GEOMETRIC CALCULATION OF THE INVARIANT INTEGRAL OF CLASSIC GROUPS

AMELIA ÁLVAREZ, CARLOS SANCHO, AND PEDRO SANCHO

Abstract. Let $G = \operatorname{Spec} A$ be a linearly reductive group, and $w_G \in A^*$ be G-invariant and $w_G(1) = 1$. We establish the algebraic harmonic analysis on G and we compute w_G when $G = Sl_n, Gl_n, O_n, Sp_{2n}$, by geometric arguments and by means of the Fourier transform.

Introduction

An affine k-group $G = \operatorname{Spec} A$ is semisimple if and only if A^* splits in the form $A^* = k \times B^*$ as k-algebras, where the first projection $\pi_1 \colon A^* \to k$ is the morphism $\pi_1(w) := w(1)$ ([A2] Theorem 2.6). The linear form $w_G := (1, 0) \in k \times B^* = A^*$ will be referred of as the *invariant integral* on G.

In the theory of invariants the calculation of the invariant integral w_G is of great interest, because it yields the calculation of the invariants of any representation. The aim of this article is the explicit calculation of w_G when $G = Sl_n, Gl_n$, O_n, Sp_{2n} ($\operatorname{char} k = 0$), by geometric arguments, and by means of the Fourier transform, defined below. Although G is not a compact group it is possible to define the invariant integral of G, the Fourier transform, the convolution product (2.8) and prove the Parseval identity (2.4), inversion formula, etc.

Let A_i^* be simple (and finite) k-algebras and $A^* = \prod_i A_i^*$. On every A_i^*, one has the non singular trace metric and its associated polarity. Hence, one obtains a morphism of A^*-modules $\phi : A = \oplus_i A_i \hookrightarrow \prod_i A_i^* = A^*$. If $G = \operatorname{Spec} A$ is a semisimple affine k-group and $* : A \to A$, $a \mapsto a^*$ is the morphism induced by the morphism $G \to G$, $g \mapsto g^{-1}$, we prove that ϕ is the morphism

$$A \to A^*,\ a \mapsto w_G(a^* \cdot -)$$

where $w_G(a^* \cdot -)(b) := w_G(a^* \cdot b)$. We shall call ϕ the Fourier transform. The product operation in A^* defines, via the Fourier transform, a product on A, which is the *convolution product* in the classical examples.

Let us consider a system of coordinates in G, that is, let us consider $G = \operatorname{Spec} A$ as a closed subgroup of a semigroup of matrices $M_n = \operatorname{Spec} B$. Then A is the quotient of B by the ideal I of the functions of M_n vanishing on G. Hence A^* is a subalgebra of B^* and one has that $k \cdot w_G = A^{*G} = \{w \in B^{*G} : w(I) = 0\}$. Moreover, B^G (which is the ring of functions of M_n/G), coincides essentially with B^{*G}, via the Fourier transform. Finally, we prove that given $w \in B^{*G}$, the condition $w(I) = 0$ is equivalent to $w(I^G) = 0$, which is a finite system of equations "in each degree".

2000 *Mathematics Subject Classification.* Primary 14L24. Secondary 14L17.

Key words and phrases. Reynolds operator, functor of algebras, semisimple group, invariants, Fourier transform.

1. Preliminary results

Let k be a commutative ring with unit. All functors considered in this paper are functors over the category of commutative k-algebras. Given a k-module E, the functor $\mathbf{E}$ defined by $\mathbf{E}(B) := E \otimes_k B$ is called a quasi-coherent k-module. The functors $E \rightsquigarrow \mathbf{E}$, $\mathbf{E} \rightsquigarrow \mathbf{E}(k)$ establish an equivalence between the category of k-modules and the category of quasi-coherent k-modules ([A], 1.12). In particular, $\mathrm{Hom}_k(\mathbf{E}, \mathbf{E}') = \mathrm{Hom}_k(E, E')$.

If F, H are two functors of k-modules, we shall denote by $\mathbf{Hom}_k(F, H)$ the functor of k-modules

$$\mathbf{Hom}_k(F, H)(B) := \mathrm{Hom}_B(F_{|B}, H_{|B})$$

where $F_{|B}$ is the functor F restricted to the category of commutative B-algebras. The functor $\mathbf{E}^* = \mathbf{Hom}_k(\mathbf{E}, \mathbf{k})$ is the functor of points of $\mathrm{Spec}\, S^{\cdot} E$ and we call $\mathbf{E}^*$ a k-modules scheme. $\mathbf{E}^{**} = \mathbf{E}$ and the category of k-modules is anti-equivalent to the category of k-modules scheme ([A, 1.10,1.12]). Given a functor of k-modules F, we denote by $\bar{F}$ the k-modules scheme closure of F; that is, $\bar{F}$ is the representant on the category of k-modules schemes of the functor $\mathrm{Hom}_k(F, -)$.

Let $G = \mathrm{Spec}\, A$ be an affine k-group and let $G^\bullet$ be the functor of points of G, that is, $G^\bullet(B) = \mathrm{Hom}_{k-alg}(A, B)$. We will denote by $k[G^\bullet]$ the functor over the category of commutative k-algebras defined by $k[G^\bullet](B) = \{$ formal finite B-linear combinations of points of G with values in $B\}$. One has a natural morphism $G^\bullet \to \mathbf{A}^*$, because $G^\bullet(B) = \mathrm{Hom}_{k-alg}(A, B) \subset \mathrm{Hom}_k(A, B) = \mathbf{A}^*(B)$, which extends to a unique morphism of functors of k-algebras $k[G^\bullet] \to \mathbf{A}^*$. By ([A], 3.3) one has that $k[G^\bullet]^* = \mathbf{A}$ and that $\mathbf{A}^*$ is the algebra (and module) scheme closure of $k[G^\bullet]$. Moreover, $\mathbf{A}^*$ represents the functor $\mathrm{Hom}_{k-alg}(k[G^\bullet], -)$ in the category of dual functors of algebras ([A], 5.3).

The category of G-modules is equivalent to the category of (quasi-coherent) $\mathbf{A}^*$-modules ([A], 5.5). Therefore, if k is a field, G is semisimple (that is, linearly reductive) if and only if $\mathbf{A}^*$ is a semisimple k-algebras scheme, i.e. $A^* = \prod_i A_i^*$, where A_i^* are simple (and finite) k-algebras ([A], 6.8). If k is an algebraically closed field, then A_i^* is an algebra of matrices by Wedderburn's theorem.

2. Algebraic harmonic analysis

Let k be a field, that we assume algebraically closed for simplicity.

On $A^* = M_n(k)$ one has the non singular metric: $T_2'(T, S) :=$ the trace of the matrix $T \circ S$. If T_2 is the metric of the trace of the algebra A^*, then $T_2 = n \cdot T_2'$. Let us denote by $\phi_A, \phi_A' : A \simeq A^*$ the polarities associated to T^2 and T'^2.

Now, let A_i^* be k-algebras of matrices, $A^* = \prod A_i^*$ and $A = \oplus A_i$. Giving a metric T_2' on A is equivalent to defining its associated polarity $A \overset{\phi'}{\to} A^*$. Since there are (iso)morphisms $\phi'_{A_i} : A_i \to A_i^*$, one has the obvious injection $\phi' : \oplus A_i \to \prod A_i^*$. If $\dim_k A_i$ is prime with the characteristic of k, replacing T_2' by T_2 we can define in the same way a canonical morphism $\phi : \oplus A_i \to \prod A_i^*$ and $\phi = (1/\sqrt{\dim_k A_i})_i \cdot \phi'$.

