INVERSE LIMIT OF GRASSMANNIANS

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Introduction

Grothendieck defined functorially the Grassmannian of the vector subspaces of finite codimension of a vector space, and proved that this functor was representable by a scheme. However, for the construction of the Hilbert scheme of the closed varieties of a projective variety (see [G]), and the moduli of formal curves and soliton theory, etc., it is convenient to consider Grassmannians in a more general situation. M. Sato and Y. Sato, [SS] and Segal and Wilson, [SW] considere, given a vector subspace $E' \subset E$, the topology on E, for which the set of the vector subspaces $F \subset E$ such that $F + E'/F \cap E'$ are finite dimensional vector spaces is a neighbourhood basis at 0. Then they prove that the subset of vector subspaces F of the completion of E, such that $E/F + E'$ and $F \cap E'$ are finite dimensional vector spaces is a scheme. Álvarez-Muñoz-Plaza, [AMP] define and prove these results functorially.

In this paper we prove that, if E is the inverse limit of linear maps $\{E_i \to E_j\}$, with finite dimensional kernels and cokernels, and if we consider the Grassmannian of finite codimensional vector spaces of E_i, $G(E_i)$, then we get an inverse system of rational morphisms $\{G(E_i) \dashrightarrow G(E_j)\}$, which is representable by a scheme, whose points represent the vector subspaces of $F \subset E$, such that the morphism $F \to E_i$ is injective and the cokernel is a finite dimensional vector space (1.10). We obtain the results of Álvarez-Muñoz-Plaza [AMP], as a consequence of 1.10.

1. Grassmannians of inverse limit of vector spaces

1. Teorema Grothendieck, [GD]: *Let E be a k-vector space. The functor on the category of noetherian k-algebras*

$$Gr(E)(\operatorname{Spec} A) := \left\{ \begin{array}{c} A\text{-submodules } M \subset E_A = E \otimes_k A \text{ whose cokernel} \\ \text{is a projective finitely generated module} \end{array} \right\}$$

is representable by a scheme and is covered by the affine open subsets

$$\mathbf{Hom}_k(E_{\infty-n}, E_n) := \operatorname{Spec} S^{\cdot}(E_{\infty-n} \otimes E_n^*)$$

where $E_n \subseteq E$, $\dim_k E_n < \infty$ and $E_{\infty-n}$ is any vector subspace of E such that $E_n \oplus E_{\infty-n} = E$ (such open subsets are defined as being the set of vector supplementary subspaces of E_n with respect to E).

Moreover, $Gr(E) = \coprod_{n\in\mathbb{N}} Gr_n(E)$, where

$$Gr_n(E)(\operatorname{Spec} A) := \left\{ \begin{array}{c} A\text{-submodules } M \subset E_A = E \otimes_k A \text{ whose cokernel} \\ \text{is a locally free finitely generated module of rank } n \end{array} \right\}$$

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2. Definición: Given V and W submodules of a module E, we will say that V and W are supplementary vector spaces with respect to E if $E = V \oplus W$. In particular, both V and W are direct summands of E.

3. Proposición : *Let L be a free A-module and $L' \subset L$ a finitely generated submodule. Then L' is a direct summand of L if and only if L/L' is a projective module, if and only if L' is a submodule on fibers for every closed point of* $\operatorname{Spec} A$, *and if and only if $L' \to L$ is a universally injective morphism.*

Therefore, the fact that a finitely generated submodule of L is a direct summand is a local question in the Zariski topology, and it is also a question on fibers for every closed point.

Proof. We must only prove that if L' is a submodule on fibers for every closed point of $\operatorname{Spec} A$, then L/L' is a projective module. Because L' is finitely generated, it injects into a free finitely generated submodule of L that is a direct summand of it, therefore we can assume that L is also finitely generated. Then, L/L' is of finite presentation and it will be projective if and only if so is locally for every closed point of $\operatorname{Spec} A$. Let A be a local ring of maximal $\mathfrak{m}$. By assumption $\operatorname{Tor}^1_A(L/L', A/\mathfrak{m}) = 0$, then L/L' is projective. □

4. Notación: We denote $(\operatorname{Spec} A)^{\cdot}$ as the functor $(\operatorname{Spec} A)^{\cdot}(X) = \operatorname{Hom}_{k-\text{sch}}(X, \operatorname{Spec} A)$.

