GEOMETRIC CALCULATION OF THE INVARIANT INTEGRAL
OF CLASSIC GROUPS

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Abstract. Let $G = \text{Spec} \, A$ be a linearly reductive group, and $w_G \in A^*$ be $G$-invariant and $w_G(1) = 1$. We establish the algebraic harmonic analysis on $G$ and we compute $w_G$ when $G = \text{Sl}_n, \text{Gl}_n, \text{O}_n, \text{Sp}_{2n}$, by geometric arguments and by means of the Fourier transform.

Introduction

An affine $k$-group $G = \text{Spec} \, A$ is semisimple if and only if $A^*$ splits in the form $A^* = k \times B^*$ as $k$-algebras, where the first projection $\pi_1: A^* \to k$ is the morphism $\pi_1(w) := w(1)$ ([A2] Theorem 2.6). The linear form $w_G := (1, 0) \in k \times B^* = A^*$ will be referred of as the invariant integral on $G$.

In the theory of invariants the calculation of the invariant integral $w_G$ is of great interest, because it yields the calculation of the invariants of any representation. The aim of this article is the explicit calculation of $w_G$ when $G = \text{Sl}_n, \text{Gl}_n, \text{O}_n, \text{Sp}_{2n}$ (char $k = 0$), by geometric arguments, and by means of the Fourier transform, defined below. Although $G$ is not a compact group it is possible to define the invariant integral of $G$, the Fourier transform, the convolution product (2.8) and prove the Parseval identity (2.4), inversion formula, etc.

Let $A_i^*$ be simple (and finite) $k$-algebras and $A^* = \prod_i A_i^*$. On every $A_i^*$, one has the non singular trace metric and its associated polarity. Hence, one obtains a morphism of $A^*$-modules $\phi: A = \oplus_i A_i \hookrightarrow \prod_i A_i^* = A^*$. If $G = \text{Spec} \, A$ is a semisimple affine $k$-group and $*: A \to A$, $a \mapsto a^*$ is the morphism induced by the morphism $G \to G, g \mapsto g^{-1}$, we prove that $\phi$ is the morphism

$$A \to A^*, \ a \mapsto w_G(a^* \cdot -)$$

where $w_G(a^* \cdot -)(b) := w_G(a^* \cdot b)$. We shall call $\phi$ the Fourier transform. The product operation in $A^*$ defines, via the Fourier transform, a product on $A$, which is the convolution product in the classical examples.

Let us consider a system of coordinates in $G$, that is, let us consider $G = \text{Spec} \, A$ as a closed subgroup of a semigroup of matrices $M_n = \text{Spec} \, B$. Then $A$ is the quotient of $B$ by the ideal $I$ of the functions of $M_n$ vanishing on $G$. Hence $A^*$ is the subalgebra of $B^*$ and one has that $k \cdot w_G = A^{*G} = \{ w \in B^{*G} : w(I) = 0 \}$. Moreover, $B^{G}$ (which is the ring of functions of $M_n/G$), coincides essentially with $B^{*G}$, via the Fourier transform. Finally, we prove that given $w \in B^{*G}$, the condition $w(I) = 0$ is equivalent to $w(I^G) = 0$, which is a finite system of equations “in each degree”.


Key words and phrases. Reynolds operator, functor of algebras, semisimple group, invariants, Fourier transform.
1. Preliminary results

Let $k$ be a commutative ring with unit. All functors considered in this paper are functors over the category of commutative $k$-algebras. Given a $k$-module $E$, the functor $E$ defined by $E(B) := E \otimes_k B$ is called a quasi-coherent $k$-module. The functors $E \to E, E \to (E)$ establish an equivalence between the category of $k$-modules and the category of quasi-coherent $k$-modules ([A], 1.12). In particular, $\text{Hom}_k(E, E') = \text{Hom}_k(E, E')$.

If $F, H$ are two functors of $k$-modules, we shall denote by $\text{Hom}_k(F, H)$ the functor of $k$-modules

$$\text{Hom}_k(F, H)(B) := \text{Hom}_B(F|_B, H|_B)$$

where $F|_B$ is the category of $k$-modules scheme. The functor $E^* = \text{Hom}_k(E, k)$ is the functor of points of $Spec\, E$ and we call $E^*$ a $k$-modules scheme. $E^{**} = E$ and the category of $k$-modules is anti-equivalent to the category of $k$-modules scheme ([A, 1.10,1.12]). Given a functor of $k$-modules $F$, we denote by $F'$ the $k$-modules scheme closure of $F'$; that is, $F'$ is the representant on the category of $k$-modules schemes of the functor $\text{Hom}_k(F, -)$.

Let $G = \text{Spec}\, A$ be an affine $k$-group and let $G^*$ be the functor of points of $G$, that is, $G^*(B) = \text{Hom}_{k-alg}(A, B)$. We will denote by $k[G^*]$ the functor over the category of $k$-algebras defined by $k[G^*](B) = \{\text{formal finite } B\text{-linear combinations of points of } G \text{ with values in } B\}$. One has a natural morphism $G^* \to A^*$, because $G^*(B) = \text{Hom}_{k-alg}(A, B) \subset \text{Hom}_k(A, B) = A^*(B)$, which extends to a unique morphism of functors of $k$-algebras $k[G^*] \to A^*$. By ([A, 3.3]) one has that $k[G^*]' = A$ and that $A^*$ is the algebra (and module) scheme closure of $k[G^*]$. Moreover, $A^*$ represents the functor $\text{Hom}_{k-alg}(k[G^*], -)$ in the category of dual functors of algebras ([A], 5.3).

The category of $G$-modules is equivalent to the category of (quasi-coherent) $A^*$-modules ([A], 5.5). Therefore, if $k$ is a field, $G$ is semisimple (that is, linearly reductive) if and only if $A^*$ is a semisimple $k$-algebras scheme, i.e. $A^* = \prod A_i^*$, where $A_i^*$ are simple (and finite) $k$-algebras ([A], 6.8). If $k$ is an algebraically closed field, then $A_i^*$ is an algebra of matrices by Wedderburn’s theorem.

2. Algebraic harmonic analysis

Let $k$ be a field, that we assume algebraically closed for simplicity.

On $A^* = M_n(k)$ one has the non singular metric: $T_2(T, S) := \text{the trace of the matrix } T \circ S$. If $T_2$ is the metric of the trace of the algebra $A^*$, then $T_2 = n \cdot T_2^2$.

Let us denote by $\phi_A, \phi_A^*: A \approx A^*$ the polarities associated to $T^2$ and $T^2$.

Now, let $A_i^*$ be $k$-algebras of matrices, $A^* = \prod A_i^*$ and $A = \oplus A_i$. Giving a metric $T_2^2$ on $A$ is equivalent to defining its associated polarity $A \overset{\phi}{\to} A^*$. Since there are (iso)morphisms $\phi_{A_i}^*: A_i \to A_i^*$, one has the obvious injection $\phi^* : \oplus A_i \to \prod A_i^*$. If dim$_k A_i$ is prime with the characteristic of $k$, replacing $T_2^2$ by $T_2^2$ we can define in the same way a canonical morphism $\phi : \oplus A_i \to \prod A_i^*$ and $\phi = (1/\sqrt{\text{dim}_k A_i}) \phi_i^*$.

