Gamma(V, \mathcal{N})/\Gamma(V, p^n\mathcal{N}) \]
and by Lemma 3, $\Gamma(V, p^n\mathcal{N}) = I^n\Gamma(V, \mathcal{N})$. In conclusion, $\Gamma(V, \hat{\mathcal{N}}) = \Gamma(V, \mathcal{N})\hat{\to}$. One concludes that $\hat{\mathcal{N}}$ is flasque because $\mathcal{N}$ is flasque and the $I$-adic completion preserves surjections. The same arguments prove the second part of the statement.

**Proposition 5.** If $\mathcal{M}$ is affinely $p$-acyclic, then $\hat{\mathcal{M}}$ is a flasque resolution of $\hat{\mathcal{M}}$.

**Proof.** We already know that $\hat{\mathcal{M}}$ is flasque. Let us prove now that $\mathcal{M}_1$ is affinely $p$-acyclic. From the exact sequence
\[ 0 \to \mathcal{M}/p^n\mathcal{M} \to C^0(\mathcal{M}/p^n\mathcal{M}) \to \mathcal{M}_1/p^n\mathcal{M}_1 \to 0 \]
one has that $\mathcal{M}_1/p^n\mathcal{M}_1$ is acyclic on any affine open subset. Moreover, taking sections on an affine open subset $U = \text{Spec} A$, one obtains the exact sequence (let us denote $I = p(U)$)
\[ 0 \to \Gamma(U, \mathcal{M}) \otimes_A A/I^n \to \Gamma(U, C^0\mathcal{M}) \otimes_A A/I^n \to \Gamma(U, \mathcal{M}_1/p^n\mathcal{M}_1) \to 0 \]
and then $\Gamma(U, \mathcal{M}_1) \otimes_A A/I^n = \Gamma(U, \mathcal{M}_1/p^n\mathcal{M}_1)$, i. e. $\mathcal{M}_1$ is affinely $p$-acyclic.

Now, taking inverse limit in the above exact sequence (and taking into account that the $I$-adic completion preserves surjections) one obtains the exact sequence
\[ 0 \to \Gamma(U, \hat{\mathcal{M}}) \to \Gamma(U, C^0\mathcal{M}) \to \Gamma(U, \hat{\mathcal{M}}_1) \to 0 \]
Therefore the sequence $0 \to \hat{\mathcal{M}} \to C^0\hat{\mathcal{M}} \to \hat{\mathcal{M}}_1 \to 0$ is exact. Conclusion follows easily.

**Remark 6.** In the proof of the preceding proposition it has been proved that if $\mathcal{M}$ is affinely $p$-acyclic, then $\hat{\mathcal{M}}$ is acyclic on any affine subset.

**Lemma 7.** Let $A$ be a noetherian ring and $I \subset A$ an ideal. If $0 \to M' \to M \to N \to 0$ is an exact sequence of $A$-modules and $N$ is finitely generated, then the $I$-adic completion $0 \to \hat{M}' \to \hat{M} \to \hat{N} \to 0$ is exact.

**Proof.** Let $L \subset M$ be a finite submodule surjecting on $N$ and $L' = L \cap M'$ which is also finite because $A$ is noetherian. The exact sequences
\[ 0 \to L \to M \to M/L \to 0, \quad 0 \to L' \to M' \to M'/L' \to 0, \quad 0 \to L' \to L \to N \to 0 \]
remain exact after $I$-adic completion, because $L$ and $L'$ are finite (this is a consequence of Artin-Rees lemma (10.10 [A])). Since $M/L \simeq M'/L'$ one concludes.
Theorem 8 (on formal functions). Let $f : X \to Y$ be a proper morphism of locally noetherian schemes, $\mathfrak{p}$ a coherent sheaf of ideals on $Y$ and $\mathfrak{p}\mathcal{O}_X$ the ideal induced in $X$. For any coherent module $\mathcal{M}$ on $X$, the natural morphisms (where completions are made by $\mathfrak{p}$ and $\mathfrak{p}\mathcal{O}_X$ respectively)

$$R^if_*\mathcal{M} \to R^i f_*(\hat{\mathcal{M}})$$

are isomorphisms. If $Y = \text{Spec } A$, then

$$H^i(X, \mathcal{M})^\wedge = H^i(X, \hat{\mathcal{M}})$$

Proof. The question is local on $Y$, so we may assume that $Y = \text{Spec } A$ is affine. It suffices to show that $H^i(X, \mathcal{M})^\wedge = H^i(X, \hat{\mathcal{M}})$. It is clear that $\mathfrak{p}\mathcal{O}_X$ is generated by its global sections. As usual, we denote $I = \Gamma(X, \mathfrak{p}\mathcal{O}_X)$.

