HOMOGENEOUS HILBERT SCHEME

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Abstract. Let S be a locally noetherian scheme and R an $\mathbb{N}$-graded $\mathcal{O}_S$-algebra of finite type. We say that $X = \operatorname{Spec} R$ is a homogeneous variety over S. In this paper we prove that the functor

$$\underline{\mathrm{HomHilb}}_{X/S} \colon \begin{bmatrix} \text{Locally noetherian} \\ S\text{-schemes} \end{bmatrix} \rightsquigarrow \text{Sets}$$

$$T \rightsquigarrow \begin{array}{l} \text{closed subschemes of } X \times_S T \\ \text{flat and homogeneous over } T \end{array}$$

is representable by an S-scheme which is a disjoint union of locally projective schemes over S. The proof is very simple and it only makes use of the theory of graded modules and standard flatness criteria. From this, one obtains an elementary construction (which does not make use of cohomology) of the ordinary Hilbert scheme of a locally projective S-scheme.

Introduction

Let S be a locally noetherian scheme and R an $\mathbb{N}$-graded $\mathcal{O}_S$-algebra of finite type. We say that $X = \operatorname{Spec} R$ is a homogeneous variety over S. In this paper we prove that the functor

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is representable by an S-scheme which is a disjoint union of locally projective schemes over S (Theorem 1.9). This scheme is called "Homogeneous Hilbert scheme of X". The proof is very simple and it only makes use of the theory of graded modules and standard flatness criteria.

Let $\mathbb{P}(X) = \operatorname{Proj} R$. From the construction of the Homogeneous Hilbert scheme of X one obtains easily the construction of the Hilbert scheme of $\mathbb{P}(X)$. This construction is very simple and does not make use of cohomology.

Grothendieck introduced the Hilbert scheme of a projective variety in [G]. The central point of the standard constructions of the Hilbert scheme is based on the following version of Serre's Vanishing Theorem:

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Theorem 1: Let $p(n) \in \mathbb{Q}[n]$. There exists $r \in \mathbb{N}$ such that for every closed subscheme $C \subset \mathbb{P}^m$ of Hilbert polynomial $p(n)$,

(1) $H^i(\mathbb{P}^m, \mathcal{O}_C(n)) = 0$ and $H^i(\mathbb{P}^m, \mathfrak{p}_C(n)) = 0$, for all $i > 0$ and $n \geq r$.

(2) The multiplicative morphisms

$$H^0(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(n)) \times H^0(\mathbb{P}^m, \mathcal{O}_C(r)) \to H^0(\mathbb{P}^m, \mathcal{O}_C(n+r))$$

$$H^0(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(n)) \times H^0(\mathbb{P}^m, \mathfrak{p}_C(r)) \to H^0(\mathbb{P}^m, \mathfrak{p}_C(n+r))$$

are surjective, for all $n \geq 0$.

(3) $\mathfrak{p}_C(r)$ is generated by its global sections.

This theorem is proved with the aid of m-regular sheaves [Mu], or in an elaborate proof in [V] or [K]. However, it is easily proved as a consequence of the standard Serre's and Grauert's vanishing theorems, once one has constructed the Hilbert scheme as we have done.

In this paper the main theorem (1.5) is:

Theorem 2: Let $p(n) \in \mathbb{Q}[n]$. There exists $r \in \mathbb{N}$ such that, for every homogeneous ideal $I \subseteq k[x_0, \ldots, x_m]$ with Samuel polynomial $p(n)$, one has

(1) I is generated by its homogeneous components of degree $\leq r$.

(2) $\dim_k[k[x_0, \ldots, x_m]/I + \mathfrak{m}^n] = p(n)$, for all $n \geq r$, where $\mathfrak{m} = (x_0, \ldots, x_m)$.

This theorem is easily proved by induction on the degree of $p(n)$ and it does not make use of cohomology.

The construction of the Homogeneous Hilbert scheme of X is reduced, by standard arguments, to the case $X = \mathbb{A}^m$. Now, the construction of the Homogeneous Hilbert scheme of $\mathbb{A}^m$ (with Hilbert polynomial $p(n)$) is done in two ways. First, we construct it as the inverse limit of the "homogeneous" Hilbert scheme of $\operatorname{Spec} k[x_1, \ldots, x_m]/(x_1, \ldots, x_m)^{n+1}$ (this limit stabilizes for $n >> 0$, hence one obtains the *equations* of the Homogeneous Hilbert scheme). The second construction is as follows: let us denote $k[x_1, \ldots, x_m]_i$ the homogeneous polynomials of degree i. Then we construct the Homogeneous Hilbert scheme (with Hilbert polynomial $p(n)$) as the Grassmannian of subvector bundles of $\overset{r}{\underset{i=0}{\oplus}} k[x_1, \ldots, x_m]_i$ of codimension $p(r)$, which vanish on a homogeneous closed subscheme of $\mathbb{A}^m$ with Samuel function $p(n)$. For this, we use the Grothendieck's "flatenning stratification theorem" ([Mu] lecture 8). In fact, we give a slight generalization of this theorem (2.2), which is proved immediately by the only use of the theory of graded modules and elementary flatness criteria.