Given $w = (w_i)_i \in \oplus A_i^*$ and $w' = (w'_i)_i \in \prod A_i^*$, we can define $T_2^2(w, w') := \sum_i T_2^2(w_i, w_i')$. Given $a \in A$ and $a' \in A^*$, then $a(w') = T_2^2(\phi(a), w')$. Let $tr' : \oplus A_i^* \to k$, be defined by $tr'(w) = T_2^2(w, 1)$. Given $a \in A$ and the unity $1 \in A^*$, then

$$a(1) = T_2^2(\phi(a), 1) = tr'(\phi(a))$$
Likewise, \( a(w) = T_2'(\phi'(a), w) = \text{tr}'(\phi'(a) \cdot w) \), for every \( a \in A \) and \( w \in A^* \), and when \( T_2 \) is a non-singular metric, \( a(w) = T_2(\phi(a), w) = \text{tr}(\phi(a) \cdot w) \), for every \( a \in A \) and \( w \in A^* \) and 

\[
a(1) = T_2(\phi(a), 1) = \text{tr}(\phi(a))
\]

**Properties.** The morphism \( \phi' : A \rightarrow A^* \) satisfies the following properties:

- it is a morphism of left and right \( A^* \)-modules, because each polarity \( A_i \rightarrow A_i^* \) is a morphism of left and right \( A_i^* \)-modules;
- it is a symmetric metric: \( \phi'(a)(b) = \phi'(b)(a) \), because the metric \( T_2' \) on each \( A_i^* \) is so;
- it is an injective morphism; that is, the metric associated to \( \phi' \) is non singular, because the metric \( T_2' \) on each \( A_i^* \) is so;
- the image of \( \phi' \) is dense in \( A^* \). More generally, the natural morphism \( \oplus_i E_i \twoheadrightarrow \prod_i E_i, \dim_k E_i < \infty \), is dense, i.e. \( \overline{\oplus_i E_i} \rightarrow \prod_i E_i \) is surjective. In other words: Observe that every \( k \)-modules subscheme of \( \prod_i E_i \) is the orthogonal of a quasi-coherent submodule of \( \oplus_i E_i^* \). If we consider in \( \prod_i E_i \) the topology of closed sets generated by the zeroes of the vectors of \( \oplus_i E_i^* \), then \( \overline{\oplus_i E_i} = \prod_i E_i \);
- the image of \( \phi' : A \rightarrow A^* \) is the maximal quasi-coherent \( A^* \)-submodule of \( A^* \). Let \( w = (w_i)_{i \in I} \in \prod_i A_i^* \) be an element such that \( w_i \neq 0 \) for infinitely many indices \( i \). The elements \((\ldots, 0, w_i, 0, \ldots) = (\ldots, 0, 1, 0, \ldots) \cdot w \) belong to the \( A^* \)-submodule \( < w > \) generated by \( w \). Then \( < w > \) is an \( A^* \)-module of infinite dimension and \( w \) cannot belong to any quasi-coherent \( A^* \)-submodule (if it belonged to some quasi-coherent \( A^* \)-submodule, then it would be included in some of its \( A^* \)-submodules of finite dimension, by [A, 4.7]).

**Proposition 2.1.** Every morphism \( f : A \rightarrow E^* \) of \( A^* \)-modules lifts to a unique morphism of \( A^* \)-modules \( f' : A^* \rightarrow E^* \) (such that \( f = f' \circ \phi' \)). Hence,

\[
\text{Hom}_{A^*}(A, E^*) = \text{Hom}_{A^*}(A^*, E^*) = E^*
\]

**Proof.** By the last property we have

\[
\text{Hom}_{A^*}(A, E^*) = \text{Hom}_{A^*}(E, A^*) = \text{Hom}_{A^*}(E, A) = \text{Hom}_{A^*}(A^*, E^*)
\]

\[\square\]

**Notation 2.2.** For any \( k \)-algebra \( C \) we shall denote by \( Z(C) \) the center of \( C \).

**Theorem 2.3.** Let \( A_i^* \) be simple \( k \)-algebras, and \( A^* = \prod_i A_i^* \). Let \( S^2 \) be a symmetric metric on \( A \). If \( S^2 \) is \( A^* \)-linear (i.e., \( S^2(w \cdot a, a') = S^2(a, a' \cdot w) \) for every \( a, a' \in A \) and \( w \in A^* \)), then \( S^2 \) coincides, up to a factor of \( Z(A^*) \), with the metric \( T^2 \).

**Proof.** The polarity \( \varphi : A \rightarrow A^* \) associated to \( S_2 \) is a morphism of left and right \( A^* \)-modules. Therefore, \( f' \) maps each summand \( A_i \) into each \( A_i^* \). Let \( \phi'_i : A_i \rightarrow A_i^* \) be the restrictions of \( \phi' \) and \( \varphi \) to each \( A_i \). The morphism \( \varphi_i \circ \phi_i^{-1} : A_i^* \rightarrow A_i^* \) is a morphism of left and right \( A^* \)-modules. Now, a morphism of left \( A_i^* \)-modules of \( A_i^* \) is an homothety by an element of \( A_i^* \); if this homothety is also a morphism of right \( A_i^* \)-modules, then it is an homothety by an element of \( Z(A_i^*) \). Thus, \( \varphi_i = z_i \circ \phi_i \), where \( z_i \in Z(A_i^*) \). Since \( Z(A^*) = \prod_i Z(A_i^*) \), the proof is easily completed. \[\square\]

Let \( E \) be a linear representation of a \( k \)-group \( G = \text{Spec} \ A \). The associated character \( \chi_E \in A \) is defined by \( \chi_E(g) = \text{trace} \) of the linear endomorphism \( E \rightarrow E \), \( e \mapsto g \cdot e \), for every \( g \in G \) and \( e \in E \).
Let $G = \text{Spec } A$ be a semisimple group. One has $A^* = \prod_i \text{End}_k(E_i)$, where $\{E_i\}$ runs over all the irreducible representations of $G$ (up to isomorphisms). The polarity $A_i^* = \text{End}_k(E_i) \to A_i$ associated to the metric $T_i^2$ maps the unit $1_i$ of $A_i^*$ to the character $\chi_{E_i}$ because $\text{tr}(1_i \cdot g) = \chi_{E_i}(g)$. Therefore, $\phi'(\chi_{E_i}) = 1_i$. Since $1_i \cdot 1_j = \delta_{ij} \cdot 1_i$, one obtains that

$$T^2(\chi_{E_i}, \chi_{E_j}) = 0$$

if $i \neq j$ and $T^2(\chi_{E_i}, \chi_{E_i}) = \dim_k E_i$. Moreover, $A$ is generated by the $\chi_{E_i}$ as an $A^*$-module, because $\text{Im } \phi$ is generated by the $1_i$.

Let $E_0 = k$ be the trivial representation of $G$ and $w_G := 1_0 \in A^*$ be the “invariant integral of $G$”. The invariant integral of $G$ is characterized for being $G$-invariant and normalized, that is, $w_G(1) = 1$ ([A2, 2.10]). A dual functor $F$ is a functor of $G$-modules if and only if is a functor of $A^*$-modules and $F^G = w_G \cdot F$ (see [A2, 2.3, 3.3]).

One has that $w_G \cdot 1_j = 0$, if $E_j$ is not the trivial representation, and $w_G \cdot 1_0 = 1_0$. Hence, $w_G \cdot \chi_{E_j} = 0$ if $E_j$ is not the trivial representation, and $w_G \cdot \chi_{E_0} = \chi_{E_0} = 1$. Moreover, since $\chi_{E \otimes E'} = \chi_E + \chi_{E'}$, one has

$$w_G \cdot \chi_E = \dim_k E^G.$$

Let $*: A \to A$, $a \mapsto a^*$, be the morphism induced by the morphism $G \to G$, $g \mapsto g^{-1}$. If $E$ is a representation of $G$, we shall consider $E^*$ as a left $G$-module by $(g * w)(e) = w(g^{-1} \cdot e)$. One has that $\chi_E^* = \chi_{E^*}$, because the trace of $g^{-1} \in G$ operating on $E$ is equal to the trace of $g$ operating on $E^*$ (which operates by the inverse transposed of $g$).