Given $w = (w_i)_i \in \oplus_i A_i^*$ and $w' = (w_i')_i \in \prod_i A_i^*$, we can define $T_2'(w, w') := \sum_i T_2'(w_i, w_i')$. Given $a \in A$ and $w' \in A^*$, then $a(w') = T_2'(\phi'(a), w')$. Let $tr' \colon \oplus_i A_i^* \to k$, be defined by $tr'(w) = T_2'(w, 1)$. Given $a \in A$ and the unity $1 \in A^*$, then

$$a(1) = T_2'(\phi'(a), 1) = tr'(\phi'(a))$$

Likewise, $a(w) = T'_2(\phi'(a), w) = tr'(\phi'(a) \cdot w)$, for every $a \in A$ and $w \in A^*$, and when T_2 is a non-singular metric, $a(w) = T_2(\phi(a), w) = tr(\phi(a) \cdot w)$, for every $a \in A$ and $w \in A^*$ and

$$a(1) = T_2(\phi(a), 1) = tr(\phi(a))$$

Properties. The morphism $\phi' : A \to A^*$ satisfies the following properties:

- it is a morphism of left and right A^*-modules, because each polarity $A_i \to A_i^*$ is a morphism of left and right A_i^*-modules;
- it is a symmetric metric: $\phi'(a)(b) = \phi'(b)(a)$, because the metric T'_2 on each A_i^* is so;
- it is an injective morphism; that is, the metric associated to ϕ' is non singular, because the metric T'_2 on each A_i^* is so ;
- the image of ϕ' is dense in A^*. More generally, the natural morphism $\oplus_i E_i \to \prod_i E_i$, $\dim_k E_i < \infty$, is dense, i.e. $\overline{\oplus_i \mathbf{E_i}} \to \prod_i \mathbf{E_i}$ is surjective. In other words: Observe that every k-modules subscheme of $\prod_i \mathbf{E_i}$ is the orthogonal of a quasi-coherent submodule of $\oplus_i \mathbf{E_i^*}$. If we consider in $\prod_i E_i$ the topology of closed sets generated by the zeroes of the vectors of $\oplus_i E_i^*$, then $\overline{\oplus_i E_i} = \prod_i E_i$.
- the image of $\phi' \colon \mathbf{A} \to \mathbf{A}^*$ is the maximal quasi-coherent $\mathbf{A}^*$-submodule of $\mathbf{A}^*$. Let $w = (w_i)_{i \in I} \in \prod A_i^*$ be an element such that $w_i \neq 0$ for infinitely many indices i. The elements $(\ldots, 0, w_i, 0, \ldots) = (\ldots, 0, 1, 0, \ldots) \cdot w$ belong to the $\mathbf{A}^*$-submodule $< w >$ generated by w. Then $< w >$ is an $\mathbf{A}^*$-module of infinite dimension and w cannot belong to any quasi-coherent $\mathbf{A}^*$-submodule (if it belonged to some quasi-coherent $\mathbf{A}^*$-submodule, then it would be included in some of its $\mathbf{A}^*$-submodules of finite dimension, by [A, 4.7]).

Proposition 2.1. *Every morphism $f \colon \mathbf{A} \to \mathbf{E}^*$ of $\mathbf{A}^*$-modules lifts to a unique morphism of $\mathbf{A}^*$-modules $f' \colon \mathbf{A}^* \to \mathbf{E}^*$ (such that $f = f' \circ \phi'$). Hence,*

$$\mathrm{Hom}_{\mathbf{A}^*}(\mathbf{A}, \mathbf{E}^*) = \mathrm{Hom}_{\mathbf{A}^*}(\mathbf{A}^*, \mathbf{E}^*) = E^*$$

Proof. By the last property we have

$$\mathrm{Hom}_{\mathbf{A}^*}(\mathbf{A}, \mathbf{E}^*) = \mathrm{Hom}_{\mathbf{A}^*}(\mathbf{E}, \mathbf{A}^*) = \mathrm{Hom}_{\mathbf{A}^*}(\mathbf{E}, \mathbf{A}) = \mathrm{Hom}_{\mathbf{A}^*}(\mathbf{A}^*, \mathbf{E}^*)$$

□

Notation 2.2. *For any k-algebra C we shall denote by $Z(C)$ the center of C.*

Theorem 2.3. *Let $A_i{}^*$ be simple k-algebras, and $A^* = \prod_i A_i^*$. Let S^2 be a symmetric metric on A. If S^2 is A^*-linear (i.e., $S^2(w \cdot a, a') = S^2(a, a' \cdot w)$ for every $a, a' \in A$ and $w \in A^*$), then S^2 coincides, up to a factor of $Z(A^*)$, with the metric T'^2.*

Proof. The polarity $\varphi : A \to A^*$ associated to S_2 is a morphism of left and right A^*-modules. Therefore, φ maps each summand A_i into each A_i^*. Let $\phi'_i, \varphi_i : A_i \to A_i^*$ be the restrictions of ϕ' and φ to each A_i. The morphism $\varphi_i \circ {\phi'_i}^{-1} : A_i^* \to A_i^*$ is a morphism of left and right A^*-modules. Now, a morphism of left A_i^*-modules of A_i^* is an homothety by an element of A_i^*; if this homothety is also a morphism of right A_i^*-modules, then it is an homothety by an element of $Z(A_i^*)$. Thus, $\varphi_i = z_i \cdot \phi'_i$, where $z_i \in Z(A_i^*)$. Since $Z(A^*) = \prod_i Z(A_i^*)$, the proof is easily completed. □

Let E be a linear representation of a k-group $G = \operatorname{Spec} A$. The associated character $\chi_E \in A$ is defined by $\chi_E(g) =$ trace of the linear endomorphism $E \to E$, $e \mapsto g \cdot e$, for every $g \in G$ and $e \in E$.

Let $G = \operatorname{Spec} A$ be a semisimple group. One has $A^* = \prod_I \operatorname{End}_k(E_i)$, where $\{E_i\}$ runs over all the irreducible representations of G(up to isomorphisms). The polarity $A_i^* = \operatorname{End}_k(E_i) \to A_i$ associated to the metric T_2' maps the unit 1_i of A_i^* to the character χ_{E_i}, because $\operatorname{tr}(1_i \cdot g) = \chi_{E_i}(g)$. Therefore, $\phi'(\chi_{E_i}) = 1_i$. Since $1_i \cdot 1_j = \delta_{ij} \cdot 1_i$ one obtains that

$$T'^2(\chi_{E_i}, \chi_{E_j}) = 0$$

if $i \neq j$ and $T'^2(\chi_{E_i}, \chi_{E_i}) = \dim_k E_i$. Moreover, A is generated by the χ_{E_i} as an A^*-module, because $\operatorname{Im}\phi$ is generated by the 1_i.

Let $E_0 = k$ be the trivial representation of G and $w_G := 1_0 \in A^*$ be the "invariant integral of G". The invariant integral of G is characterized for being G-invariant and normalized, that is, $w_G(1) = 1$ ([A2, 2.10]). A dual functor F is a functor of G-modules if and only if is a functor of $\mathbf{A}^*$-modules and $F^G = w_G \cdot F$ (see [A2, 2.3, 3.3]).

One has that $w_G \cdot 1_j = 0$, if E_j is not the trivial representation, and $w_G \cdot 1_0 = 1_0$. Hence, $w_G \cdot \chi_{E_j} = 0$ if E_j is not the trivial representation, and $w_G \cdot \chi_{E_0} = \chi_{E_0} = 1$. Moreover, since $\chi_{E \oplus E'} = \chi_E + \chi_{E'}$, one has

$$w_G \cdot \chi_E = \dim_k E^G.$$

Let $* : A \to A$, $a \mapsto a^*$, be the morphism induced by the morphism $G \to G$, $g \mapsto g^{-1}$. If E is a representation of G, we shall consider E^* as a left G-module by $(g * w)(e) = w(g^{-1} \cdot e)$. One has that $\chi_E^* = \chi_{E^*}$, because the trace of $g^{-1} \in G$ operating on E is equal to the trace of g operating on E^* (which operates by the inverse transposed of g).

Theorem 2.4. *Let $G = \operatorname{Spec} A$ be a semisimple group and let $w_G \in A^*$ be its invariant integral. The morphism*

$$A \to A^*, \; a \mapsto w_G(a^* \cdot -)$$

where $w_G(a^* \cdot -)(a') := w_G(a^* \cdot a')$ coincides with ϕ.

Proof. Let us first prove that $\mathbf{A} \to \mathbf{A}^*$, $a \mapsto w_G(a^* \cdot -)$, is a morphism of left G-modules: For all point g of G,

$$\begin{aligned} w_G((g \cdot a)^* \cdot -) &\overset{*}{=} w_G((a^* \cdot g^{-1}) \cdot -) \overset{**}{=} w_G(((a^* \cdot g^{-1}) \cdot -) \cdot g) = w_G(a^* \cdot (- \cdot g)) \\ &= g \cdot (w_G(a^* \cdot -)) \end{aligned}$$

where $\overset{*}{=}$ is due to $(g \cdot a)^*(g') = a(g'^{-1} \cdot g) = a((g^{-1} \cdot g')^{-1}) = a^* \cdot g^{-1}(g')$, and $\overset{**}{=}$ is due to $g \cdot w_G = w_G$.