5. Lema : *Given a vector subspace $E' \subset E$ of finite dimension, it holds that the set of vector subspaces of $E'' \subset E$, of finite codimension such that $E' \cap E'' = 0$, is an open subset of the Grassmannian of subspaces of finite codimension of E.*

Proof. The subfunctor $U \subset Gr(E)$ from the statement is defined by

$$U(\operatorname{Spec} A) := \left\{ \begin{array}{c} M \in Gr(E)(\operatorname{Spec} A) \text{ such that } M \hookrightarrow E_A/E'_A \\ \text{is injective with projective cokernel} \end{array} \right\}$$

We must prove that $U \times_{Gr(E)} (\operatorname{Spec} A)^{\cdot}$ is an open subfunctor of $(\operatorname{Spec} A)^{\cdot}$. Given $\operatorname{Spec} A \to Gr(E)$, that is, an A-submodule $M \subset E_A$ with finite projective cokernel, then by Proposition 1.3, $U \times_{Gr(E)} (\operatorname{Spec} A)^{\cdot}$ is the open subset of the points of $\operatorname{Spec} A$ where the morphism $E'_A \hookrightarrow E_A/M$ is injective on fibers. □

Every linear application between vector spaces $L : E_1 \to E_2$, whose kernels and cokernels are of finite dimension, induces a rational morphism between their Grassmannians $L_G : Gr(E_1) \dashrightarrow Gr(E_2)$, $V \mapsto L(V)$. This morphism is defined on the open subset $Gr'(E_1)$ of $Gr(E_1)$ made up by the vector subspaces V of E_1 that do not intersect with $\ker L$, more precisely, by the submodules V such that any of their supplementary submodules contain $\ker L$, that is equivalent to saying that on fibers $\ker L$ and V intersect in zero. Moreover, $Gr'_n(E_1) = Gr_n(E_1) \cap Gr'(E_1)$ is the maximal open subset of $Gr_n(E_1)$ where L_G is defined (for $n > \dim_k \ker L$): Let us assume , for simplicity, that E_1 is vector space of finite dimension, $V \subset E_1$ satisfies $V \cap \ker L = \langle w \rangle$ and L_G is defined on V. Let be $r+1$ vectors of E_1, $v_1, v'_1, v_2, \dots, v_r$, such that are linearly independents in $E_1/\ker L$ and that $V =< w, v_2, \dots, v_r >$. We would have that

$$L_G(< w, v_2, \dots, v_r >) \overset{\lambda \to 0}{=} L_G(< w + \lambda v_1, v_2, \dots, v_r >) = L_G(< v_1, \dots, v_r >)$$

and

$$L_G(< w, v_2, \dots, v_r >) \overset{\lambda \to 0}{=} L_G(< w + \lambda v'_1, v_2, \dots, v_r >) = L_G(< v'_1, \dots, v_r >)$$

that is, $L_G(<v_1,\dots,v_r>) = L_G(<v'_1,\dots,v_r>)$. But the morphism $E_1/\ker L \overset{\overline{L}}{\hookrightarrow} E_2$ is injective, so that $\overline{L}(<\bar{v}_1,\dots,\bar{v}_r>) \neq \overline{L}(<\bar{v}'_1,\dots,\bar{v}_r>)$, which leads to contradiction.

$L_G : Gr'_n(E_1) \to Gr(E_2)$ is an affine morphism (for $n > \dim_k \ker L$), because if U is the affine open subset of $Gr(E_2)$ of vector subspaces of E_2 that are supplementary vector spaces of V', a subspace of E of finite dimension, then $L_G^{-1}(U)$ is the affine open subset of $Gr(E_1)$ of vector subspaces of E_1 that are supplementary vector spaces of $L^{-1}(V')$.

For $n \leq \dim_k \ker L$, L_G is constant over $Gr_n(E_1)$ and its image is $\operatorname{Im} L \in Gr(E_2)$.

6. Definición : Let $f : X \dashrightarrow Y$ be a rational morphism between schemes, being Y a separated one, and let $g : T \to X$, $h : T \to Y$ be two morphisms of schemes. We shall say that $f \circ g = h$ if g values in the definition domain of f and it holds, now with sense, that $f \circ g = h$.

We shall say that a scheme $X = \lim\limits_{\overleftarrow{i\in I}} X_i$ is the inverse limit of the family of rational morphisms $\{X_i \dashrightarrow X_j\}_{i>j}$ if it holds that

$$\operatorname{Hom}_{\mathrm{k-sch}}(Y,X) = \lim_{\overrightarrow{j\in I}} \left[\lim_{\overleftarrow{i>j}} \operatorname{Hom}_{\mathrm{k-sch}}(Y,X_i) \right]$$

That is, giving a morphism $Y \to X$ is equivalent to giving a family of morphisms $Y \to X_i$ for all i greater than some j, satisfying the corresponding commutative diagrams, and we say two of these families are equal if the morphisms that value in the same X_i are equal.