**Theorem 2.4.** Let $G = \text{Spec } A$ be a semisimple group and let $w_G \in A^*$ be its invariant integral. The morphism

$$A \to A^*, \ a \mapsto w_G(a^* \cdot -)$$

where $w_G(a^* \cdot -)(a') := w_G(a^* \cdot a')$ coincides with $\phi$.

**Proof.** Let us first prove that $A \to A^*, \ a \mapsto w_G(a^* \cdot -)$, is a morphism of left $G$-modules: For all point $g$ of $G$,

$$w_G((g \cdot a)^* \cdot -) = w_G((a^* \cdot g^{-1}) \cdot -) = w_G((a^* \cdot g^{-1} \cdot -) \cdot g) = w_G(a^* \cdot (\cdot g))$$

$$= g \cdot (w_G(a^* \cdot -))$$

where $\hat{=} \text{ is due to } (g \cdot a)^* (g') = a(g'^{-1} \cdot g) = a((g^{-1} \cdot g')^{-1}) = a^* \cdot g^{-1} (g')$, and $\hat{=} \text{ is due to } g \cdot w_G = w_G$.

Let us show now that it is symmetric: $w_G(a^* \cdot a') = w_G((a \cdot a'^*)^*) = w_G(a \cdot a'^*)$, because $*(w_G) = w_G$.

By Theorem 2.3, it only remains to prove the orthonormality of $\{\chi_{E_i}\}$, where $E_i$ are irreducible:

$$w_G(\chi_{E_i}^* \cdot \chi_{E_j}) = w_G(\chi_{E_i^* \otimes E_j}) = w_G(\chi_{E_i^* \otimes E_j}) = \dim_k \text{Hom}_G(E_i, E_j) = \delta_{ij}.$$

\qed

If we consider in $A$, the metric $T^2$, defined by $T^2(a, a') = w_G(a^* \cdot a)$, then $\phi$ is the polarity associated to $T^2$. If we consider in $A' = \oplus_i A_i^* \subset \text{Im } \phi$, the metric $T_2$, then $\phi: A \to A'$ is an isometry.
Proposition 2.5. Let \( G = \text{Spec} \ A \) be a semisimple group, \( D \) be a left \( G \)-invariant vector field on \( G \) and \( D_e \) be the value of the vector field at the identity element \( e \in G \). Then
\[
\phi(D(a)) = D_e \cdot \phi(a)
\]

Proof. \( D(a) = D_e \cdot a \) because
\[
(D_e \cdot a)(g) = a(g \cdot D_e) = a(D_g) = D_g(a) = D(a)(g)
\]
for all \( g \in G \). Therefore, \( \phi(D(a)) = \phi(D_e \cdot a) = D_e \cdot \phi(a) \). \( \square \)

Let \( \pi_1 : A^* \to k \), be defined by \( \pi_1(w) = w(1) \). Then, \( \pi_1 \circ \phi = w_G \), because \( \pi_1 \circ \phi(1) = \pi_1(w_G) = 1 \) and \( \pi_1 \circ \phi \) is \( G \)-invariant, that is, a morphism of \( A^* \)-modules.

If \( G = \text{Spec} \ A \) is semisimple and \( E \) is an irreducible linear representation of \( G \), then \( \dim_k E \) is prime with the characteristic of \( k \), because \( \phi' \) is injective and \( \phi'(1) = (\dim_k E_i)_{i \in I} \cdot \phi \) (where \( I \) runs over the set of the irreducible linear representations up to isomorphisms).

Given \( a, b \in A \), one has that \( T^2(a, b) = \phi(a)(b) = w_G(a^* \cdot b) \). If we denote \( w_G = \int dg \), then
\[
T^2(a, b) = \int a(g^{-1}) \cdot b(g) \ dg.
\]

Let \( G = \text{Spec} \ A \) be a semisimple group, \( E \) a \( G \)-module and \( E_i \) a simple \( G \)-module. Let us consider the \( G \)-module decomposition \( E = E' \oplus F \), where \( E' \) is the homogeneous component of \( E \) isomorphic to \( \bigoplus^n E_i \). Now, we want to compute the morphism \( E \to E \) which is the identity over \( E' \) and nulle over \( F \). In particular, we could obtain the decomposition of \( E \) as direct sum of homogeneous modules.

Let \( 1_i = (0, \ldots, \hat{1}_i, \ldots, 0) \in \prod_{E_i} \text{End}_k E_j = A^* \). We have to calculate the morphism \( E \to E, e \mapsto 1_i \cdot e \). Recall that \( 1_i = \phi(n_i \cdot \chi_{E_i}) \), \( n_i = \dim_k E_i \). The dual morphism of the multiplication morphism \( E^* \otimes A^* \to E^* \) is the comultiplication morphism \( \mu : E \to E \otimes A \). If \( \{ e_i^* \} \) is a basis of \( E \), and \( \mu(e) = \sum_i e_i^* \otimes a_i \), then
\[
g \cdot e = \sum_i a_i(g) e_i', \quad \text{for all} \ g \in G.\]
Hence,
\[
1_i \cdot e = \sum_i a_i(1_i) \cdot e_i' = \sum_i a_i(\phi(n_i \cdot \chi_{E_i})) \cdot e_i' = \sum_i n_i \cdot w_G(a_i \cdot \chi_{E_i}^*) \cdot e_i' = n_i \cdot \sum_i \int a_i \cdot \chi_{E_i}^* dg \cdot e_i'.
\]

Notation 2.6. Given an affine scheme \( X = \text{Spec} \ A \) will denote \( A_X = A \).

Proposition 2.7. Let \( G = \text{Spec} \ A \) be a semisimple group, \( H \subseteq G \) a normal subgroup and \( \pi : G \to G/H \) the quotient morphism. Let \( i^*: A_G \to A_H \) and \( \pi : A_G^* \to A_G^*/\pi \) be the natural morphisms. Then, with the obvious notations,
\[
w_H(i^*(a)) = \text{tr}_{G/H}(\pi(\phi_G(a)))
\]
for all \( a \in A_G \).

Proof. The set of irreducible representations of \( G/H \) is equal to the set of irreducible representations of \( G \) which are \( H \)-invariant. The natural projection \( A_G^* = \prod E_i^* \to \prod_{E_i = E_i^H} \text{End}_k(E_i) = A^*_G/H \) coincides with \( \pi \).

The diagram
\[ A_G = \bigoplus_i \text{End}_k(E_i)^* \xrightarrow{\phi_G} \prod_i \text{End}_k(E_i) \xrightarrow{\pi} A_G^* \rightarrow A_G^G \]

\[ A_G^H = \bigoplus_{E_i \in E^H} \text{End}_k(E_i)^* \xrightarrow{\phi_{G/H}} \prod_{E_i \in E^H} \text{End}_k(E_i) \xrightarrow{\pi} A_G^H \rightarrow A_{G/H}^G \]

is commutative. Then

\[ tr_{G/H}(\pi(\phi_G(a))) = tr_{G/H}(\phi_{G/H}(w_H \cdot a)) = (w_H \cdot a)(1) = a(w_H) = w_H(i^*(a)) \]

\[ \square \]

The image of the morphism \( \phi : A \hookrightarrow A^* \) is a bilateral ideal. Then it is a subring, although without unit, because \((\ldots, 1, 1, \ldots) \notin \bigoplus A^*_i = \text{Im} \phi \).