1. Homogeneous Hilbert Scheme and Hilbert Scheme

1.1. **The Main Theorem.**

Let A be a commutative ring (associative and with unit), $R = A[x_0, \ldots, x_m]$, $(x_0, \ldots, x_m)$ the irrelevant ideal and $\mathbb{P}^m_A = \operatorname{Proj} R$ the projective space of dimension m over A. Given a graded R-module M we denote by M_n the A-submodule of the homogeneous elements of degree n of M. Given $f \in R$, M_f denotes the localization of M with respect to the multiplicative set $\{1, f, f^2, \ldots\}$.

Each homogeneous ideal $I \subset R$ defines the closed subscheme $C_I = \operatorname{Proj} R/I \hookrightarrow \mathbb{P}^m_A$. Two homogeneous ideals I and I' define the same closed subscheme if and only if $I_n = I'_n$ for all $n \geq m$, for some m. Two homogeneous ideals I and I' define the same closed subscheme if and only if $I_{x_i} = I'_{x_i}$ for all i. We say that two homogeneous ideals defining the same closed scheme are equivalent. We say that a homogenous ideal I is saturated if it is the biggest of all the homogeneous ideals equivalent to I. This ideal coincides with the homogeneous ideal of all the polynomials vanishing on C_I. The quotient R/I is called the ring of homogeneous polynomials of C_I. Finally, we say that a homogeneous ideal I is proper if $C_I \neq \emptyset$.

¿From now on and until the end of this subsection, $A = k$ is a field.

Proposition 1.1. *A proper homogeneous ideal $I \subset k[x_0, \dots, x_m]$ is saturated if and only if the irrelevant ideal is not an associated prime of I.*

Proof. Let $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n \cap \mathfrak{q}$ be a shortest primary decomposition, where $\mathfrak{q}_i$ are homogeneous ideals and the radical of $\mathfrak{q}$ is $(x_0, \dots, x_n)$. I' is equivalent to I if and only if either $I' = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n \cap \mathfrak{q}'$, where the radical of $\mathfrak{q}'$ is $(x_0, \dots, x_n)$ or $I' = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$, because I is equivalent to I' if and only if $I_{x_i} = I'_{x_i}$, for all i. Hence, the saturated ideal which is equivalent to I is $\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$. □

Primary decompositions localize. Therefore, we can define the primary components (either isolated or embedded) of a noetherian scheme.

Given a saturated homogeneous ideal I and a shortest homogeneous primary decomposition $I = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_n$, we have $[I_{x_i}]_0 = [\mathfrak{q}_{1x_i}]_0 \cap \ldots \cap [\mathfrak{q}_{nx_i}]_0$, for all i. Therefore, the primary components of $C_I = \operatorname{Proj} R/I$ are equal to the projectivization of the primary components of the "cone" $\operatorname{Spec} R/I$.

Definition 1.2. *Let $I \subset R$ be a saturated homogeneous ideal. Given a homogeneous polynomial $f \in R$, we say that $H_f = \operatorname{Proj} R/(f)$ meets C_I transversally if f is not a zero-divisor in R/I (i.e., no primary component of C_I lays on H_f).*

If k is an infinite set, then there exists a hyperplane which meets C_I transversally.

Definition 1.3. *Let $I \subset R$ be a homogeneous ideal. The function $p(n) = \dim_k [R/I]_n$ (resp. $\sum_{i=0}^{n} \dim_k [R/I]_i$) is called the Hilbert function (respectively, the Samuel function) of I. If $C \subset \mathbb{P}_n$ is a projective subvariety, we shall call Hilbert function (respectively, Samuel function) of C to the Hilbert function (resp. Samuel function) of the saturated homogeneous ideal defining C.*

The Hilbert and Samuel functions are equal to certain polynomials for $n >> 0$. These polynomials are called Hilbert and Samuel polynomials.

The degree of the Hilbert polynomial of I coincides with the dimension of the projective variety C_I. All the equivalent homogeneous ideals have the same Hilbert polynomial.

Definition 1.4. *We say that the Samuel (resp. Hilbert) function of a homogeneous ideal is a polynomial from m, if the Samuel (resp. Hilbert) function coincides with the Samuel (resp. Hilbert) polynomial for all $n \geq m$.*

The Samuel function is a polynomial from m if and only if the Hilbert function is a polynomial from m.