Let us show now that it is symmetric: $w_G(a^* \cdot a') = w_G((a \cdot a'^*)^*) = w_G(a \cdot a'^*)$, because $*(w_G) = w_G$.

By Theorem 2.3, it only remains to prove the orthonormality of $\{\chi_{E_i}\}$, where E_i are irreducible:

$$w_G(\chi_{E_i}^* \cdot \chi_{E_j}) = w_G(\chi_{E_i^* \otimes E_j}) = w_G \cdot \chi_{E_i^* \otimes E_j} = \dim_k \operatorname{Hom}_G(E_i, E_j) = \delta_{ij}.$$

□

If we consider in A, the metric T^2, defined by $T^2(a, a') = w_G(a^* \cdot a)$, then ϕ is the polarity asociated to T^2. If we consider in $A' = \oplus_i A_i^* = \operatorname{Im}\phi$, the metric T_2, then $\phi \colon A \to A'$ is an isometry.

Proposition 2.5. *Let $G = \operatorname{Spec} A$ be a semisimple group, D be a left G-invariant vector field on G and D_e be the value of the vector field at the identity element $e \in G$. Then*

$$\phi(D(a)) = D_e \cdot \phi(a)$$

Proof. $D(a) = D_e \cdot a$ because

$$(D_e \cdot a)(g) = a(g \cdot D_e) = a(D_g) = D_g(a) = D(a)(g)$$

for all $g \in G$. Therefore, $\phi(D(a)) = \phi(D_e \cdot a) = D_e \cdot \phi(a)$. □

Let $\pi_1 \colon A^* \to k$, be defined by $\pi_1(w) = w(1)$. Then, $\pi_1 \circ \phi = w_G$, because $\pi_1 \circ \phi(1) = \pi_1(w_G) = 1$ and $\pi_1 \circ \phi$ is G-invariant, that is, a morphism of $\mathbf{A}^*$-modules.

If $G = \operatorname{Spec} A$ is semisimple and E is an irreducible linear representation of G, then $\dim_k E$ is prime with the characteristic of k, because ϕ' is injective and $\phi' = (\dim_k E_i)_{i \in I} \cdot \phi$ (where I runs over the set of the irreducible linear representations up to isomorphisms).

Given $a, b \in A$, one has that $T^2(a,b) = \phi(a)(b) = w_G(a^* \cdot b)$. If we denote $w_G = \int dg$, then

$$T^2(a,b) = \int a(g^{-1}) \cdot b(g)\, dg.$$

Let $G = \operatorname{Spec} A$ be a semisimple group, E a G-module and E_i a simple G-module. Let us consider the G-module descomposition $E = E' \oplus F$, where E' is the homogeneous component of E isomorphic to $\oplus^n E_i$. Now, we want to compute the morphism $E \to E$ which is the identity over E' and nulle over F. In particular, we could obtain the descomposition of E as direct sum of homogeneous modules.

Let $1_i = (0, \ldots, \overset{i}{1}, \ldots, 0) \in \prod_{j \in I} \operatorname{End}_k E_j = A^*$. We have to calculate the morphism $E \to E$, $e \mapsto 1_i \cdot e$. Recall that $1_i = \phi(n_i \cdot \chi_{E_i})$, $n_i = \dim_k E_i$. The dual morphism of the multiplication morphism $\mathbf{E}^* \otimes \mathbf{A}^* \to \mathbf{E}^*$ is the comultiplication morphism $\mu \colon E \to E \otimes A$. If $\{e'_l\}$ is a basis of E, and $\mu(e) = \sum_l e'_l \otimes a_l$, then $g \cdot e = \sum_l a_l(g) e'_l$, for all $g \in G$. Hence,

$$\begin{aligned} 1_i \cdot e &= \sum_l a_l(1_i) \cdot e'_l = \sum_l a_l(\phi(n_i \cdot \chi_{E_i})) \cdot e'_l = \sum_l n_i \cdot w_G(a_l \cdot \chi_{E_i^*}) \cdot e'_l \\ &= n_i \cdot \sum_l \int a_l \cdot \chi_{E_i^*} dg \cdot e'_l. \end{aligned}$$

Notation 2.6. *Given an affine scheme $X = \operatorname{Spec} A$ will denote $A_X = A$.*

Proposition 2.7. *Let $G = \operatorname{Spec} A$ be a semisimple group, $H \overset{i}{\subset} G$ a normal subgroup and $G \overset{\pi}{\to} G/H$ the quotient morphism. Let $i^* \colon A_G \to A_H$ and $\pi \colon A^*_G \to A^*_{G/H}$ be the natural morphisms. Then, with the obvious notations,*

$$w_H(i^*(a)) = tr_{G/H}(\pi(\phi_G(a)))$$

for all $a \in A_G$.

Proof. The set of irreducible representations of G/H is equal to the set of irreducible representations of G which are H-invariant. The natural projection $A^*_G = \prod_i \operatorname{End}_k(E_i) \to \prod_{E_i = E_i^H} \operatorname{End}_k(E_i) = A^*_{G/H}$ coincides with π.

The diagram

$$\begin{array}{ccccccc} A_G = \oplus_i \mathrm{End}_k(E_i)^* & \xrightarrow{\phi_G} & \prod_i \mathrm{End}_k(E_i) & = & A_G^* & = & A_G^* \\ \downarrow w_H\cdot & & \downarrow w_H\cdot & & \downarrow w_H\cdot & & \downarrow \pi \\ A_G^H = \oplus_{E_i=E_i^H} \mathrm{End}_k(E_i)^* & \xrightarrow{\phi_{G/H}} & \prod_{E_i=E_i^H} \mathrm{End}_k(E_i) & = & A_G^{*H} & = & A_{G/H}^* \end{array}$$

is commutative. Then

$$tr_{G/H}(\pi(\phi_G(a))) = tr_{G/H}(\phi_{G/H}(w_H \cdot a)) = (w_H \cdot a)(1) = a(w_H) = w_H(i^*(a))$$

□

The image of the morphism $\phi : A \hookrightarrow A^*$ is a bilateral ideal. Then it is a subring, although without unit, because $(\dots, 1, 1, 1, \dots) \notin \oplus A_i^* = \mathrm{Im}\,\phi$.

Definition 2.8. *The product of the subring* $\mathrm{Im}\,\phi$ *induces a product on* A*, through the identification* $A \overset{\phi}{\simeq} \mathrm{Im}\,\phi$*. This product is called the convolution product.*

Let $a, b \in A$, $w' = \phi(a)$, $w'' = \phi(b)$ and let us denote by $*$ the convolution product. Then

$$a * b = \phi^{-1}(w' \cdot w'') = w' \cdot \phi^{-1}(w'') = w' \cdot b.$$

Therefore, $(a*b)(x) = (w' \cdot b)(x) = b(x \cdot w') = (x \cdot w')(b) = w'(b \cdot x) = w_G(a^* \cdot (b \cdot x))$, for all point x of G. If we denote $w_G = \int dg$, then

$$(a * b)(x) = \int a(g^{-1}) \cdot b(x \cdot g)\, dg.$$

3. Invariant integral of Sl_n, Gl_n, O_n and of Sp_{2n}

Let k be a field of characteristic zero. The groups Gl_n, Sl_n, O_n and of Sp_{2n} are semisimple, so they have an invariant integral. This section is devoted to the explicit calculation of the invariant integral of the groups Gl_n, Sl_n, O_n and of Sp_{2n}.