**Definition 2.8.** The product of the subring \( \text{Im} \phi \) induces a product on \( A \), through the identification \( A \cong \bigoplus \text{Im} \phi \). This product is called the convolution product.

Let \( a, b \in A \), \( w' = \phi(a), w'' = \phi(b) \) and let us denote by \( * \) the convolution product. Then

\[ a * b = \phi^{-1}(w' \cdot w'') = w' \cdot \phi^{-1}(w'') = w' \cdot b. \]

Therefore, \((a * b)(x) = (w' \cdot b)(x) = b(x \cdot w') = (x \cdot w')(b) = w'(b \cdot x) = w_G(a^* \cdot (b \cdot x))\), for all point \( x \) of \( G \). If we denote \( w_G = \int dg \), then

\[ (a * b)(x) = \int a(g^{-1}) \cdot b(x \cdot g) \, dg. \]

**3. Invariant integral of \( \text{Sl}_n, \text{Gl}_n, O_n \) and of \( \text{Sp}_{2n} \)**

Let \( k \) be a field of characteristic zero. The groups \( \text{Gl}_n, \text{Sl}_n, O_n \) and of \( \text{Sp}_{2n} \) are semisimple, so they have an invariant integral. This section is devoted to the explicit calculation of the invariant integral of the groups \( \text{Gl}_n, \text{Sl}_n, O_n \) and of \( \text{Sp}_{2n} \).

Let us consider the affine algebraic \( k \)-variety \( M_n = \text{End}_k(E) \), whose points with values in a \( k \)-algebra \( B \) is the semigroup of square matrices of order \( n \) with coefficients in \( B \). Its ring of functions is \( A_{M_n} = \bigoplus_{n \in \mathbb{N}} S^n(\text{End}_k(E)^*) \). Although \( M_n \) is a semigroup scheme and not a group one, \( k[M_n^*] \) is a functor of \( k \)-algebras and its \( k \)-algebra scheme closure is \( A_{M_n}^* = \prod_{n \in \mathbb{N}} (\text{End}_k(E) \otimes_k . \otimes \text{End}_k(E))^{S_n} \). One has the explicit natural morphism

\[ M_n^* \rightarrow A_{M_n}^*, \tau \mapsto (1, \tau, \tau \otimes \tau, \tau \otimes \tau \otimes \tau, \ldots) \]

Since this morphism is a morphism of semigroups, the unique structure of functors of algebras of \( A_{M_n}^* \) is the one of the direct product of the algebras \( (\text{End}_k(E) \otimes . \otimes \text{End}_k(E))^{S_n} \). One has that the category of quasi-coherent \( A_{M_n}^* \)-modules is equivalent to the category of \( M_n \)-modules. \( A_{M_n}^* \) is a semisimple algebra (see [A2, 2.7]); hence every \( A_{M_n}^* \)-module and every \( M_n \)-module is semisimple.

The natural action of \( \text{End}_k(E) \) on \( E \otimes . \otimes E \) extends uniquely to a structure of \( A_{M_n}^* \)-module. It consists of the projection of \( \prod M_n^{S_n} \text{End}_k(E) \) onto the \( m \)-th factor,
$S^m\text{End}_k(E)$, and the action of $S^m\text{End}_k(E)$ on $E \otimes m \cdot E$ via its inclusion in $\text{End}_k(E) \otimes m \cdot \text{End}_k(E)$; that is

$$(g_1 \cdot \ldots \cdot g_m) \cdot (v_1 \otimes \ldots \otimes v_m) = \frac{1}{m!} \cdot \sum_{\sigma \in S_m} g_{\sigma(1)}(v_1) \otimes \ldots \otimes g_{\sigma(m)}(v_m).$$

The isomorphism $\phi : (\text{End}_k(E))^* \to \text{End}_k(E)$, induces, by taking symmetric algebras, a morphism $\varphi : A_{M_n} \to A_{M_n}^*$ of left and right $A_{M_n}$-modules. It coincides with $\phi$, up to an invertible factor of the center $Z(A_{M_n})$.

**Invariant integral of $Sl_n$.**

Let $A_{Sl_n} = k[x_{11}, \ldots, x_{nn}]/(\text{det}(x_{ij}) - 1)$ be the ring of functions of the special linear group. Then $A_{Sl_n}^*$ splits into a direct product of simple algebras, one of them being the one corresponding to the trivial representation. So

$$A_{Sl_n}^* = k \cdot w_{Sl_n} \times B^*.$$  

Recall that $A_{Sl_n}^{Sl_n} = k \cdot w_{Sl_n}$. Let us compute the invariants of $A_{Sl_n}^*$ by $Sl_n$.

From the inclusion $Sl_n \subset M_n$ one obtains the injective morphism $A_{Sl_n}^* \subset A_{M_n}^*$. We shall first compute the invariants of $A_{M_n}^*$ by the action of $Sl_n$ and then we shall compute the ones belonging to $A_{Sl_n}^*$. Since $A_{M_n}$ is a semisimple $Sl_n$-module, it splits into a direct sum

$$A_{M_n} = A_{M_n}^{Sl_n} \oplus (\oplus_i S_i),$$

where $S_i$ are simple $Sl_n$-submodules, but not $Sl_n$-invariant. Taking dual one obtains that

$$A_{M_n}^* = (A_{M_n}^{Sl_n})^* \times (\prod_i S_i^*) = (A_{M_n}^*)^{Sl_n} \times (\prod_i S_i^*)$$

because a group acts trivially on a module if and only if it acts trivially on its dual.

The morphism $\varphi : A_{M_n} = A_{M_n}^{Sl_n} \oplus B \rightarrow (A_{M_n}^{Sl_n})^* \times B^*$ is a morphism of $Sl_n$-modules, so that

$$\varphi(A_{M_n}^{Sl_n}) \subseteq (A_{M_n}^*)^{Sl_n} = (A_{M_n}^*)^{Sl_n}$$

and $\varphi(B) \subseteq B^*$. Since the closure, $\overline{\varphi(A_{M_n})}$, of $\varphi(A_{M_n})$ is $A_{M_n}$, one has that $\overline{\varphi(A_{M_n}^{Sl_n})} = (A_{M_n}^*)^{Sl_n}$. Then, let us compute $A_{M_n}^{Sl_n}$.