7. Teorema : *Let $E = \lim\limits_{\overleftarrow{i\in I}} E_i$ be an inverse limit of vector spaces with kernels and cokernels of finite dimension. Let us consider the projective system of rational morphisms $\{Gr(E_i) \dashrightarrow Gr(E_j)\}_{i\geq j}$. The scheme $\lim\limits_{\overleftarrow{i\in I}} Gr(E_i)$ exists.*

Proof. The functor $X \rightsquigarrow F(X) = \lim\limits_{\overrightarrow{j\in I}} \left[\lim\limits_{\overleftarrow{i>j}} \operatorname{Hom}_{\mathrm{k-sch}}(X, Gr(E_i)) \right]$ is a sheaf for the Zariski topology, then to prove that it is representable it is sufficient to prove it locally.

Fixed an index j let us consider a subspace V'_j of finite dimension of E_j, and for every morphism $E_i \to E_j$ let us consider the vector subspace V'_i of E_i inverse image of V'_j, and the affine schemes $U_i \subset Gr(E_i)$, made up by the supplementary vector spaces of V'_i with respect to E_i. The inverse limit $G_j = \lim\limits_{\overleftarrow{i\geq j}} U_i$, that is an affine scheme because $\lim\limits_{\overleftarrow{i}} \operatorname{Spec} A_i = \operatorname{Spec} \lim\limits_{\overrightarrow{i}} A_i$, turns out to be an open subfunctor of F: Given a morphism $(\operatorname{Spec} A)^{\cdot} \to F$, that is, a family of morphisms $\operatorname{Spec} A \to Gr(E_i)$ for all $i \geq j'$, where j' is an index we can assume greater than j, we have to prove that $G_j \times_F (\operatorname{Spec} A)^{\cdot}$ is an open subset of $(\operatorname{Spec} A)^{\cdot}$. The morphism $\operatorname{Spec} A \to Gr(E_{j'})$ is defined by a direct summand W of $E_{j'A}$, and $G_j \times_F (\operatorname{Spec} A)^{\cdot}$ identifies with the open subset of the points of $\operatorname{Spec} A$ where the morphism $V'_{j'_A} \to E_{j'_A}/W$ is an isomorphism.

Perhaps, we have to consider in $\lim\limits_{\leftarrow} Gr(E_i)$ one more irreducible component, a single point, which corresponds to $\{\operatorname{Im} f_i\} \in \lim\limits_{\leftarrow} Gr(E_i)$ (provided that it is well defined), where $f_i\colon E \to E_i$ are the natural morphisms. (From now on, we will always exclude from $\lim\limits_{\leftarrow} Gr(E_i)$ this single-pointed component).

Varying the indexes j and the subspaces V'_j, we have that the schemes G_j form a covering of F, because the open subsets $G_j \times_F (\operatorname{Spec} A)^{\cdot}$ cover $\operatorname{Spec} A$. □

Let E be an inverse limit of morphisms between vector spaces $\{E_i\}$ with kernels and cokernels of finite dimension. From now on, we denote $E_A = \lim\limits_{\substack{\leftarrow \\ i}} (E_{i_A})$, where A is a noetherian k-algebra. We have $E_A \otimes_A A/I = E_{A/I}$: The functor $\lim\limits_{\leftarrow} (E_i \otimes_k -) = \lim\limits_{\leftarrow} (E_{iA} \otimes_A -)$ is exact and the functor $E_A \otimes_A -$ is right exact. Both of them coincide over free A-modules of finite type, then they coincide over the finite presentation modules such as A/I.

8. Lema: *Under the above assumptions:*

(a) *Given a vector subspace of finite dimension $M \subset E$, there exists an index i such that the natural morphism $M \to E_i$ is injective.*
(b) *If $M \subset E_A$ is a finitely generated direct summand, then there exist an index i such that the morphism $M \to E_{i_A}$ is injective and M is a direct summand of E_{i_A}.*
(c) *Reciprocally, given an index i, a finitely generated module M and morphisms $M \to E_{j_A}$, for all $j \geq i$, such that corresponding diagrams are commutative, and the morphism $M \to E_{iA}$ is injective with projective cokernel, define an injection $M \hookrightarrow E_A$ such that M is direct summand of E_A.*

Proof.