Theorem 1.5. *Let $p(n) \in \mathbb{Q}[n]$. There exists $r \in \mathbb{N}$ such that, for every homogeneous ideal $I \subset k[x_0, \dots, x_n] = R$ with Samuel polynomial $p(n)$ one has:*

1. *I is generated by its homogeneous elements of degree $\leq r$. In other words, every homogeneous function of I of degree $m > r$ is an R-linear combination of homogeneous functions of I of degree r.*

(2) *The Samuel function of I is a polynomial from r.*

Proof. Under base change of k, we can assume k to be algebraically closed. We proceed by induction on $s =$ degree of $p(n)$. If $s = 0$, then $[R/I]_i = 0$ for some $i \leq p(0)$ and it suffices to take $r = p(0)$.

Let $s \geq 1$ and let us assume that the theorem holds for any polynomial $p(n)$ of degree $\leq s - 1$.

Let I' be the saturated homogeneous ideal which is equivalent to I. The Samuel polynomial of I' is $p(n) - a$, with $a \in \mathbb{N}$, and its Hilbert polynomial is $p(n) - p(n-1)$.

Let H_f be a hyperplane meeting $C_{I'}$ transversally. The rows and columns of the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I' & \xrightarrow{f\cdot} & I' & \longrightarrow & I'/fI' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R & \xrightarrow{f\cdot} & R & \longrightarrow & R/(f) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R/I' & \xrightarrow{f\cdot} & R/I' & \longrightarrow & R/(I', f) & \longrightarrow & 0
\end{array}$$

are exact (the third column is exact by the snake lemma).

($*$) From the third row we obtain that the Samuel polynomial of (I', f) is the Hilbert polynomial of I'. By induction, there exists r' such that (I', f) is generated by its homogeneous elements of degree $\leq r'$ and its Samuel function is a polynomial from r'. Therefore, the Samuel function of I' is a polynomial from r'. Moreover, I'/fI' is generated by its homogeneous elements of degree $\leq r'$. Applying Nakayama's Lemma for graded modules to the first row we obtain that I' is generated by its homogeneous elements of degree $\leq r'$.

Notice that $0 \leq p(r') - a$, so $0 \leq a \leq p(r')$. Since $\dim_k I'/I = a$, $[I]_{r'+i} = [I']_{r'+i}$ for some $0 \leq i \leq p(r')$. Hence, $[I]_{r'+p(r')} = [I']_{r'+p(r')}$, so I is generated by its homogeneous elements of degree $\leq r = r' + p(r')$ (determined by $p(n)$) and its Samuel function is a polynomial from $r = r' + p(r')$.

□

Proposition 1.6. *Let $p(n) \in \mathbb{Q}[n]$. There exists $r \in \mathbb{N}$ such that, for every saturated homogenous ideal $I' \subset k[x_0, \dots, x_m]$ with Hilbert polynomial $p(n)$ one has*

(1) *I' is generated by its homogeneous elements of degree $\leq r$.*
(2) *The Hilbert function of I' is a polynomial from r.*

Proof. Argue as in ($*$) (in the proof 1.5) replacing the sentence "by induction" by the sentence "by 1.5". □

Remark 1.7. *The r in 1.5 coincides with the one in 1.6, and the r assigned to $p(n)$ is bigger than the one assigned to $p(n) - p(n-1)$.*

1.2. **Homogeneous Hilbert Scheme.**

Definition 1.8. *If S is a scheme and R is an $\mathbb{N}$-graded $\mathcal{O}_S$-algebra of finite type, we say that* $\operatorname{Spec} R$ *is a homogeneous variety over S.*

Remark: Let $X = \operatorname{Spec} R$ be a homogeneous affine variety over $\operatorname{Spec} A$. Assume that R is A-flat and $\operatorname{Spec} A$ is connected. Then, R_n is a locally free finite A-module and the map

$$\operatorname{Spec} A \to \mathbb{N}$$
$$x \mapsto \dim_{k(x)}[R_n \otimes_A k(x)]$$

is constant. Hence it makes sense to talk of the Hilbert and Samuel functions and polynomials of X, and it coincides with the Hilbert and Samuel functions of the fiber of any point $x \in \operatorname{Spec} A$.

Theorem 1.9. *Let S be a locally noetherian scheme and X a homogeneous variety over S. The functor*

$$\underline{\mathrm{HomHilb}}_{X/S} \colon \left[\begin{array}{l}\textit{Locally noetherian}\\ S\textit{-schemes}\end{array}\right] \rightsquigarrow \textit{Sets}$$

$$T \rightsquigarrow \begin{array}{l}\textit{closed subschemes of } X \times_S T\\ \textit{which are flat and homogeneous over } T\end{array}$$

is representable by an S-scheme which is a disjoint union of locally projective schemes over S.