Let $C\mathcal{M}$ be the Godement resolution of $\mathcal{M}$. Then $\hat{C}\mathcal{M}$ is a flasque resolution of $\hat{\mathcal{M}}$ (by Proposition 5) and $\Gamma(X, \hat{C}\mathcal{M}) = \Gamma(X, C\mathcal{M})^\wedge$ (by Proposition 4, (3)). Then we have to prove that the natural map

$$H^i(X, \mathcal{M})^\wedge = [H^i\Gamma(X, C\mathcal{M})]^\wedge \to H^i(\Gamma(X, C\mathcal{M})^\wedge) = H^i(\Gamma(X, \hat{C}\mathcal{M})) = H^i(X, \hat{\mathcal{M}})$$

is an isomorphism. Let us denote by $d_i$ the differential of the complex $\Gamma(X, C\mathcal{M})$ on degree $i$. Completing the exact sequences

$$0 \to \ker d_i \to \Gamma(X, C\mathcal{M}) \to \im d_i \to 0$$

we obtain the exact sequences

$$0 \to \overline{\ker d_i} \to \Gamma(X, \hat{C}\mathcal{M}) \to \overline{\im d_i} \to 0$$

because, as we shall see below, the $I$-adic topology of $\Gamma(X, C\mathcal{M})$ induces in $\ker d_i$ the $I$-adic topology. Hence

$$H^i(X, \mathcal{M})^\wedge = (\ker d_i / \im d_{i-1})^\wedge \xrightarrow{\text{Lemma ?}} \overline{\ker d_i / \im d_{i-1}} = H^i(X, \hat{\mathcal{M}})$$

Let $\mathcal{M}_i$ be the kernel of $C^i\mathcal{M} \to C^{i+1}\mathcal{M}$ (recall that $C^i\mathcal{M} = C^0\mathcal{M}_i$). Let us prove that the $I$-adic topology of $\Gamma(X, C\mathcal{M})$ induces the $I$-adic topology on $\ker d_i = \Gamma(X, \mathcal{M}_i)$. Intersecting the equality $I^n\Gamma(X, C^0\mathcal{M}_i) = \Gamma(X, C^n(\mathfrak{p}\mathcal{M}_i))$ with $\Gamma(X, \mathcal{M}_i)$, one obtains that the induced topology on $\Gamma(X, \mathcal{M}_i)$ is given by the filtration $\{\Gamma(X, \mathfrak{p}^n\mathcal{M}_i)\}$. Hence it suffices to show that this filtration is $I$-stable. Since $\mathfrak{p}^n\mathcal{M}_i = (\mathfrak{p}\mathcal{M}_i)_i$ (see the proof of 4.1.), it is enough to prove that the filtration $\{\Gamma(X, \mathfrak{p}^n\mathcal{M}_i)\}$ is $I$-stable; this is equivalent to show that $\oplus_{n=0}^\infty \Gamma(X, (\mathfrak{p}\mathcal{M}_i)_i)$ is a $D_1\mathcal{A}$-module generated by a finite number of homogeneous components, where $D_1\mathcal{A} = \oplus_{n=0}^\infty I^n$. By the exact sequence

$$\oplus_{n=0}^\infty \Gamma(X, C^{n-1}(\mathfrak{p}\mathcal{M}_i)) \to \oplus_{n=0}^\infty \Gamma(X, (\mathfrak{p}\mathcal{M}_i)_i) \to \oplus_{n=0}^\infty H^i(X, \mathfrak{p}^n\mathcal{M}) \to 0$$

it suffices to see the statement for the first and the third members. For the first one is obvious because $\Gamma(X, C^{n-1}(\mathfrak{p}\mathcal{M}_i)) = I^n\Gamma(X, C^{n-1}\mathcal{M}_i)$. For the third one, it suffices to see that it is a finite $D_1\mathcal{A}$-module. Let $X' = X \times_A D_1\mathcal{A}$, $\pi : X' \to X$ the natural projection and $\mathcal{M}' = \oplus_{n=0}^\infty \mathfrak{p}^n\mathcal{M}$ the obvious $C_{X'}$-module. Since $H^i(X', \mathcal{M}')$ is a finite $D_1\mathcal{A}$-module, one concludes from the equalities $H^i(X', \mathcal{M}') = H^i(X, \pi_*\mathcal{M}') = \oplus_{n=0}^\infty H^i(X, \mathfrak{p}^n\mathcal{M})$, because $\pi_*\mathcal{M}' = \oplus_{n=0}^\infty \mathfrak{p}^n\mathcal{M}$.
Remark 9. Reading carefully the above proof, it is not difficult to see that one has already showed that $H^i(X, \mathcal{M}) = \lim_{\rightarrow} H^i(X, \mathcal{M}/p^n\mathcal{M})$.

REFERENCES


Departamento de Matemáticas, Universidad de Salamanca, Plaza de la Merced 1-4, 37008 Salamanca, Spain
E-mail address: fsancho@usal.es

Departamento de Matemáticas, Universidad de Extremadura, Avenida de Elvas s/n, 06071 Badajoz, Spain
E-mail address: sancho@unex.es