(a) Given i, if the kernel of $f_i\colon M \to E_i$ is not null, there exists an index $j > i$ such that $\ker f_j \underset{\neq}{\subset} \ker f_i$. As they are of finite dimension, a time comes in which $\ker f_k = 0$.
(b) As M is a finitely generated direct summand of E_A, on fibers M injects into E_A. Since on fibers M and E_A are vector spaces, because of last section there exist an index j for every point of $\operatorname{Spec} A$ such that the morphism $M \to E_{j_A}$ is injective in a neighborhood of the point. We can choose an index i greater than the previous ones such that the morphism $M \to E_{i_A}$ is injective on all fibers, then by Proposition 1.3 we have that M is a direct summand of E_{i_A}.
(c) We have $E_{iA} = M \oplus N_i$. Let N_j be the inverse image of N_i by the morphism $E_j \to E_i$. We have $E_{jA} = M \oplus N_j$, and $E_A = M \oplus \lim\limits_{\leftarrow} N_j$.

□

Then we have obtained the following theorem.

9. Teorema : *Let* $E = \lim\limits_{\substack{\leftarrow \\ i}} E_i$ *be an inverse limit of vector spaces of finite dimension. The functor*

$$Gr_{\text{fin}}(E)(\operatorname{Spec} A) := \{\textit{Finitely generated direct summands of } E_A\}$$

is representable by $\lim\limits_{\substack{\leftarrow \\ i}} Gr(E_i)$.

More generally,

10. Teorema : *Let* $E = \lim\limits_{\substack{\leftarrow \\ i}} E_i$ *be an inverse limit of morphisms between vector spaces with kernels and cokernels of finite dimension. The functor*

$$GR(E)(\operatorname{Spec} A) := \left\{ \begin{array}{c} A\textit{-submodules}\, M \subset E_A \textit{ such that for} \\ \textit{some } i, M \textit{ injects into}\, E_{i_A} \textit{ with a projective} \\ \textit{finitely generated module as cokernel} \end{array} \right\}$$

is representable by $\lim\limits_{\substack{\leftarrow \\ i}} Gr(E_i)$.

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Given an inverse limit of vector spaces $E = \lim\limits_{\substack{\leftarrow \\ i}} E_i$, replacing each E_i with the intersection of the images of the morphisms $f_{ji} : E_j \to E_i$ for $j \geq i$, that is, replacing each E_i with the image of the morphism $E \to E_i$, we obtain that E is the inverse limit of epimorphisms of vector spaces, although the kernels are not necessarily finitely generated except they were previously.

In the hypothesis of the theorem let us assume then that the morphisms $f_i : E \to E_i$ are surjective. Let us consider the topology of vector spaces in E defined by the basis of neighborhoods of $0 \in E$ made up by the kernels $\{\ker f_i\}_i$. It holds that $M \in GR(E)(\operatorname{Spec} A)$ is a discrete topological space endowed with the initial topology. As well, if we write $E_{i_A} = M \oplus N_i$ then $E_A = M \oplus N$, where $N = \lim\limits_{\substack{\leftarrow \\ j>i}} f_{ji_A}^{-1}(N_i)$. It holds that the quotient topology of $E_A/M = N$ is the one defined by $\overline{\ker f_{i_A}}$, that are direct summands with finitely generated supplementary vector spaces, since $N/\overline{\ker f_{i_A}} = E_{i_A}/M$ is a projective finitely generated module. Then, for noetherian k-algebras A,

$$GR(E)(\operatorname{Spec} A) = \left\{ \begin{array}{c} \text{Direct summands, } M \subset E_A \text{ such that the initial topology of } M \text{ is the} \\ \text{discrete one and a basis of neighborhoods of the quotient topology of } E_A/M \\ \text{are some submodules with cokernel a projective finitely generated module}^1 \end{array} \right.$$

([1] Actually, it is enough to say that for an index i, $\overline{\ker f_{i_A}} = \ker f_{i_A} \subset E_A/M$ has a projective finitely generated cokernel).

In [AMP], to define the topology of E it is fixed a vector subspace and it is considered all subspaces that "differs from it" in a vector space of finite dimension. In a more particular situation and with a more elaborate terminology, without using inverse limits, $GR(E)$ is the functor considered by Álvarez-Muñoz -Plaza in [AMP].

References

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