Proof. Since $\underline{\mathrm{HomHilb}}_{X/S}$ is a sheaf in the Zariski Topology, we can assume that $S = \operatorname{Spec} A$ is noetherian and connected, and $X = \operatorname{Spec} R$ (where R is an $\mathbb{N}$-graded A-álgebra of finite type). Then X is a homogeneous subvariety of an affine space $\mathbb{A}^{m+1} = \operatorname{Spec} A[x_0, \dots, x_m]$. Since $\underline{\mathrm{HomHilb}}_{X/S}$ is a closed subfunctor of $\underline{\mathrm{HomHilb}}_{\mathbb{A}^{m+1}/S}$, it suffices to show that $\underline{\mathrm{HomHilb}}_{\mathbb{A}^{m+1}/S}$ is representable.

Now, $\underline{\mathrm{HomHilb}}_{\mathbb{A}^{m+1}/S} = \coprod_{\varphi(n)} \underline{\mathrm{HomHilb}}^{\varphi(n)}_{\mathbb{A}^{m+1}/S}$, where $\underline{\mathrm{HomHilb}}^{\varphi(n)}_{\mathbb{A}^{m+1}/S}$ is the functor of homogeneous closed subschemes of $\mathbb{A}^{m+1}$ with Samuel function $\varphi(n)$. Let us prove that $\underline{\mathrm{HomHilb}}^{\varphi(n)}_{\mathbb{A}^{m+1}/S}$ is representable. Let $r \in \mathbb{N}$ be as in Theorem 1.5 (where $p(n)$ is the Samuel polynomial defined by $\varphi(n)$). We give two proofs of the representability of $\underline{\mathrm{HomHilb}}^{\varphi(n)}_{\mathbb{A}^{m+1}/S}$:

First proof. Let $B^n := A[x_0, \dots, x_m]/(x_0, \dots, x_m)^{n+1}$ and $\phi(i) := \varphi(i) - \varphi(i-1)$. Let $G(B^n_i)$ be the Grassmannian of "subvector bundles of $[B^n]_i$ of codimension $\phi(i)$". Let $G'(B^n) = \prod_{i=0}^n G(B^n_i)$, which is a closed subscheme of the Grassmannian of B^n, $Grass(B^n)$. Let $G''(B^n)$ be the moduli of A-flat closed subschemes of $\operatorname{Spec} B^n$, which is a closed subscheme of $Grass(B^n)$. Finally, let $G(B^n) = G'(B^n) \cap G''(B^n)$ be "the Grassmannian of homogeneous ideals $I \subseteq B^n$ such that $\operatorname{rank}[B^n/I]_i = \phi(i)$, for all $i \leq n$".

Consider the sequence of natural epimorphisms $\cdots \to B^{n+1} \to B^n \to \cdots \to B^r$. Then we have an obvious sequence of proper morphisms

$$\cdots \to G(B^{n+1}) \to G(B^n) \to \cdots \to G(B^r)$$

whose inverse limit is the functor $\underline{\mathrm{HomHilb}}^{\varphi(n)}_{\mathbb{A}^{m+1}/S}$.

Let $C^i \subseteq G(B^i)$ be the intersection of the schematic images of the morphisms $G(B^n) \to G(B^i)$, for all $n \geq i$, which is the schematic image of a morphism $G(B^n) \to G(B^i)$ for $n >> 0$, by standard noetherian arguments.

The inverse limit of $\{G(B^n)\}_{n \in \mathbb{N}}$ is equal to the inverse limit of $\{C^n\}_{n \in \mathbb{N}}$.

$C^{i+1} \to C^i$ is a surjective map of sets. Each geometric point of C^i, with $i \geq r$, "lifts to a unique geometric point of the homogeneous Hilbert functor". In fact, given a geometric point of C^i, there exists a geometric point of C^{i+1} in the fiber. Again, there exists a geometric point of C^{i+2} in the fiber of that geometric point of C^{i+1}. Continuing in this fashion we obtain a geometric point of the homogeneous Hilbert functor, i.e., a homogeneous ideal $\underset{s\geq 0}{\oplus} I_s \subset k[x_0,\dots,x_m]$ that is generated by I_r (by Theorem 1.5).

Consequently, the geometric points of C^{i+1} coincide with the geometric points of C^i, for all $i \geq r$. The morphisms of functors $C^{i+1} \to C^i$ are injective, for all $i \geq r$, because the homogeneous elements of degree $i+1$ of an ideal are generated by the homogeneous elements of degree i, since this is true in fibers.

By Zariski's Main Theorem ([H] III. 11.5) $C^{i+1} = C^i$ for all $i \geq r$, and it coincides with the inverse limit.

Remark 1: For $n >> r$ we have an epimorphism $G(B^n) \to C^r$ and C^r is the schematic image of π. Again, this morphism (of functors) is injective, and $C^r = G(B^n)$. Then $\underline{\mathrm{HomHilb}}^{\varphi(n)}_{\mathbb{A}^{m+1}/S}$ is represented by $G(B^n)$.