From the exact sequence of groups

$$1 \rightarrow Sl_n \subset Gl_n \rightarrow Gl_n/Sl_n = G_m \rightarrow 1$$

it follows easily that $k[x_{11}, \ldots, x_{nn}]^{Sl_n} = k[\text{det}(x_{ij})]$ and therefore

$$(k[x_{11}, \ldots, x_{nn}]^{Sl_n})^{Sl_n} = k \cdot k \cdot \varphi(\text{det}(x_{ij})) \cdot \ldots \cdot k \cdot \varphi(\text{det}(x_{ij})^r) \times \ldots$$

Let us denote by $\delta_{ij}$ the matrix of null coefficients, except for the $ij$-th coefficient that is 1. As a linear map on $k[x_{ij}]$, $\delta_{ij}$ coincides with $\frac{\partial}{\partial x_{ij}} |_0$. One has that

$$\varphi(x_{ij}) = \frac{\partial}{\partial x_{ij}} |_0, \quad \varphi(x_{i1j} \cdot \ldots \cdot x_{imjn}) = \frac{1}{m!} \frac{\partial}{\partial x_{i1j1}} \cdot \ldots \cdot \frac{\partial}{\partial x_{imjn}} |_0.$$  

Since $\text{det}(x_{ij})^r$ is an homogeneous polynomial of $rn$-th degree,

$$\varphi(\text{det}(x_{ij})^r) = \frac{1}{(rn)!} \text{det}' \left( \frac{\partial}{\partial x_{ij}} \right) |_0.$$
Let $D = \det \left( \frac{\partial}{\partial x_{ij}} \right)$ be the Cayley operator, and let us denote $D_0^n = D^n |_0$. One has that
\[(k[x_{i1}, \ldots , x_{nn}]^*)^{SL_n} = k \times k \cdot D_0 \times \ldots \times k \cdot D_0^n \times \ldots \]

Let us compute now the $w \in (k[x_{i1}, \ldots , x_{nn}]^*)^{SL_n}$ vanishing on the ideal $I = (\det(x_{ij}) - 1)$. Since $w$ is $SL_n$-invariant, one has that $w \cdot w_{SL_n} = \tilde{w}$. Therefore, $\tilde{w}(I) = 0$ if and only if $w(w_{SL_n} \cdot I) = \tilde{w}(I^{SL_n}) = 0$. Now, $I^{SL_n} = (\det^n(x_{ij}) - \det^{n+1}(x_{ij}))_n \subset k[\det(x_{ij})]$. Hence, if $\tilde{w}$ is null on the functions $\det^n(x_{ij}) - \det^{n+1}(x_{ij})$, for every $n \geq 0$, and $\tilde{w}(1) = 1$, then $\tilde{w} = w_{SL_n}$. Consequently,
\[w_{SL_n} = \sum_i \frac{D_0^i}{D_0^i(\det(x_{ij}))}.\]

It only remains to determine the value of $D_0^n(\det^r(x_{ij})) \in k$.

**Lemma 3.1.** ([D, Th. 2.1]) $D(\det^r(x_{ij})) = \mu_r \cdot \det^{r-1}(x_{ij})$, where $\mu_r = r \cdot (r + 1) \cdot \ldots \cdot (r + n - 1) = \frac{(r + n - 1)!}{(r - 1)!}$.

Then $D^r(\det^r(x_{ij})) = \mu_r \cdot \mu_{r-1} \cdot \ldots \cdot \mu_1$.

Every linear representation of $SL_n$ is a submodule of direct sum of $AS_{SL_n}$ (the regular representation). Moreover, $AS_{SL_n}$ is a quotient of the ring of functions of $M_n$. Finally, the ring of functions of $M_n = \text{End}_k(E)$ is included in a direct sum of $E \otimes \otimes^m \otimes E$. Let us compute then the invariants of $SL_n$ operating on these vector spaces.

**Proposition 3.2.** ([S, Th. 19.2]) Let $SL_n$ be the special linear group of an $n$-dimensional vector space $E$. Let us consider the natural action of $SL_n$ on $E \otimes \otimes \otimes \otimes E$, $g \cdot (v_1 \otimes \ldots \otimes v_m) = g \cdot v_1 \otimes \ldots \otimes g \cdot v_m$. One has that:
1. $(E \otimes \otimes \otimes E)^{SL_n} = \Lambda^n E$.
2. $(E \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes E)^{SL_n} = \sum_{\sigma \in S_{nm}} \sigma(\Lambda^n E \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes E)$, where $\sigma \in S_{nm}$ acts on $E \otimes \otimes \otimes \otimes E$ by permuting the factors.
3. $(E \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes E)^{SL_n} = 0$ if $m$ is not a multiple of $n$.

**Proof.**
1. We must calculate $w_{SL_n} \cdot (E \otimes \otimes \otimes \otimes E) = D_0 \cdot (E \otimes \otimes \otimes \otimes E)$. Fixed a basis $(e_1, \ldots , e_n)$ of $E$, let us observe that $D_0 |_{0 \otimes e}$ corresponds to the matrix $\delta_{ij}$ that maps $e_j$ to $e_i$ and the rest of the $e_k$ to zero. Then it is clear that $D_0 \cdot (e_{i_1} \otimes \ldots \otimes e_{i_n}) = e_{i_1} \wedge \ldots \wedge e_{i_n}$ and (1) is proved.

2. The $r = nm$-th degree component of $w_{SL_n}$ is, up to scalars, $D^m$; that is, it coincides, up to scalars, with $\sum_{\sigma \in S_{nm}} \sigma \circ (D \otimes \otimes \otimes D) \circ \sigma^{-1}$. Then,
\[(E \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes E)^{SL_n} \subseteq \sum_{\sigma \in S_{nm}} \sigma(\Lambda^n E \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes E).

The inverse inclusion is obvious.
3. $w_{SL_n} \in \bigoplus^r S^r \text{End}_k(E)$ and its $r$-th degree “components” are null when $r$ is not a multiple of $n$.  


We can compute the dimension of \((E \otimes \cdots \otimes E)^{S_n}\):

\[
\dim_k (E \otimes \cdots \otimes E)^{S_n} = w_{S_n} \cdot \chi_{E \otimes \cdots \otimes E} = \frac{D^m}{D^m(\det^m)} (\chi_E^{nm}) = \frac{D^m}{D^m(\det^m)} \left((x_1 + x_2 + \cdots + x_{nm})^m\right) = \frac{D^m}{D^m(\det^m)} \left(x_1^m \cdots x_{nm}^m \cdot \frac{(mn)!}{m!^n}\right) = (mn)!.
\]

**Invariant integral of \(GL_n\).**

Let \(A_{GL_n} = k[x_{11}, \ldots, x_{nn}, \frac{1}{\det(x_{ij})}]\) be the ring of functions of the linear group. One has that \((A_{GL_n}^{*})^{GL_n} = k \cdot w_{GL_n}\) and \((A_{GL_n}^{*})^{G_m} = ((A_{GL_n}^{*})^{G_m})^{S_n}\), where \(G_m\) is the multiplicative group. We shall first compute \((A_{GL_n}^{*})^{G_m} = (A_{GL_n}^{*})^{*}\) and then we shall look for the \(S_n\)-invariant ones among them.

\(A_{GL_n}\) is a \(\mathbb{Z}\)-graded algebra, whose \(i\)-th degree component we denote by \(A_i\). Given \(\lambda \in G_m\) and \(a_i \in A_i\), \(\lambda \cdot a_i = \lambda^i \cdot a_i\). Therefore, \(A_{GL_n}^{G_m} = A_0\) and \(w \in (A_{GL_n}^{*})^{G_m}\) if and only if it factors through the obvious quotient \(A_{GL_n} \to A_0\). This quotient morphism is a morphism of \(GL_n\)-modules. Now we must compute the linear forms \(w : A_0 \to k\) that are \(S_n\)-invariant. One has

\[
A_0 = \bigcup_{r \in \mathbb{N}} A^r_{\det^r(x_{ij})}, \quad A^r := \{\text{homogeneous polynomials of } n \cdot r\text{-th degree}\}.
\]