Let us consider the affine algebraic k-variety $M_n = \mathrm{End}_k(E)$, whose points with values in a k-algebra B is the semigroup of square matrices of order n with coefficients in B. Its ring of functions is $A_{M_n} = \underset{n\in\mathbb{N}}{\oplus} S^n(\mathrm{End}_k(E)^*)$. Although M_n is a semigroup scheme and not a group one, $k[M_n^\bullet]$ is a functor of k-algebras and its k-algebras scheme closure is $\mathbf{A}^*_{M_n} = \prod_{n\in\mathbb{N}} (\mathbf{End}_k(\mathbf{E}) \otimes_k \overset{n}{\cdots} \otimes_k \mathbf{End}_k(\mathbf{E}))^{S_n}$. One has the explicit natural morphism

$$M_n^\bullet \to \mathbf{A}^*_{M_n},\ \tau \mapsto (1, \tau, \tau \otimes \tau, \tau \otimes \tau \otimes \tau, \dots)$$

Since this morphism is a morphism of semigroups, the unique structure of functors of algebras of $\mathbf{A}^*_{M_n}$ is the one of the direct product of the algebras $(\mathbf{End}_k(\mathbf{E}) \otimes \overset{n}{\cdots} \otimes \mathbf{End}_k(\mathbf{E}))^{S_n}$. One has that the category of quasi-coherent $\mathbf{A}^*_{M_n}$-modules is equivalent to the category of M_n-modules. $\mathbf{A}^*_{M_n}$ is a semisimple algebra (see [A2, 2.7]); hence every $\mathbf{A}^*_{M_n}$-module and every M_n-module is semisimple.

The natural action of $\mathrm{End}_k(E)$ on $E \otimes \overset{m}{\cdots} \otimes E$ extends uniquely to a structure of $\mathbf{A}^*_{M_n}$-module. It consists of the projection of $\prod_m S^m \mathbf{End}_k(\mathbf{E})$ onto the m-th factor,

$S^m\mathbf{End}_k(\mathbf{E})$, and the action of $S^m\mathbf{End}_k(\mathbf{E})$ on $E\otimes \overset{m}{\cdots}\otimes E$ via its inclusion in $\mathbf{End}_k(E)\otimes \overset{m}{\cdots}\otimes \mathbf{End}_k(E)$; that is

$$(g_1\cdot\ldots\cdot g_m)\cdot(v_1\otimes\ldots\otimes v_m)=\frac{1}{m!}\cdot\sum_{\sigma\in S_m} g_{\sigma(1)}(v_1)\otimes\ldots\otimes g_{\sigma(m)}(v_n).$$

The isomorphism $\phi_{(\mathrm{End}_k(E))^*}:(\mathrm{End}_k(E))^*\to\mathrm{End}_k(E)$, induces, by taking symmetric algebras, a morphism $\varphi: A_{M_n}\hookrightarrow A^*_{M_n}$, of left and right $A^*_{M_n}$-modules. It coincides with ϕ, up to an invertible factor of the center $Z(A^*_{M_n})$.

Invariant integral of Sl_n.

Let $A_{Sl_n}=k[x_{11},\ldots,x_{nn}]/(det(x_{ij})-1)$ be the ring of functions of the special linear group. Then $A^*_{Sl_n}$ splits into a direct product of simple algebras, one of them being the one corresponding to the trivial representation. So

$$A^*_{Sl_n}=k\cdot w_{Sl_n}\times B^*.$$

Recall that $\mathbf{A}^{*\;Sl_n}_{Sl_n}=\mathbf{k}\cdot w_{Sl_n}$. Let us compute the invariants of $\mathbf{A}^*_{Sl_n}$ by Sl_n.

From the inclusion $Sl_n\subset M_n$ one obtains the injective morphism $A^*_{Sl_n}\subset A^*_{M_n}$. We shall first compute the invariants of $A^*_{M_n}$ by the action of Sl_n and then we shall compute the ones belonging to $A^*_{Sl_n}$. Since A_{M_n} is a semisimple Sl_n-module, it splits into a direct sum

$$A_{M_n}=A^{Sl_n}_{M_n}\oplus(\bigoplus_i S_i),$$

where S_i are simple Sl_n-submodules, but not Sl_n-invariant. Taking dual one obtains that

$$A^*_{M_n}=(A^{Sl_n}_{M_n})^*\times(\prod_i S_i^*)=(A^*_{M_n})^{Sl_n}\times(\prod_i S_i^*)$$

because a group acts trivially on a module if and only if it acts trivially on its dual.

The morphism $\varphi:\mathbf{A}_{M_n}=\mathbf{A}^{Sl_n}_{M_n}\oplus\mathbf{B}\to(\mathbf{A}^{Sl_n}_{M_n})^*\times\mathbf{B}^*$ is a morphism of Sl_n-modules, so that

$$\varphi(A^{Sl_n}_{M_n})\subseteq(A^{Sl_n}_{M_n})^*=(A^*_{M_n})^{Sl_n}$$

and $\varphi(B)\subseteq B^*$. Since the closure, $\overline{\varphi(A_{M_n})}$, of $\varphi(A_{M_n})$ is $A^*_{M_n}$, one has that $\overline{\varphi(A^{Sl_n}_{M_n})}=(A^*_{M_n})^{Sl_n}$. Then, let us compute ${A_{M_n}}^{Sl_n}$.

From the exact sequence of groups

$$\begin{array}{ccccccccc} 1 & \to & Sl_n & \subset & Gl_n & \to & Gl_n/Sl_n=G_m & \to & 1\\ & & & & T & \mapsto & det(T) & & \end{array}$$

it follows easily that $k[x_{11},\ldots,x_{nn}]^{Sl_n}=k[det(x_{ij})]$ and therefore

$$(k[x_{11},\ldots,x_{nn}]^*)^{Sl_n}=k\times k\cdot\varphi(det(x_{ij}))\times\ldots,\times k\cdot\varphi(det(x_{ij})^r)\times\ldots$$

Let us denote by δ_{ij} the matrix of null coefficients, except for the ij-th coefficient that is 1. As a linear map on $k[x_{ij}]$, δ_{ij} coincides with $\dfrac{\partial}{\partial x_{ij}}\mid_0$. One has that

$$\varphi(x_{ij})=\frac{\partial}{\partial x_{ji}}\mid_0,\quad \varphi(x_{i_1j_1}\cdot\ldots\cdot x_{i_mj_m})=\frac{1}{m!}\frac{\partial}{\partial x_{j_1i_1}}\cdot\ldots\cdot\frac{\partial}{\partial x_{j_mi_m}}\mid_0.$$

Since $det(x_{ij})^r$ is an homogeneous polynomial of rn-th degree,

$$\varphi(det(x_{ij})^r)=\frac{1}{(rn)!}det^r\left(\frac{\partial}{\partial x_{ij}}\right)\mid_0.$$

Let $D = det\left(\dfrac{\partial}{\partial x_{ij}}\right)$ be the Cayley operator, and let us denote $D_0^r = D^r\mid_0$. One has that

$$(k[x_{11},\ldots,x_{nn}]^*)^{Sl_n} = k \times k\cdot D_0 \times \ldots \times k\cdot D_0^r \times \ldots$$

Let us compute now the $\tilde{w} \in (k[x_{11},\ldots,x_{nn}])^{*Sl_n}$ vanishing on on the ideal $I = (det(x_{ij}) - 1)$. Since $\tilde{w}$ is Sl_n-invariant, one has that $\tilde{w}\cdot w_{Sl_n} = \tilde{w}$. Therefore, $\tilde{w}(I) = 0$ if and only if $\tilde{w}(w_{Sl_n}\cdot I) = \tilde{w}(I^{Sl_n}) = 0$. Now, $I^{Sl_n} = \langle det^n(x_{ij}) - det^{n+1}(x_{ij})\rangle_n \subset k[det(x_{ij})]$. Hence, if $\tilde{w}$ is null on the functions

$$det^n(x_{ij}) - det^{n+1}(x_{ij}),$$

for every $n \geq 0$, and $\tilde{w}(1) = 1$, then $\tilde{w} = w_{Sl_n}$. Consequently,

$$w_{Sl_n} = \sum_i \frac{D_0^i}{D_0^i(det^i(x_{ij}))}.$$

It only remains to determine the value of $D_0^i(det^i(x_{ij})) \in k$.

Lemma 3.1. ([D, 2.1]) *$D(det^r(x_{ij})) = \mu_r det^{r-1}(x_{ij})$, where $\mu_r = r\cdot(r+1)\cdot\ldots\cdot(r+n-1) = \dfrac{(r+n-1)!}{(r-1)!}$.*

Then

$$D^r(det^r(x_{ij})) = \mu_r\cdot\mu_{r-1}\cdot\ldots\cdot\mu_1.$$

Every linear representation of Sl_n is a submodule of direct sum of A_{Sl_n} (the regular representation). Moreover, A_{Sl_n} is a quotient of the ring of functions of M_n. Finally, the ring of functions of $M_n = \mathrm{End}_k(E)$ is included in a direct sum of $E\otimes \overset{m}{\ldots}\otimes E$. Let us compute then the invariants of Sl_n operating on these vector spaces.