Second proof. Let G_i be the Grassmannian of subvector bundles of $A[x_0,\dots,x_m]_i$ of codimension $\phi(i) := \varphi(i) - \varphi(i-1)$. Let $G = \prod_{i=0}^{r} G_i$, $\mathcal{O}_G$ the sheaf of rings of G and $I \subset \overset{r}{\underset{i=0}{\oplus}} \mathcal{O}_G[x_0,\dots,x_m]_i$ the universal subvector bundle defined by the morphism $\mathrm{Id}\colon G \to G$. Let $(I) \subset \mathcal{O}_G[x_0,\dots,x_m]$ be the sheaf of ideals generated by I. Let $R = \mathcal{O}_G[x_0,\dots,x_m]/(I)$ and $\phi(n) := \varphi(n) - \varphi(n-1)$. Then the functor F_ϕ of Theorem 2.2 coincides with $\underline{\mathrm{HomHilb}}^{\varphi(n)}_{\mathbb{A}^{m+1}/S}$ and one concludes by Theorem 2.2.

Remark 2: Using 3. of 2.2, we obtain again the above remark 1.

□

1.3. **Hilbert scheme.**

First of all, let us characterize the flat parameterizations of closed subschemes of $\mathbb{P}^m$.

Let A be a noetherian ring and C a closed subscheme of $\mathbb{P}^m_A$. Let B be the ring of homogeneous functions of C and $A \to A'$ a morphism of rings. The ring of homogeneous functions of $C_{A'} = C \times_A A' \subseteq \mathbb{P}^m_{A'}$ is a quotient of $B \otimes_A A'$, and it is equal to $B \otimes_A A'$ if $A \to A'$ is a flat morphism.

Given a morphism of schemes $X \to Y$ and a closed point $y = \operatorname{Spec} k \hookrightarrow Y$, we denote $X_y = X \times_Y y$ the fiber of y.

Proposition 1.10. *Let A be a noetherian local ring and $x \in \operatorname{Spec} A$ the closed point. Let $p(n) \in \mathbb{Q}[n]$ and r be as in proposition 1.6. Let $C \subseteq \mathbb{P}^m_A$ be a closed subscheme such that the Hilbert polynomial of C_x is $p(n)$. Then C is flat over* $\operatorname{Spec} A \iff$ *the A-module of the homogeneous functions of degree n of C is a free A-module of rank $p(n)$, for all $n \geq r$.*

In particular, if C is flat over A (not necessarily local) and $\operatorname{Spec} A$ *is connected, it makes sense to talk of the Hilbert polynomial of C and it coincides with the Hilbert polynomial of the fiber C_x of any point $x \in$* $\operatorname{Spec} A$.

Proof. If the reader assumes the vanishing theorems of Grauert and Serre ([H] III. 8.8, 12.11), then he needs only to read the section 2 of the following proof; however, in order to emphasize that only elementary theory of graded modules is required, we give a proof that does not make use of those vanishing theorems.

By faithfully base change we can assume that the residual field of A has infinite elements.

Let I be the saturated homogeneous ideal of functions vanishing on C and $B := A[x_0, \dots, x_m]/I$ the ring of homogeneous functions of C.

$\Rightarrow$) 1. Let us prove the statement for $n >> 0$. Let H_f be a hyperplane of $\mathbb{P}^m_A$ which meets C_x transversally. The closed subscheme $C' = C \cap H_f$ is flat over A and f/x_i is not a zero divisor in the affine open scheme $C - H_{x_i}$, for the flatness criteria ([M] theorem 22.5). Moreover, f is not a divisor of zero in B: If $f \cdot p_n = 0$, then $\frac{f}{x_i} \cdot \frac{p_n}{x_i^n} = 0$ for all i. Hence, $\frac{p_n}{x_i^n} = 0$ for all i and $p_n = 0$.

The Hilbert polynomial of C'_x is $p(n) - p(n-1)$. Let B' be the ring of homogeneous functions of C'. We proceed by induction on the degree of $p(n)$. By induction B'_n is locally free of rank $p(n) - p(n-1)$, for all $n \geq r$. B' is a quotient of B/fB and $B'_n = [B/fB]_n$ for all $n > s$ $(s >> 0)$.

For $n > s$ the exact sequences

$$0 \to B_n \xrightarrow{f\cdot} B_{n+1} \to [B/fB]_{n+1} \to 0$$

split, hence $B_{n+1} = f[B]_n \oplus [B/fB]_{n+1}$ and $\frac{B_{n+1}}{f^{n+1}} = \frac{B_n}{f^n} \oplus [B/fB]_{n+1}$.

Denote $U = C - H_f$ and let $\mathcal{O}_C$ be sheaf of functions of C. One has that $\mathcal{O}_C(U) = \varinjlim_{n>s} \frac{B_n}{f^n} \simeq \varinjlim_{n \geq t} \frac{B_n}{f^n} \simeq \frac{B_t}{f^t} \oplus (\bigoplus_{n>t} [B/fB]_n)$, for $t > s$. Since $\mathcal{O}_C(U)$ is a flat A-module, $\frac{B_t}{f^t} \simeq B_t$ is a free A-module. Moreover, the rank of B_t is $p(t)$, for $t >> 0$, as it is deduced by taking the fiber at x.