The morphism \(w_r = w \circ \det^{-r}(x_{ij}) : A^r \to k\) is a morphism of \(S_n\)-modules, i.e., it is \(S_n\)-invariant. Since \(w_r \in (A^r)^{*} = (S^r \text{End}_{k}(E)^{*} = S^r \text{End}_{k}(E) \subset \prod S^r \text{End}_{k}(E) = k[x_{11}, \ldots, x_{nn}]^r\), and it is \(S_n\)-invariant, then it must be \(w_r = \alpha_r \cdot D^r\). If we ask for \(w(1) = 1\), then it must be \(1 = w_r(\det^r(x_{ij})) = \alpha_r \cdot D^r(\det^r(x_{ij}))\), because \(1 = \frac{\det^r(x_{ij})}{\det^r(x_{ij})} \in A_{GL_n}^{G_m}\). Consequently, \(\alpha_r = \frac{1}{D^r(\det^r(x_{ij}))}\) and

\[
w_r = \frac{D^r}{D^r(\det^r(x_{ij}))} = \frac{D^r}{\mu_1 \cdots \mu_t}.
\]

In conclusion, we have determined\(^1\) the invariant integral \(w_{GL_n}\) on \(GL_n\) as a linear form over \(A_{GL_n}\):

\[
w_{GL_n} \left( \frac{p(x_{ij})}{\det^r(x_{ij})} \right) = w_{GL_n} \left( \frac{\cdots + p_{n,s}(x_{ij}) + \cdots}{\det^r(x_{ij})} \right) = w_{s}(p_{n,s}(x_{ij})) = \frac{D^r(p_{n,s}(x_{ij}))}{D^r(\det^r(x_{ij}))}.
\]

**Invariant integral of \(O_n\).**

Let \(T_2\) be a non singular symmetric metric on a vector space \(E\) of dimension \(n\). Let \(O_n\) be the subgroup of the linear group of the symmetries of \(T_2\). In the algebraic

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\(^1\)Marcel Bokstedt checks in “Notes on Geometric Invariant Theory” (available at http://home.imf.au.dk/marcel/GIT/GIT.ps) that the integral thus defined is the Reynolds operator of the linear group, and he states that Cayley, in a sense, had already checked it.
variety $S^2E^*$ of the symmetric metrics, regardless of the basis of $E$ chosen, we can define (up to a constant multiplicative factor) the function $\text{det}$ that assigns to each metric its determinant. So, we can consider the open set $S^2E^* - (\text{det})_0$. The sequence of morphisms of varieties

$$1 \rightarrow O_n \rightarrow \text{Gl}(E) \rightarrow S^2E^* - (\text{det})_0 \rightarrow 1$$

shows that $S^2E^* - (\text{det})_0$ is the quotient variety of $\text{Gl}(E)$ by the orthogonal subgroup $O_n$ ($O_n$ acting on $\text{Gl}(E)$ by the left). Fixing a basis in $E$, we shall say that $k[x_{11}, \ldots, x_{nn}, \frac{1}{\text{det}(x_{ij})}]$ is the ring of functions of $\text{Gl}(E)$ and $k[y_{ij}, \frac{1}{\text{det}(y_{ij})}]$ is the ring of functions of $S^2E^* - (\text{det})_0$. One has the induced morphism of rings

$$k[y_{ij}, \frac{1}{\text{det}(y_{ij})}] \hookrightarrow k[x_{11}, \ldots, x_{nn}, \frac{1}{\text{det}(x_{ij})}]$$

$$y_{rs} \mapsto [(x_{ij})^T \circ T_2 \circ (x_{ij})]_{rs}$$

$$\text{det}(y_{ij}) \mapsto \text{det}(x_{ij})^2 \cdot \text{det} T_2.$$

The functions of $\text{Gl}(E)$ invariant by $O_n$ are identified with the functions of $S^2E^* - (\text{det})_0$. Therefore, via the morphism of varieties $\text{End}_k(E) \rightarrow S^2E^*$, $S \mapsto S^t \circ T_2 \circ S$, the functions of $S^2E^*$ are identified with the functions of $\text{End}_k(E)$ that are (right) invariant by $O_n$.

Let us express these equations without fixing basis. We have defined the morphism

$$\text{End}_k(E) = E^* \otimes E \rightarrow S^2E^*$$

$$w \otimes e \mapsto C^{1,2}_{2,3}(w \otimes e \otimes T_2 \otimes e \otimes w) = T_2(e, e) \cdot w \otimes w$$

that induces a morphism between the rings of functions $S^m(S^2E) \rightarrow S^m(\text{End}_k(E)^*)$, that is expressed explicitly as follows

$$S^m(S^2E) \rightarrow S^{2m}(\text{End}_k(E)^*)$$

$$s_1 \cdot \ldots \cdot s_m \mapsto T_2 \otimes \ldots \otimes T_2 \otimes s_1 \otimes \ldots \otimes s_m$$

(we think of $S^{2m}(\text{End}_k(E)^*)$ as a quotient of $(E^* \otimes E) \otimes \ldots \otimes (E^* \otimes E) = E^* \otimes \ldots \otimes E^* \otimes E \otimes \ldots \otimes E$). Equivalently, the left $O_n$-invariant functions of the variety $\text{End}_k(E)$ are the direct sum of the images of the morphisms

$$S^m(S^2E^*) \rightarrow S^{2m}(\text{End}_k(E)^*)$$

$$\omega_1 \cdot \ldots \cdot \omega_m \mapsto \omega_1 \otimes \ldots \otimes \omega_m \otimes T_2 \otimes \ldots \otimes T_2$$

(we think of $S^{2m}(\text{End}_k(E)^*)$ as a quotient of $E^{*2m} \otimes E^{2m}$). Therefore, the invariants of $S^{2m}(\text{End}_k(E)^*)$ by the action of $O_n$ by the left and by the right are
One has divisible by Two metrics are isometric (with regard to each metric \( T \)).

Proposition 3.3. Let \( A_{O_n} \) be the ring of functions of \( O_n \) and \( w_{O_n} \in A_{O_n}^* \subset A_{S_n}^* = \prod_r S^r \text{End}_k(E) \) the invariant integral on \( O_n \). The \( r \)-th component \([w_{O_n}]_r \) of \( w_{O_n} \) is

\[
[w_{O_n}]_r = \sum_{\sigma \in S_m} \lambda_{\sigma} \cdot (T_2 \otimes \ldots \otimes T_2) \otimes \sigma(T^2 \otimes \ldots \otimes T^2) \quad \text{if } r = 2m,
\]

\[
[w_{O_n}]_r = 0 \quad \text{if } r = 2m + 1.
\]

Proof.

1. \( w_{O_n} \cdot (E \otimes 2m+1 \otimes E) = [w_{O_n}]_{2m+1} \cdot E^{\otimes 2m+1} = 0 \cdot E^{\otimes 2m+1} = 0. \)

2. One has \([w_{O_n}]_{2m} = \sum_{\sigma \in S_{2m}} \lambda_{\sigma} \cdot (T_2 \otimes \ldots \otimes T_2) \otimes \sigma(T^2 \otimes \ldots \otimes T^2) \) and

\[
\frac{1}{(2m)!} \sum_{\sigma' \in S_{2m}} \sigma'(T_2 \otimes \ldots \otimes T_2) \otimes \sigma'(T^2 \otimes \ldots \otimes T^2)
\]

via the inclusion \( S^{2m} \text{End}_k(E) \subset E^{\otimes 2m} \otimes E^{\otimes 2m} \). Moreover, \( E^{\otimes 2m} \otimes E^{\otimes 2m} \) acts on \( E^{\otimes 2m} \) by contracting each linear form with the corresponding vector. Therefore,

\[
(E^{\otimes 2m})^{O_n} = w_{O_n} \cdot E^{\otimes 2m} = [w_{O_n}]_{2m} \cdot E^{\otimes 2m} \subseteq \langle \sigma(T^2 \otimes \ldots \otimes T^2) \rangle_{\sigma \in S_{2m}}.
\]

The inverse inclusion is obvious.