Proposition 3.2. ([S, Th. 19.2]) *Let Sl_n be the special linear group of an n-dimensional vector space E. Let us consider the natural action of Sl_n on $E\otimes\overset{m}{\ldots}\otimes E$, $g\cdot(v_1\otimes\ldots\otimes v_m) = g\cdot v_1\otimes\ldots\otimes g\cdot v_m$. One has that:*

1. $(E\otimes\overset{n}{\ldots}\otimes E)^{Sl_n} = \Lambda^n E$.
2. $(E\otimes\overset{nm}{\ldots}\otimes E)^{Sl_n} = \sum_{\sigma\in S_{nm}}\sigma(\Lambda^n E\otimes\overset{m}{\ldots}\otimes\Lambda^n E)$, *where $\sigma\in S_{nm}$ acts on $E\otimes\overset{nm}{\ldots}\otimes E$ by permuting the factors.*
3. $(E\otimes\overset{m}{\ldots}\otimes E)^{Sl_n} = 0$ *if m is not a multiple of n.*

Proof.

1. We must calculate $w_{Sl_n}\cdot(E\otimes\overset{n}{\ldots}\otimes E) = D_0\cdot(E\otimes\overset{n}{\ldots}\otimes E)$. Fixed a basis $\{e_1,\ldots,e_n\}$ of E, let us observe that $\frac{\partial}{\partial x_{ij}}_{|0}$ corresponds to the matrix δ_{ij} that maps e_j to e_i and the rest of the e_k to zero. Then it is clear that $D_0\cdot(e_{i_1}\otimes\ldots\otimes e_{i_n}) = e_{i_1}\wedge\ldots\wedge e_{i_n}$ and (1) is proved.
2. The $r = nm$-th degree component of w_{Sl_n} is, up to scalars, D^m; that is, it coincides, up to scalars, with $\sum\limits_{\sigma\in S_{nm}}\sigma\circ(D\otimes\overset{m}{\ldots}\otimes D)\circ\sigma^{-1}$. Then,
$$(E\otimes\overset{nm}{\ldots}\otimes E)^{Sl_n} \subseteq \sum_{\sigma\in S_{nm}}\sigma(\Lambda^n E\otimes\overset{m}{\ldots}\otimes\Lambda^n E).$$
The inverse inclusion is obvious.
3. $w_{Sl_n}\in\prod\limits_r S^r\mathrm{End}_k(E)$ and its r-th degree "components" are null when r is not a multiple of n.

□

We can compute the dimension of $(E\otimes \overset{nm}{\cdots} \otimes E)^{Sl_n}$:

$$\begin{aligned}\dim_k(E\otimes \overset{nm}{\cdots} \otimes E)^{Sl_n} &= w_{Sl_n}\cdot \chi_{E\otimes \overset{nm}{\cdots}\otimes E} = \frac{D^m}{D^m(\det^m)}(\chi_E^{nm})\\ &= \frac{D^m}{D^m(\det^m)}((x_{11}+\overset{n}{\cdots}+x_{nn})^{nm}) = \frac{D^m}{D^m(\det^m)}(x_{11}^m\cdot\ldots\cdot x_{nn}^m\cdot\frac{(mn)!}{m!^n})\\ &= \frac{(mn)!}{D^m(\det^m)}.\end{aligned}$$

Invariant integral of Gl_n.

Let $A_{Gl_n} = k[x_{11},\dots,x_{nn},\frac{1}{det(x_{ij})}]$ be the ring of functions of the linear group. One has that $(A^*_{Gl_n})^{Gl_n} = k\cdot w_{Gl_n}$ and $(A^*_{Gl_n})^{Gl_n} = ((A^*_{Gl_n})^{G_m})^{Sl_n}$, where G_m is the multiplicative group. We shall first compute $(A^*_{Gl_n})^{G_m} = (A^{G_m}_{Gl_n})^*$ and then we shall look for the Sl_n-invariant ones among them.

A_{Gl_n} is a $\mathbb{Z}$-graded algebra, whose i-th degree component we denote by A_i. Given $\lambda\in G_m$ and $a_i\in A_i$, $\lambda * a_i = \lambda^i\cdot a_i$. Therefore, $A^{G_m}_{Gl_n} = A_0$ and $w\in (A^*_{Gl_n})^{G_m}$ if and only if it factors through the obvious quotient $A_{Gl_n}\to A_0$. This quotient morphism is a morphism of Gl_n-modules. Now we must compute the linear forms $w: A_0\to k$ that are Sl_n-invariant. One has

$$A_0 = \bigcup_{r\in\mathbb{N}} \frac{A^r}{det^r(x_{ij})}, \qquad A^r := \{\text{ homogeneous polynomials of } n\cdot r\text{-th degree }\}.$$

The morphism $w_r = w\circ det^{-r}(x_{ij})\cdot : A^r\to k$ is a morphism of Sl_n-modules, i.e., it is Sl_n-invariant. Since $w_r\in (A^r)^* = (S^{r\cdot n}\mathrm{End}_k(E)^*)^* = S^{r\cdot n}\mathrm{End}_k(E)\subset \prod_i S^i\mathrm{End}_k(E) = k[x_{11},\dots,x_{nn}]^*$, and it is Sl_n-invariant, then it must be $w_r = \alpha_r\cdot D^r$. If we ask for $w(1)=1$, then it must be $1 = w_r(det^r(x_{ij})) = \alpha_r\cdot D^r(det^r(x_{ij}))$, because $1 = \dfrac{det^r(x_{ij})}{det^r(x_{ij})}\in A^{G_m}_{Gl_n}$. Consequently, $\alpha_r = \dfrac{1}{D^r(det^r(x_{ij}))}$ and

$$w_r = \frac{D^r}{D^r(det^r(x_{ij}))} = \frac{D^r}{\mu_r\cdot\ldots\cdot\mu_1}.$$

In conclusion, we have determined[1] the invariant integral w_{Gl_n} on Gl_n as a linear form over A_{Gl_n}:

$$\begin{aligned} w_{Gl_n}\left(\frac{p(x_{ij})}{det^s(x_{ij})}\right) &= w_{Gl_n}\left(\frac{\ldots + p_{n\cdot s}(x_{ij}) + \ldots}{det^s(x_{ij})}\right) = w_s(p_{n\cdot s}(x_{ij}))\\ &= \frac{D^s(p_{n\cdot s}(x_{ij}))}{D^s(det^s(x_{ij}))}.\end{aligned}$$

Invariant integral of O_n.

Let T_2 be a non singular symmetric metric on a vector space E of dimension n. Let O_n be the subgroup of the linear group of the symmetries of T_2. In the algebraic

[1] Marcel Bökstedt checks in *"Notes on Geometric Invariant Theory"* (available at `http://home.imf.au.dk/marcel/GIT/GIT.ps`) that the integral thus defined is the Reynolds operator of the linear group, and he states that Cayley, in a sense, had already checked it.

variety $S^2\mathrm{E}^*$ of the symmetric metrics, regardless of the basis of E chosen, we can define (up to a constant multiplicative factor) the function det that assigns to each metric its determinant. So, we can consider the open set $S^2\mathrm{E}^* - (det)_0$. The sequence of morphisms of varieties

$$\begin{array}{ccccccccc} 1 & \to & O_n & \to & Gl(E) & \to & S^2\mathrm{E}^* - (det)_0 & \to & 1 \\ & & & & S & \mapsto & S^t \circ T_2 \circ S & & \end{array}$$

shows that $S^2\mathrm{E}^* - (det)_0$ is the quotient variety of $Gl(E)$ by the orthogonal subgroup O_n (O_n acting on $Gl(E)$ by the left). Fixing a basis in E, we shall say that $k[x_{11}, \ldots, x_{nn}, \dfrac{1}{det(x_{ij})}]$ is the ring of functions of $Gl(E)$ and $k[y_{i\leq j}, \dfrac{1}{det(y_{ij})}]$ is the ring of functions of $S^2\mathrm{E}^* - (det)_0$. One has the induced morphism of rings

$$\begin{array}{ccc} k[y_{i\leq j}, \dfrac{1}{det(y_{ij})}] & \hookrightarrow & k[x_{11}, \ldots, x_{nn}, \dfrac{1}{det(x_{ij})}] \\ y_{rs} & \mapsto & [(x_{ij})^t \circ T_2 \circ (x_{ij})]_{rs} \\ det(y_{ij}) & \mapsto & det(x_{ij})^2 \cdot det\, T_2. \end{array}$$

The functions of $Gl(E)$ invariant by O_n are identified with the functions of $S^2\mathrm{E}^* - (det)_0$. Therefore, via the morphism of varieties $\mathrm{End}_k(E) \to S^2\mathrm{E}^*$, $S \mapsto S^t \circ T_2 \circ S$, the functions of $S^2\mathrm{E}^*$ are identified with the functions of $\mathrm{End}_k(E)$ that are (right) invariant by O_n.