2. Let us prove that B_n is a free module of rank $p(n)$, for $n \geq r$, by decreasing induction on n:

Let B^x be the ring of homogeneous functions of C_x. The dimension of B^x_n is $p(n)$, for $n \geq r$. Let B'^x be the ring of functions of C'_x. The dimension of B'^x_n is $p(n) - p(n-1)$, for $n \geq r$.

Consider the diagram of exact rows (for $n \geq r$)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B_n & \xrightarrow{f\cdot} & B_{n+1} & \longrightarrow & B_{n+1}/fB_n & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B^x_n & \xrightarrow{f\cdot} & B^x_{n+1} & \longrightarrow & B^x_{n+1}/fB^x_n & \longrightarrow & 0 \end{array}$$

Suppose B_{n+1} is a free module of rank $p(n+1)$. Then $B_{n+1} \otimes_A k(x) = B^x_{n+1}$ (where $k(x)$ is the residual field of A). Therefore, tensoring the superior exact row by $\otimes_A k(x)$, we obtain that $(B_{n+1}/fB_n) \otimes_A k(x)$ coincides with B^x_{n+1}/fB^x_n, which has dimension $p(n+1) - p(n)$. Then B_{n+1}/fB_n coincides with the free module B'_{n+1} and B_n is a free module of rank $p(n)$.

$\Leftarrow$) Let H_f be a hyperplane meeting C_x transversally.

For $m \geq r$, B_m is a free A-module of rank $p(m)$. Since the Hilbert function is a polynomial from r, $B_m \otimes_A k(x)$ coincides with the module of homogeneous functions of degree m of C_x. For $m \geq r$ the morphisms $B_m \xrightarrow{f\cdot} B_{m+1}$ are injective, because these morphisms are injective at the fiber over x. Write $U = C - H_f$. One has $\mathcal{O}_C(U) = \varinjlim_{m>r} \frac{B_m}{f^m} \simeq \varinjlim_{m>r} B_m$, hence it is flat. Since the open schemes U (varying f) cover C, we conclude that C is flat over $\operatorname{Spec} A$.

□

Theorem 1.11. *Let S be a locally noetherian scheme and $X \to S$ a locally projective morphism. The "Hilbert functor"*

$$\underline{\mathrm{Hilb}}_{X/S}\colon \begin{bmatrix} \text{Locally noetherian} \\ S\text{-schemes} \end{bmatrix} \rightsquigarrow \text{Sets}$$

$$T \rightsquigarrow T\text{-flat closed subschemes of } X_T$$

is representable by an S-scheme which is a disjoint union of locally projective S-schemes.

Proof. Since the Hilbert functor is a sheaf in the Zariski topology, we can assume that $S = \operatorname{Spec} A$ is an affine, connected, noetherian scheme and $X \to S$ is a projective morphism.

Consider a closed immersion $X \hookrightarrow \mathbb{P}^m_A$. It suffices to prove that the Hilbert functor of $\mathbb{P}^m_A$ is representable, because the Hilbert functor of X is a closed subfunctor of the Hilbert functor of $\mathbb{P}^m_A$.

Now $\underline{\mathrm{Hilb}}_{\mathbb{P}^m_A/S} = \coprod_{p(n)} \underline{\mathrm{Hilb}}^{p(n)}_{\mathbb{P}^m_A/S}$, where $\underline{\mathrm{Hilb}}^{p(n)}_{\mathbb{P}^m_A/S}$ is the functor of closed subschemes of $\mathbb{P}^m_A$ with Hilbert polynomial $p(n)$.

Let $r \in \mathbb{N}$ be as in 1.6. Let us define $\phi(n)$ by $\phi(n) = 0$ if $n < r$ and $\phi(n) = p(n)$ if $n \geq r$. Let $\varphi(n) = \sum_{i=0}^{n} \phi(n)$.

We have a natural morphism $\pi\colon \underline{\mathrm{HomHilb}}^{\varphi(n)}_{\mathbb{A}^{m+1}/S} \to \underline{\mathrm{Hilb}}^{p(n)}_{\mathbb{P}^m/S}$, which assigns to each homogeneous variety its projectivization, i.e., to each homogeneous ideal I the projective subscheme C_I. This morphism has an inverse $\underline{\mathrm{Hilb}}^{p(n)}_{\mathbb{P}^m/S} \to \underline{\mathrm{HomHilb}}^{\varphi(n)}_{\mathbb{A}^{m+1}/S}$ which assigns to each projective subscheme C the homogeneous ideal

$$I = \underset{n\geq r}{\oplus} H^0(\mathbb{P}^m, \mathfrak{p}_C(n)), \qquad (I_n = 0 \text{ for } n < r).$$