Let us consider the morphism \( S^2 E \leftrightarrow \text{End}_k(E), T^2 \mapsto T^2 \circ T_2 \), that assigns to each metric \( T^2 \) the associated endomorphism to the pair of metrics \( T^2, T^2 \).

Two metrics are isometric (with regard to \( T_2 \)) if and only if their associated endomorphisms are equivalent, and every endomorphism (up to conjugations) is the associated endomorphism of a symmetric metric and \( T_2 \) (\( [E] \)). As a result one has that the invariant functions of \( \text{End}_k(E) \) (by the action by conjugation of the linear group) are invariant functions of \( S^2 E \) by the orthogonal group. Conversely, let us see that \( A_{S^2E}^{O_n} \subseteq A_{\text{End}_k(E)}^{Gl_n} \). Let \( d(\lambda_{ij}) \) be the discriminant of the characteristic polynomial of the matrix \( \lambda_{ij} \) and let \( U = \text{End}_k(E) - (d)_0 \) be the open subset of \( \text{End}_k(E) \) of the diagonalizable endomorphisms with different eigenvalues. It is clear that \( (U \cap S^2 E)/O_n = U/\text{Gl}_n = \text{Spec} k[a_1, \ldots, a_n]_d \), where \( a_s(\lambda_{ij}) \) are the coefficients of the characteristic polynomial of the matrix \( \lambda_{ij} \). Given \( f \in A_{S^2E}^{O_n} \), one has that \( f = p(a_1, \ldots, a_n)/d' \), where we can assume that \( p(a_1, \ldots, a_n) \) is not divisible by \( d \). However, if \( r > 0 \), then \( p(a_1, \ldots, a_n) \) must vanish on all the diagonal matrices with repeated eigenvalues, then \( p(a_1, \ldots, a_n) \) is a multiple of \( d \), which is impossible. Therefore, \( f = p(a_1, \ldots, a_n) \in A_{\text{End}_k(E)}^{Gl_n} \).

Let \( f \) be the composite morphism \( \text{End}_k(E) \rightarrow S^2 E \rightarrow \text{End}_k(E), T \mapsto TT^2 T' \mapsto TT^2 T'T_2 \). The invariant functions of \( \text{End}_k(E) \) by the action by conjugation of the
linear group coincide, via $f^*$, with the functions of $\text{End}_k(E)$ that are left and right invariant by the action of the orthogonal group. Since $k[S_m] = (\langle \text{End}_k(E) \rangle)^{\otimes m} \cdot \text{Gl}_n$, if we denote $\sigma \mapsto \tilde{\sigma}$, where $\tilde{\sigma}(e_1 \otimes \ldots \otimes e_m) = e_{\sigma(1)} \otimes \ldots \otimes e_{\sigma(m)}$, then $(S^m \cdot \text{End}_k(E))^{\otimes m} \cdot \text{Gl}_n = \langle \tilde{\sigma} \rangle_{\sigma \in S_m} = Z(k[S_m])$. Now, $(\langle \text{End}_k(E) \rangle)^* = \text{End}_k E$ (via $\phi|_{\langle \text{End}_k(E) \rangle}^*$, or via the obvious isomorphism $E \otimes E^* = E^* \otimes E$), and one has in the same way that $((S^m \cdot \text{End}_k(E))^{*})^{\otimes m} \cdot \text{Gl}_n = \langle \tilde{\sigma} \rangle_{\sigma \in S_m}$. Finally,
\[ \tilde{\sigma} f^* \mapsto \sum_{m \geq 0} T^{(1)}_2 \otimes \ldots \otimes T^{(m)}_2 \rightarrow \sum_{m \geq 0} T^{(1)}_2 \otimes \ldots \otimes T^{(m)}_2 \rightarrow \sum_{m \geq 0} T^{(1)}_2 \otimes \ldots \otimes T^{(m)}_2 =: a_{\sigma} \]
where $T^{(1)}_2 \otimes \ldots \otimes T^{(m)}_2 \rightarrow \tau(T^2 \otimes \ldots \otimes T^2)$ and $\tau \in S_{2m}$ is the permutation $\tau(2i - 1) = 2i - 1$ and $\tau(2i) = 2\sigma(i)$.

So we have calculated the forms $\tilde{w} \in A^*_{M_n}$ that are left and right $O_n$-invariant. To compute the invariant integral $w_{O_n}$ on $O_n$ it remains to impose that $\tilde{w}(I) = 0$, where $I$ is the ideal of functions of $M_n$ vanishing on $O_n$. Now, since $w_{O_n} \cdot \tilde{w} \cdot w_{O_n} = \tilde{w}$, one has that $\tilde{w}(I) = \tilde{w}(w_{O_n} \cdot I \cdot w_{O_n})$ and $w_{O_n} \cdot I \cdot w_{O_n}$ are the functions of $M_n$ that are left and right $O_n$-invariant and vanish on $O_n$. These ones are identified, via $f^*$, with the ideal $I'$ of the functions of $M_n$ that are invariant by the action by conjugation of the linear group and vanish on $I d \in M_n$.

One has $A^*_{M_n} = \oplus_{m \in \mathbb{C}} S^{m} \otimes \ldots \otimes S^{m} \cdot \otimes I d$. If $\sigma$ is a product of $r$ disjoint cycles (including the cycles of order 1), it is easy to check that $\tilde{\sigma}(I d) = n^r$. Let us denote $w_{\sigma} := \sum_{m \geq 0} \sum_{[\sigma] \in S_m/\sim} \lambda_{[\sigma]} \cdot w_{\sigma} \cdot \tilde{\sigma}(I d)$ satisfying $w_{O_n}(I) = 0$, we have to solve, for every $m$, the system of equations (varying $[\sigma'] \in S_m/\sim$)
\[
\left( \sum_{[\sigma] \in S_m/\sim} \lambda_{[\sigma]} \cdot w_{\sigma} \cdot \tilde{\sigma}(I d) \right) \left( \frac{a_{\sigma'}}{\tilde{\sigma}(I d)} \right) = 1.
\]
If we denote $\lambda_{\sigma \sigma'} = \frac{w_{\sigma}(a_{\sigma'})}{\tilde{\sigma}(I d)} \cdot \tilde{\sigma}(I d)$, then
\[
(\lambda_{[\sigma]}[\sigma] \in S_m/\sim) = \left( \lambda_{\sigma \sigma'}[\sigma], [\sigma'] \in S_m/\sim \right) \cdot (1, \ldots, 1).
\]
Let us give the first three terms of $w_{O_n}$:
\[
w_{O_n} = 1 + \sum_{m \geq 0} \sum_{[\sigma] \in S_m/\sim} \lambda_{[\sigma]} \cdot w_{\sigma} \cdot \tilde{\sigma}(I d) \left( \frac{a_{\sigma'}}{\tilde{\sigma}(I d)} \right)
\]
\[
+ \sum_{m \geq 0} \sum_{[\sigma] \in S_m/\sim} \lambda_{[\sigma]} \cdot w_{\sigma} \cdot \tilde{\sigma}(I d) \left( \frac{a_{\sigma'}}{\tilde{\sigma}(I d)} \right)
\]
\[
+ \ldots.
\]

**Invariant integral of $Sp_{2n}$.**

Let $H_2$ be a non-singular skew-symmetric metric on a vector space $E$ of dimension $2n$. Let $Sp_{2n}$ be the subgroup of the linear group of the symmetries of $H_2$. In the algebraic variety $\Lambda^3 E^* \otimes E^*$ of the skew-symmetric metrics, regardless of the basis of $E$ chosen, we can define (up to a constant multiplicative factor) the function
det that assigns to each metric its determinant. So, we can consider the open set \( \Lambda^2 E^* - (\det)_0 \). The sequence of morphisms of varieties

\[
1 \to Sp_{2n} \to GL(E) \to \Lambda^2 E^* - (\det)_0 \to 1
\]

shows that \( \Lambda^2 E^* - (\det)_0 \) is the quotient variety of \( GL(E) \) by the simplectic subgroup \( Sp_{2n} \) (acting on \( GL(E) \) by the left).