Let us express these equations without fixing basis. We have defined the morphism

$$\begin{array}{ccl} \mathrm{End}_k(E) = \mathrm{E}^* \otimes \mathrm{E} & \longrightarrow & S^2\mathrm{E}^* \\ w \otimes e & \mapsto & C^{1,2}_{2,3}(w \otimes e \otimes T_2 \otimes e \otimes w) \\ & & = T_2(e,e) \cdot w \otimes w \end{array}$$

that induces a morphism between the rings of functions $S^{\cdot}(S^2E) \to S^{\cdot}(\mathrm{End}_k(E)^*)$, that is expressed explicitly as follows

$$\begin{array}{ccc} S^m(S^2E) & \longrightarrow & S^{2m}(\mathrm{End}_k(E)^*) \\ s_1 \cdot \ldots \cdot s_m & \mapsto & \overline{T_2 \otimes \overset{m}{\ldots} \otimes T_2 \otimes s_1 \otimes \ldots \otimes s_m} \end{array}$$

(we think of $S^{2m}(\mathrm{End}_k(E)^*)$ as a quotient of $(E^* \otimes E) \otimes \overset{2m}{\ldots} \otimes (E^* \otimes E) = E^* \otimes \overset{2m}{\ldots} \otimes E^* \otimes E \otimes \overset{2m}{\ldots} \otimes E)$. Equivalently, the left O_n-invariant functions of the variety $\mathrm{End}_k(\mathrm{E})$ are the direct sum of the images of the morphisms

$$\begin{array}{ccc} S^m(S^2E^*) & \longrightarrow & S^{2m}(\mathrm{End}_k(E)^*) \\ \omega_1 \cdot \ldots \cdot \omega_m & \mapsto & \overline{\omega_1 \otimes \ldots \otimes \omega_m \otimes T^2 \otimes \overset{m}{\ldots} \otimes T^2} \end{array}$$

(we think of $S^{2m}(\mathrm{End}_k(E)^*)$ as a quotient of $E^{*2m} \otimes E^{2m}$). Therefore, the invariants of $S^{2m}(\mathrm{End}_k(E)^*)$ by the action of O_n by the left and by the right are

$$\langle\overline{\sigma(T_2\otimes \overset{m}{\cdots} \otimes T_2)\otimes\sigma'(T^2\otimes\overset{m}{\cdots}\otimes T^2)}\rangle_{\sigma,\sigma'\in S_{2m}}$$
$$=\langle\overline{(T_2\otimes \overset{m}{\cdots} \otimes T_2)\otimes\sigma'(T^2\otimes\overset{m}{\cdots}\otimes T^2)}\rangle_{\sigma'\in S_{2m}}.$$

Let A_{O_n} be the ring of functions of O_n and $w_{O_n}\in A^*_{O_n}\subset A^*_{M_n}=\prod\limits_r S^r\mathrm{End}_k(E)$ the invariant integral on O_n. The r-th component $[w_{O_n}]_r$ of w_{O_n} is

$$[w_{O_n}]_r=\sum_{\sigma\in S_{2m}}\lambda_\sigma\cdot\overline{(T_2\otimes\overset{m}{\cdots}\otimes T_2)\otimes\sigma(T^2\otimes\overset{m}{\cdots}\otimes T^2)}\quad \text{ if } r=2m,$$
$$[w_{O_n}]_r=0\quad \text{ if } r=2m+1.$$

Proposition 3.3. ([G, Th. 4.3.3]) *Let us consider the natural action of O_n on $E\otimes\overset{r}{\cdots}\otimes E$, $g\cdot(e_1\otimes\cdots\otimes e_r)=g\cdot e_1\otimes\ldots\otimes g\cdot e_r$. One has that:*
(1) $(E\otimes\overset{2m+1}{\cdots}\otimes E)^{O_n}=0$.
(2) $(E\otimes\overset{2m}{\cdots}\otimes E)^{O_n}=\langle\sigma(T^2\otimes\overset{m}{\cdots}\otimes T^2)\rangle_{\sigma\in S_{2m}}$.

Proof.
(1) $w_{O_n}\cdot(E\otimes\overset{2m+1}{\cdots}\otimes E)=[w_{O_n}]_{2m+1}\cdot E^{\otimes 2m+1}=0\cdot E^{\otimes 2m+1}=0$.
(2) One has $[w_{O_n}]_{2m}=\sum\limits_{\sigma\in S_{2m}}\lambda_\sigma\cdot\overline{(T_2\otimes\overset{m}{\cdots}\otimes T_2)\otimes\sigma(T^2\otimes\overset{m}{\cdots}\otimes T^2)}$ and

$$\overline{(T_2\otimes\overset{m}{\cdots}\otimes T_2)\otimes\sigma(T^2\otimes\overset{m}{\cdots}\otimes T^2)}$$
$$=\frac{1}{(2m)!}\sum_{\sigma'\in S_{2m}}\sigma'(T_2\otimes\overset{m}{\cdots}\otimes T_2)\otimes\sigma'\sigma(T^2\otimes\overset{m}{\cdots}\otimes T^2)$$

via the inclusion $S^{2m}\mathrm{End}_k(E)\subset E^{*\otimes 2m}\otimes E^{\otimes 2m}$. Moreover, $E^{*\otimes 2m}\otimes E^{\otimes 2m}$ acts on $E^{\otimes 2m}$ by contracting each linear form with the corresponding vector. Therefore,

$$(E^{\otimes 2m})^{O_n}=w_{O_n}\cdot E^{\otimes 2m}=[w_{O_n}]_{2m}\cdot E^{\otimes 2m}\subseteq\langle\sigma(T^2\otimes\overset{m}{\cdots}\otimes T^2)\rangle_{\sigma\in S_{2m}}.$$

The inverse inclusion is obvious.

□

Let us consider the morphism $S^2E\hookrightarrow\mathrm{End}_k(E)$, $T'^2\mapsto T'^2\circ T_2$, that assigns to each metric T'^2 the associated endomorphism to the pair of metrics T^2,T'^2. Two metrics are isometric (with regard to T_2) if and only if their associated endomorphisms are equivalent, and every endomorphism (up to conjugations) is the associated endomorphism of a symmetric metric and T_2 ([E]). As a result one has that the invariant functions of $\mathrm{End}_k(E)$ (by the action by conjugation of the linear group) are invariant functions of S^2E by the orthogonal group. Conversely, let us see that $A^{O_n}_{S^2E}\subseteq A^{Gl_n}_{\mathrm{End}_k(E)}$. Let $d(\lambda_{ij})$ be the discriminant of the characteristic polynomial of the matrix (λ_{ij}) and let $U=\mathrm{End}_k(E)-(d)_0$ be the open subset of $\mathrm{End}_k(E)$ of the diagonalizable endomorphisms with different eigenvalues. It is clear that $(U\cap S^2E)/O_n=U/Gl_n=\mathrm{Spec}\,k[a_1,\ldots,a_n]_d$, where $a_s(\lambda_{ij})$ are the coefficients of the characteristic polynomial of the matrix (λ_{ij}). Given $f\in A^{O_n}_{S^2E}$, one has that $f=p(a_1,\ldots,a_n)/d^r$, where we can assume that $p(a_1,\ldots,a_n)$ is not divisible by d. However, if $r>0$, then $p(a_1,\ldots,a_n)$ must vanish on all the diagonal matrices with repeated eigenvalues, then $p(a_1,\ldots,a_n)$ is a multiple of d, which is impossible. Therefore, $f=p(a_1,\ldots,a_n)\in A^{Gl_n}_{\mathrm{End}_k(E)}$.