Since $\underline{\mathrm{HomHilb}}^{\varphi(n)}_{\mathbb{A}^{m+1}/S}$ is representable, so is $\underline{\mathrm{Hilb}}^{p(n)}_{\mathbb{P}^m/S}$. □

In the remarks 1 and 2 of 1.9, we have obtained the equations of $\underline{\mathrm{HomHilb}}^{\varphi(n)}_{\mathbb{A}^{m+1}/S}$. In particular, one obtains the equations of $\underline{\mathrm{Hilb}}^{p(n)}_{\mathbb{P}^m/S}$. This last result was obtained in [CK]

2. Flattening Stratification Theorem

Given a prime ideal $\mathfrak{p}_x \subset A$ and an A-module M, M_x denotes the localization of M by the multiplicative set $A - \mathfrak{p}_x$.

Lemma 2.1. *Let A be a noetherian ring and let R be an $\mathbb{N}$-graded A-algebra of finite type. If $M = \underset{n\in\mathbb{N}}{\oplus} M_n$ is a graded R-module of finite type, then $U := \{x \in \operatorname{Spec} A\colon M_x$ is a flat A_x-module$\}$ is a open subscheme of $\operatorname{Spec} A$.*

Proof. Let us prove that U satisfies the two conditions of the Nagata's topological flatness criterion ([M] theorem 24.2):

1. $\bar{x} \cap U \neq \emptyset \Rightarrow x \in U$ ($\bar{x}$ denotes the closure of x in $\operatorname{Spec} A$).
2. If $x \in U$ then $\bar{x} \cap U$ is an open neighborhood of $x \in \bar{x}$.

It is obvious that U satisfies (1). Let us see that it satisfies (2). Let $x \in U$ and $y \in \bar{x}$. Denote $\bar{A} = A/\mathfrak{p}_x$. Then,

M_y is a flat A_y-module $\iff$ $(M_n)_y$ is a flat A_y-module for all $n \in \mathbb{N}$ $\iff$ $\mathrm{Tor}_1((M_n)_y, \bar{A}) = 0$ and $(M_n)_y/\mathfrak{p}_x(M_n)_y$ is a flat $\bar{A}$-module for all $n \in \mathbb{N}$ $\iff$ $\mathrm{Tor}_1(M_y, \bar{A}) = 0$ and $M_y/\mathfrak{p}_x M_y$ is a flat $\bar{A}$-module.

Now, $\mathrm{Tor}_1(M, \bar{A})_x = \mathrm{Tor}_1(M_x, \bar{A}) = 0$, hence $\mathrm{Tor}_1(M, \bar{A})$ is zero on a open neighborhood of x (because $\mathrm{Tor}_1(M, \bar{A})$ is a finite R-module). Moreover, $M/\mathfrak{p}_x M$ is a flat $\bar{A}$-module over an open neighborhood of x in $\bar{x}$, by the theorem of generic flatness ([M] theorem 24.1). Conclusion follows. □

Given two maps $\phi_1, \phi_2 \colon \mathbb{N} \to \mathbb{N}$, we say that $\phi_1 < \phi_2$ if there exists $n \in \mathbb{N}$ such that $\phi_1(n) < \phi_2(n)$ and $\phi_1(m) \leq \phi_2(m)$, for all $m > n$. The set of maps from $\mathbb{N}$ to $\mathbb{N}$ which are polynomials for $n >> 0$ is a totally ordered set with this relation.

Let A be a noetherian ring, R an $\mathbb{N}$-graded A-algebra of finite type and $\phi \colon \mathbb{N} \to \mathbb{N}$ a map. Let us define a functor F_ϕ from the category of rings (or affine schemes) to the category of sets in the following way: for any ring B

$$F_\phi(B) = \left\{ \begin{array}{l} \text{Morphisms } \operatorname{Spec} B \to \operatorname{Spec} A \text{ such that } [R \otimes_A B]_n \text{ is a} \\ \text{locally free } B\text{-module of rank } \phi(n), \text{ for any } n \in \mathbb{N}. \end{array} \right\}$$

Theorem 2.2. *1. F_ϕ is representable by a subscheme (an open subscheme of a closed subscheme) of* $\operatorname{Spec} A$.

2. There exists only a finite number of maps $\phi_1 < \phi_2 < \cdots < \phi_m$ such that F_{ϕ_i} is not the empty functor. Moreover, there exists a filtration of closed subschemes $\operatorname{Spec} A = C_0 \supset C_1 \supset \cdots \supset C_m = \emptyset$ *such that $C_{i-1} - C_i$ is topologically equal to the scheme representing F_{ϕ_i}.*

3. For $s >> 0$ the functor

$$F^s_\phi(B) = \left\{ \begin{array}{l} \textit{Morphisms } \operatorname{Spec} B \to \operatorname{Spec} A \textit{ such that } [R \otimes_A B]_n \textit{ is a} \\ \textit{locally free module of rank } \phi(n), \textit{ for all } n \leq s. \end{array} \right\}$$

coincides with the functor F_ϕ.