The invariant functions of \( GL(E) \) by \( Sp_{2n} \) are identified with the functions of \( \Lambda^2 E^* - (\det)_0 \). Therefore, via the morphism of varieties \( End_k(E) \to \Lambda^2 E^* \), \( S \mapsto S' \circ H_2 \circ S \), the functions of \( \Lambda^2 E^* \) are identified with the functions of \( End_k(E) \) that are (right) invariant by \( Sp_{2n} \). The morphism between the rings of functions \( S(\Lambda^2 E) \to S(End_k(E)^*) \) is expressed explicitly as follows

\[
S^m(\Lambda^2 E) \to S^{2m}(End_k(E)^*)
\]

Equivalently, the left \( Sp_{2n} \)-invariant functions of the variety \( End_k(E) \) are the direct sum of the images of the morphisms

\[
S^m(\Lambda^2 E^*) \to S^{2m}(End_k(E)^*)
\]

(we think of \( S^{2m}(End_k(E)^*) \) as a quotient of \( E^* \otimes E^{2m} \)). Therefore, the invariants of \( S^{2m}(End_k(E)^*) \) by the action of \( Sp_{2n} \) by the left and by the right are

\[
(\sigma(H_2 \otimes \ldots \otimes H_2) \otimes \sigma'(H^2 \otimes \ldots \otimes H^2))_{\sigma, \sigma' \in S_{2m}}
\]

Let \( A_{Sp_{2n}} \) be the ring of functions of \( Sp_{2n} \) and \( w_{Sp_{2n}} \in A_{Sp_{2n}}^* \subset A_{M_{2n}}^* \), namely the invariant integral on \( Sp_{2n} \). The \( r \)-th component \( [w_{Sp_{2n}}]_r \) of \( w_{Sp_{2n}} \) is

\[
[w_{Sp_{2n}}]_r = \sum_{\sigma \in S_{2m}} \lambda_\sigma \cdot (H_2 \otimes \ldots \otimes H_2) \otimes \sigma(H^2 \otimes \ldots \otimes H^2) \quad \text{if } r = 2m,
\]

\[
[w_{Sp_{2n}}]_r = 0 \quad \text{if } r = 2m + 1.
\]

**Proposition 3.4.** ([G, Th. 4.3.3]) Let us consider the natural action of \( Sp_{2n} \) on \( E \otimes \ldots \otimes E \), \( g \cdot (e_1 \otimes \ldots \otimes e_r) = g \cdot e_1 \otimes \ldots \otimes g \cdot e_r \). One has that:

1. \( (E \otimes 2m+1 \otimes E)^{Sp_{2n}} = 0 \).
2. \( (E \otimes 2m \otimes E)^{Sp_{2n}} = (\sigma(H^2 \otimes \ldots \otimes H^2))_{\sigma \in S_{2m}} \).

Let us consider the morphism \( \Lambda^2 E \to End_k(E) \), \( H^2 \mapsto H^2 \circ H_2 \), that assigns to each metric \( H^2 \) the associated endomorphism to the pair of metrics \( H^2, H^2 \). Two skew-symmetric metrics are isometric (with regard to \( H_2 \)) if and only if their associated endomorphisms are equivalent, and an endomorphism \( T \) (up to conjugations) is the associated endomorphism of a skew-symmetric metric and \( H_2 \) if and
only if every elementary divisor of \( T \) appears twice ([E]). Let \( C \subset \text{End}_k(E) \) the closed set of such endomorphisms. Then \( \Lambda^2 E/Sp_{2n} = C/\text{Gl}(E) \). Let us write

\[
E = E' \oplus E', \quad H_2 = \begin{pmatrix} 0 & -I_d \\ I_d & 0 \end{pmatrix}
\]

The diagram

\[
\begin{array}{ccc}
\text{End}_k E' & \hookrightarrow & \Lambda^2 E \\
T & \mapsto & \begin{pmatrix} 0 & T \\ -T & 0 \end{pmatrix} & \mapsto & \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}
\end{array}
\]

shows that \( \text{End}_k E'/\text{Gl}(E') = \Lambda^2 E/\text{Sp}_{2n} = C/\text{Gl}(E) \). The ring of invariant functions of \( \text{End}_k(E') \) (by the action by conjugation of the linear group \( \text{Gl}(E') \)) is isomorphic to the ring of invariant functions of \( \text{End}_k(E) \) (by the action by conjugation of the linear group \( \text{Gl}(E) \)), then the ring of invariant functions of \( \Lambda^2 E \) (by the action of \( \text{Sp}_{2n} \)) is isomorphic to the ring of invariant functions of \( \text{End}_k(E) \) (by the action by conjugation of the linear group \( \text{Gl}(E) \)).

Let \( f \) be the composite morphism \( \text{End}_k(E) \to \Lambda^2 E \to \text{End}_k(E), T \mapsto TH^2T' \mapsto THT^2T'H_2 \). The invariant functions of \( \text{End}_k(E) \) by the action of conjugation of the linear group coincide, via \( f^* \), with the functions of \( \text{End}_k(E) \) that are left and right invariant by the action of the symplectic group. Now we can calculate the invariant integral of \( \text{Sp}_{2n} \) as we have calculated the invariant integral of \( O_n \).

Let us denote \( w_\sigma := H_2 \otimes \ldots \otimes H_2 \otimes H^2_{\alpha(1)} \otimes \ldots \otimes H^2_{\alpha(m)} \in \text{S}^{2m} \text{End}_k(E) \)

and \( a_\sigma := \frac{H_2 \otimes \ldots \otimes H_2 \otimes H^2_{\alpha(1)} \otimes \ldots \otimes H^2_{\alpha(m)} \in \text{S}^{2m} \text{End}_k(E)^*} {\lambda_{\alpha(1)} \cdots \lambda_{\alpha(m)}} \). Let \( \lambda_{\alpha(1)} \cdots \lambda_{\alpha(m)} = (\lambda_{\alpha(1)})^{-1} \cdots (\lambda_{\alpha(m)})^{-1} \). Then, \( w_{\text{Sp}_{2n}} = 1 + \sum m > 0 \sum \left| \sigma \right| \in S_m/\sim \lambda_{\sigma} \cdot w_\sigma \cdot \frac{\sigma}{\sigma(I_d)} \)

Let us give the first three terms of \( w_{\text{Sp}_{2n}}, m = 2n \):

\[
w_{\text{Sp}_{2n}} = 1 + \frac{H^2_2 \otimes H^2_2}{m^2 - m + 3} \cdot \frac{H_2 \otimes H_2 \otimes H^2_2 \otimes H^2_2 + (3m - 6) \cdot H_2 \otimes H_2 \otimes H^2_{12} \otimes H^2_{21}}{m^4 - m^3 + m^2 + 3m}
\]

\[+ \ldots \]

REFERENCES


GEOMETRIC CALCULATION OF THE INVARIANT INTEGRAL OF CLASSIC GROUPS

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