Let f be the composite morphism $\mathrm{End}_k(E)\to S^2\mathrm{E}\to\mathrm{End}_k(E)$, $T\mapsto TT^2T^t\mapsto TT^2T^tT_2$. The invariant functions of $\mathrm{End}_k(E)$ by the action by conjugation of the linear group coincide, via f^*, with the functions of $\mathrm{End}_k(E)$ that are left and right invariant by the action of the orthogonal group. Since $k[S_m] = ((\mathrm{End}_k(E))^{\otimes m})^{Gl_n}$, $\sigma \mapsto \tilde{\sigma}$, where $\tilde{\sigma}(e_1 \otimes \ldots \otimes e_m) = e_{\sigma(1)} \otimes \ldots \otimes e_{\sigma(m)}$, then $(S^m\mathrm{End}_k(E))^{Gl_n} = \langle \bar{\tilde{\sigma}} \rangle_{\sigma \in S_m} = Z(k[S_m])$. Now, $(\mathrm{End}_k(E))^* = \mathrm{End}_k E$ (via $\phi_{(\mathrm{End}_k(E))^*}$, or via the obvious isomorphism $E \otimes E^* = E^* \otimes E$), and one has in the same way that $((S^m\mathrm{End}_k(E))^*)^{Gl_n} = \langle \bar{\tilde{\sigma}} \rangle_{\sigma \in S_m}$. Finally,

$$\tilde{\sigma} \overset{f^*}{\mapsto} \overline{T_2 \otimes \overset{m}{\cdots} \otimes T_2 \otimes T^2_{1\sigma(1)} \otimes \overset{m}{\cdots} \otimes T^2_{m\sigma(m)}} =: a_\sigma$$

where $T^2_{1\sigma(1)} \otimes \overset{m}{\cdots} \otimes T^2_{m\sigma(m)} := \tau(T^2 \otimes \ldots \otimes T^2)$ and $\tau \in S_{2m}$ is the permutation $\tau(2i-1) = 2i-1$ and $\tau(2i) = 2\sigma(i)$.

So we have calculated the forms $\tilde{w} \in A^*_{M_n}$ that are left and right O_n-invariant. To compute the invariant integral w_{O_n} on O_n it remains to impose that $\tilde{w}(I) = 0$, where I is the ideal of functions of M_n vanishing on O_n. Now, since $w_{O_n} \cdot \tilde{w} \cdot w_{O_n} = \tilde{w}$, one has that $\tilde{w}(I) = \tilde{w}(w_{O_n} \cdot I \cdot w_{O_n})$ and $w_{O_n} \cdot I \cdot w_{O_n}$ are the functions of M_n that are left and right O_n-invariant and vanish on O_n. These ones are identified, via f^*, with the ideal I' of the functions of M_n that are invariant by the action by conjugation of the linear group and vanish on $Id \in M_n$.

One has $A^{Gl_n}_{M_n} = \oplus_{m \in \mathbb{N}} < \bar{\tilde{\sigma}} >_{[\sigma] \in S_m/\sim} \subset A_{M_n}$ and $w_{O_n} \cdot I \cdot w_{O_n}$ is identified with

$$I' = \underset{m}{\oplus} < 1 - \frac{\bar{\tilde{\sigma}}}{\bar{\tilde{\sigma}}(Id_1)} >_{[\sigma] \in S_m/\sim}$$

where $Id_1 = Id \otimes \overset{m}{\cdots} \otimes Id$. If σ is a product of r disjoint cycles (including the cycles of order 1), it is easy to check that $\bar{\tilde{\sigma}}(Id_1) = n^r$.

Let us denote $w_\sigma := \overline{T_2 \otimes \overset{m}{\cdots} \otimes T_2 \otimes T^2_{1\sigma(1)} \otimes \overset{m}{\cdots} \otimes T^2_{m\sigma(m)}} \in S^{2m}\mathrm{End}_k(E)$. In order to find $w_{O_n} = 1 + \sum\limits_{m>0} \sum_{[\sigma] \in S_m/\sim} \lambda_{[\sigma]} \cdot \dfrac{w_\sigma}{\bar{\tilde{\sigma}}(Id_1)}$ satisfying $w_{O_n}(I) = 0$, we have to solve, for every m, the system of equations (varying $[\sigma'] \in S_m/\sim$)

$$\left(\sum_{[\sigma] \in S_m/\sim} \lambda_{[\sigma]} \cdot \frac{w_\sigma}{\bar{\tilde{\sigma}}(Id_1)} \right) \left(\frac{a_{\sigma'}}{\bar{\tilde{\sigma}}'(Id_1)} \right) = 1.$$

If we denote $\lambda_{\sigma\sigma'} = \dfrac{w_\sigma(a_{\sigma'})}{\bar{\tilde{\sigma}}(Id_1) \cdot \bar{\tilde{\sigma}}'(Id_1)}$, then

$$(\lambda_{[\sigma]})_{[\sigma] \in S_m/\sim} = (\lambda_{\sigma\sigma'})^{-1}_{[\sigma],[\sigma'] \in S_m/\sim} \cdot (1, \ldots, 1).$$

Let us give the first three terms of w_{O_n}:

$$\begin{aligned} w_{O_n} &= 1 + \frac{\overline{T_2 \otimes T^2}}{n} \\ &+ \frac{(3n^2+3n+3) \cdot \overline{T_2 \otimes T_2 \otimes T^2 \otimes T^2} + (-3n-6) \cdot \overline{T_2 \otimes T_2 \otimes T^2_{12} \otimes T^2_{21}}}{n^4 + n^3 + n^2 - 3n} \\ &+ \ldots \end{aligned}$$

Invariant integral of Sp_{2n}.

Let H_2 be a non-singular skew-symmetric metric on a vector space E of dimension $2n$. Let Sp_{2n} be the subgroup of the linear group of the symmetries of H_2. In the algebraic variety $\Lambda^2\mathrm{E}^*$ of the skew-symmetric metrics, regardless of the basis of E chosen, we can define (up to a constant multiplicative factor) the function

det that assigns to each metric its determinant. So, we can consider the open set $\Lambda^2\mathrm{E}^* - (det)_0$. The sequence of morphisms of varieties

$$\begin{array}{ccccccccc} 1 & \to & Sp_{2n} & \to & Gl(E) & \to & \Lambda^2\mathrm{E}^* - (det)_0 & \to & 1 \\ & & & & S & \mapsto & S^t \circ H_2 \circ S & & \end{array}$$

shows that $\Lambda^2\mathrm{E}^* - (det)_0$ is the quotient variety of $Gl(E)$ by the simplectic subgroup Sp_{2n} (Sp_{2n} acting on $Gl(E)$ by the left).

The invariant functions of $Gl(E)$ by Sp_{2n} are identified with the functions of $\Lambda^2\mathrm{E}^* - (det)_0$. Therefore, via the morphism of varieties $\mathrm{End}_k(E) \to \Lambda^2\mathrm{E}^*$, $S \mapsto S^t \circ H_2 \circ S$, the functions of $\Lambda^2\mathrm{E}^*$ are identified with the functions of $\mathrm{End}_k(E)$ that are (right) invariant by Sp_{2n}. The morphism between the rings of functions $S^\cdot(\Lambda^2 E) \to S^\cdot(\mathrm{End}_k(E)^*)$ is expressed explicitly as follows

$$\begin{array}{ccc} S^m(\Lambda^2 E) & \longrightarrow & S^{2m}(\mathrm{End}_k(E)^*) \\ s_1 \cdot \ldots \cdot s_m & \mapsto & \overline{H_2 \otimes \overset{m}{\cdots} \otimes H_2 \otimes s_1 \otimes \ldots \otimes s_m} \end{array}$$

Equivalently, the left Sp_{2n}-invariant functions of the variety $\mathrm{End}_k(\mathrm{E})$ are the direct sum of the images of the morphisms

$$\begin{array}{ccc} S^m(\Lambda^2 E^*) & \longrightarrow & S^{2m}(\mathrm{End}_k(E)^*) \\ \omega_1 \cdot \ldots \cdot \omega_m & \mapsto & \overline{\omega_1 \otimes \ldots \otimes \omega_m \otimes H^2 \otimes \overset{m}{\cdots} \otimes H^2} \end{array}$$