Proof. 1. Let $\varphi \colon \operatorname{Spec} B \to \operatorname{Spec} A$ belong to $F_\phi(S)$. Let us denote by J_n the $(\phi(n) - 1)$-th Fitting ideal of R_n. Then $J_n \cdot B = 0$. Let $C := \bigcap_n (J_n)_0 \subseteq \operatorname{Spec} A$. It is easy to prove (by the theory of Fitting ideals) that φ factors through C. Hence, we can assume that $C = \operatorname{Spec} A$.

Let I_n be the $\phi(n)$-th Fitting ideal of R_n. If $x \in \operatorname{Im} \varphi$, then $x \in \operatorname{Spec} A - (I_n)_0$, for all n; hence, $(R_n)_x$ is a flat A_x-module (again by the theory of Fitting ideals). By the previous lemma, R is a flat A-module over a connected open neighborhood of x. Then R_n is a locally free module of rank $\phi(n)$ over all the points of this connected open neighborhood, for all n. In conclusion, the set of points of $\operatorname{Spec} A$ such that each R_n is a free module of rank $\phi(n)$, for all n, is an open set which represents F_ϕ.

2. For each point $x \in \operatorname{Spec} A$, either closed or not, let us define the function $\phi_x(n) := \dim_{k(x)} R_n \otimes_A k(x)$. It is obvious that $x \in X_{\phi_x}$. By standard noetherian arguments there exist functions $\phi_1, \ldots, \phi_m$ such that $\operatorname{Spec} A = \coprod_i X_{\phi_i}$. Reordering, we can write $\phi_1 < \ldots < \phi_m$. Then, there exist $n_2, \ldots, n_m \in \mathbb{N}$ such that $\phi_1(n_i) < \phi_i(n_i)$. So, given $x \in X_{\phi_1}$, there exist open neighborhoods U_i of x such that

$U_i \cap X_{\phi_i} = \emptyset$. Hence, X_{ϕ_1} is an open set. Let $C_1 = \operatorname{Spec} A - X_{\phi_1}$. Restricting F_{ϕ_2} to C_1, one constructs C_2 in the same way, and so on.

Finally, let us prove 3. We have $\operatorname{Spec} A = X_{\phi_1} \coprod \overset{m}{\cdots} \coprod X_{\phi_m}$. If $\phi \neq \phi_i$, for all i, there exists $s \in \mathbb{N}$ such that $\phi_{|Y} \neq \phi_{i|Y}$ for all i, with $Y = \{0, 1, \ldots, s\}$. Hence $F^s_\phi = \emptyset = F_\phi$. If $\phi = \phi_i$, let X_t denote the subscheme of $\operatorname{Spec} A$ representing the functor F^t_ϕ, $t \in \mathbb{N}$. Let r be a natural number such that $\phi_{i|Y'} \neq \phi_{j|Y'}$, $i \neq j$, where $Y' = \{0, 1, \ldots, r\}$. Then we have a sequence of subschemes $X_r \supseteq X_{r+1} \supseteq X_{r+2} \supseteq \cdots$, which are topologically equal. By noetherianity of A, $X_s = X_{s+1} = \cdots$ for $s >> 0$. Therefore, $X_\phi = X_s$ and $F_\phi = F^s_\phi$.

$\square$

References

[CK] I. Ciocan-Fontanine, M. Kapranov, *Derived Hilbert schemes.* Journal: J. Amer. Math. Soc. **15** (2002), 787-815.

[G] A. Grothendieck, *Fondements de la géométrie algébrique (Extraits du Séminaire Bourbaki.* Springer-Verlag (1957-1962)

[GD] A. Grothendieck, J. Dieudonné, *Eléments de géométrie algébrique* **I**. Springer-Verlag (1971)

[H] R. Hartshorne, *Alegebraic Geometry*, GTM, Vol **52**, Springer-Verlag, New York 1977.

[K] J. Kollar, *Rational Curves on Algebraic Varieties Series.* Ergeb. Math. Grenzgeb. (3), Vol. **32**. 1st ed. 1996. Corr. 2nd printing, 1999, VIII, 321 pp. Springer-Verlag.

[M] H. Matsumura, *Commutative ring theory.* Cambridge University Press (1989)

[Mu] D. Mumford, *Lecture on curves on an algebraic surface.* Annals of Mathematics Studies **59**. Princeton University Press (1966).

[V] E. Viehweg, *Quasi-projective Moduli for Polarized Manifolds*, Ergeb. Math. Grenzgeb. (3), Vol **30**, Springer-Verlag, Berlin 1995.

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