(we think of $S^{2m}(\mathrm{End}_k(E)^*)$ as a quotient of $E^{*2m} \otimes E^{2m}$). Therefore, the invariants of $S^{2m}(\mathrm{End}_k(E)^*)$ by the action of Sp_{2n} by the left and by the right are

$$\begin{aligned} &\langle \overline{\sigma(H_2 \otimes \overset{m}{\cdots} \otimes H_2) \otimes \sigma'(H^2 \otimes \overset{m}{\cdots} \otimes H^2)} \rangle_{\sigma,\sigma' \in S_{2m}} \\ &= \langle \overline{(H_2 \otimes \overset{m}{\cdots} \otimes H_2) \otimes \sigma'(H^2 \otimes \overset{m}{\cdots} \otimes H^2)} \rangle_{\sigma' \in S_{2m}}. \end{aligned}$$

Let $A_{Sp_{2n}}$ be the ring of functions of Sp_{2n} and $w_{Sp_{2n}} \in A^*_{Sp_{2n}} \subset A^*_{M_{2n}} = \prod_r S^r \mathrm{End}_k(E)$ the invariant integral on Sp_{2n}. The r-th component $[w_{Sp_{2n}}]_r$ of $w_{Sp_{2n}}$ is

$$\begin{aligned} &[w_{Sp_{2n}}]_r = \sum_{\sigma \in S_{2m}} \lambda_\sigma \cdot \overline{(H_2 \otimes \overset{m}{\cdots} \otimes H_2) \otimes \sigma(H^2 \otimes \overset{m}{\cdots} \otimes H^2)} \quad \text{if } r = 2m, \\ &[w_{Sp_{2n}}]_r = 0 \quad \text{if } r = 2m+1. \end{aligned}$$

Proposition 3.4. ([G, Th. 4.3.3]) *Let us consider the natural action of* Sp_{2n} *on* $E \otimes \overset{r}{\cdots} \otimes E$, $g \cdot (e_1 \otimes \cdots \otimes e_r) = g \cdot e_1 \otimes \ldots \otimes g \cdot e_r$. *One has that:*

(1) $(E \otimes \overset{2m+1}{\cdots} \otimes E)^{Sp_{2n}} = 0$.
(2) $(E \otimes \overset{2m}{\cdots} \otimes E)^{Sp_{2n}} = \langle \sigma(H^2 \otimes \overset{m}{\cdots} \otimes H^2) \rangle_{\sigma \in S_{2m}}$.

Let us consider the morphism $\Lambda^2 E \hookrightarrow \mathrm{End}_k(E)$, $H'^2 \mapsto H'^2 \circ H_2$, that assigns to each metric H'^2 the associated endomorphism to the pair of metrics H^2, H'^2. Two skew-symmetric metrics are isometric (with regard to H_2) if and only if their associated endomorphisms are equivalent, and an endomorphism T (up to conjugations) is the associated endomorphism of a skew-symmetric metric and H_2 if and only if every elementary divisor of T appears twice ([E]). Let $C \subset \mathrm{End}_k(E)$ the closed set of such endomorphisms. Then $\Lambda^2 E/Sp_{2n} = C/Gl(E)$. Let us write

$$E = E' \oplus E', \quad H_2 = \begin{pmatrix} 0 & -Id \\ Id & 0 \end{pmatrix}$$

The diagram

$$\begin{array}{ccccccc} \mathrm{End}_k E' & \hookrightarrow & \Lambda^2 E & \hookrightarrow & C & \hookrightarrow & \mathrm{End}_k E \\ T & \mapsto & \begin{pmatrix} 0 & T \\ -T & 0 \end{pmatrix} & \mapsto & \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} & & \end{array}$$

shows that $\mathrm{End}_k E'/Gl(E') = \Lambda^2 E/Sp_{2n} = C/Gl(E)$. The ring of invariant functions of $\mathrm{End}_k(E')$ (by the action by conjugation of the linear group $Gl(E')$) is isomorphic to the ring of invariant functions of $\mathrm{End}_k(E)$ (by the action by conjugation of the linear group $Gl(E)$), then the ring of invariant functions of $\Lambda^2 E$ (by the action of Sp_{2n}) is isomorphic to the ring of invariant functions of $\mathrm{End}_k(E)$ (by the action by conjugation of the linear group $Gl(E)$).

Let f be the composite morphism $\mathrm{End}_k(E) \to \Lambda^2 \mathrm{E} \to \mathrm{End}_k(E)$, $T \mapsto TH^2T^t \mapsto TH^2T^tH_2$. The invariant functions of $\mathrm{End}_k(E)$ by the action by conjugation of the linear group coincide, via f^*, with the functions of $\mathrm{End}_k(E)$ that are left and right invariant by the action of the symplectic group. Now we can calculate the invariant integral of Sp_{2n} as we have calculated the invariant integral of O_n.

Let us denote $w_\sigma := \overline{H_2 \otimes \overset{m}{\cdots} \otimes H_2 \otimes H^2_{1\sigma(1)} \otimes \overset{m}{\cdots} \otimes H^2_{m\sigma(m)}} \in S^{2m}\mathrm{End}_k(E)$ and $a_\sigma := \overline{H_2 \otimes \overset{m}{\cdots} \otimes H_2 \otimes H^2_{1\sigma(1)} \otimes \overset{m}{\cdots} \otimes H^2_{m\sigma(m)}} \in S^{2m}\mathrm{End}_k(E)^*$. Let $\lambda_{\sigma\sigma'} = \dfrac{w_\sigma(a_{\sigma'})}{\bar{\bar{\sigma}}(Id_1) \cdot \bar{\bar{\sigma}}'(Id_1)}$ and $(\lambda_{[\sigma]})_{[\sigma]\in S_m/\sim} = (\lambda_{\sigma\sigma'})^{-1}_{[\sigma],[\sigma']\in S_m/\sim} \cdot (1,\dots,1)$. Then, $w_{Sp_{2n}} = 1 + \sum\limits_{m>0} \sum_{[\sigma]\in S_m/\sim} \lambda_{[\sigma]} \cdot \dfrac{w_\sigma}{\bar{\bar{\sigma}}(Id_1)}$

Let us give the first three terms of w_{Sp_m}, $m = 2n$:

$$\begin{aligned} w_{Sp_m} &= 1 + \frac{\overline{H_2 \otimes H^2}}{m} \\ &+ \frac{(3m^2 - 3m + 3) \cdot \overline{H_2 \otimes H_2 \otimes H^2 \otimes H^2} + (3m-6) \cdot \overline{H_2 \otimes H_2 \otimes H^2_{12} \otimes H^2_{21}}}{m^4 - m^3 + m^2 + 3m} \\ &+ \dots \end{aligned}$$

References

[A] Álvarez, A., Sancho, C., Sancho, P., *Algebra schemes and their representations*, J. Algebra **296/1** (2006) 110-144.

[A2] Álvarez, A., Sancho, C., Sancho, P., *Reynolds operator*, Preprint (2006).

[B] Brion, M., Schwarz, G.W., *Théorie des invariants et géometrie des variétés quotients*, Travaux en cours, vol. 61, Hermann, Paris, 2000.

[D] Dolgachev, I., *Lectures on invariant theory*, London Mathematical Society Lecture Notes Series, vol. 296, Cambridge University Press, Cambridge, 2003.

[G] Goodman, R., Wallach, N.R., *Representations and Invariants of the Classical Groups*, Cambridge University Press, New York, 1998.

[E] Ermolaev, Ju. B., *Simultaneous reduction of a pair of bilinear forms to canonical form*, Dokl. Akad. Nauk SSSR 132 257-259 (Russian); translated as Soviet Math. Dokl. 1 1960 523-525.

[S] Sancho de Salas, C., *Grupos algebraicos y teoría de invariantes*, Sociedad Matemática Mexicana, México, 2001.

Departamento de Matemáticas, Universidad de Extremadura, Avenida de Elvas s/n, 06071 Badajoz, Spain
E-mail address: aalarma@unex.es

Departamento de Matemáticas, Universidad de Salamanca, Plaza de la Merced 1-4, 37008 Salamanca, Spain
E-mail address: mplu@usal.es

Departamento de Matemáticas, Universidad de Extremadura, Avenida de Elvas s/n, 06071 Badajoz, Spain
E-mail address: sancho@unex.es