

REYNOLDS OPERATOR ON FUNCTORS

AMELIA ÁLVAREZ, CARLOS SANCHO, AND PEDRO SANCHO

ABSTRACT. Let $G = \text{Spec } A$ be an affine R -monoid scheme. We prove that the category of dual functors (over the category of commutative R -algebras) of G -modules is equivalent to the category of dual functors of \mathcal{A}^* -modules. We prove that G is invariant exact if and only if $A^* = R \times B^*$ as R -algebras and the first projection $A^* \rightarrow R$ is the unit of A . If \mathbb{M} is a dual functor of G -modules and $w_G := (1, 0) \in R \times B^* = A^*$, we prove that $\mathbb{M}^G = w_G \cdot \mathbb{M}$ and $\mathbb{M} = w_G \cdot \mathbb{M} \oplus (1 - w_G) \cdot \mathbb{M}$; hence, the Reynolds operator can be defined on \mathbb{M} .

INTRODUCTION

Let R be a commutative ring with unit. An R -module M can be considered as a functor of \mathcal{R} -modules over the category of commutative R -algebras, which we will denote by \mathcal{M} and we will call a quasi-coherent module, by defining $\mathcal{M}(S) := M \otimes_R S$. If \mathbb{M} and \mathbb{N} are functors of \mathcal{R} -modules, we will denote by $\mathbb{H}om_{\mathcal{R}}(\mathbb{M}, \mathbb{N})$ the functor of \mathcal{R} -modules

$$\mathbb{H}om_{\mathcal{R}}(\mathbb{M}, \mathbb{N})(S) := \text{Hom}_S(\mathbb{M}|_S, \mathbb{N}|_S)$$

where $\mathbb{M}|_S$ is the functor \mathbb{M} restricted to the category of commutative S -algebras. The functor $\mathbb{M}^* := \mathbb{H}om_{\mathcal{R}}(\mathbb{M}, \mathcal{R})$ is said to be a dual functor. For example, \mathcal{M} , \mathcal{M}^* and $\mathbb{H}om_{\mathcal{R}}(\mathcal{M}, \mathcal{M}')$ are dual functors (see [1, 1.10] and Proposition 1.1).

An affine R -monoid scheme $G = \text{Spec } A$ can be considered as a functor of monoids over the category of commutative R -algebras: $G(S) := \text{Hom}_{R\text{-alg}}(A, S)$, which is known as the functor of points of G .

It is well known that the theory of linear representations of algebraic groups can be developed, via their associated functors, as a theory of (abstract) groups and their linear representations. That is, the category of modules is equivalent to the category of quasi-coherent modules and the category of rational G -modules is equivalent to the category of quasi-coherent G -modules. Moreover, it is sometimes necessary to consider some natural vector spaces via their associated functors. For example, if M and M' are two (rational) linear representations of $G = \text{Spec } A$, then $\text{Hom}_R(M, M')$ is not a (rational) linear representation of G , although G operates naturally on $\mathbb{H}om_{\mathcal{R}}(\mathcal{M}, \mathcal{M}')$; A^* is not a (rational) linear representation of $G = \text{Spec } A$, although G operates naturally on \mathcal{A}^* .

\mathcal{A}^* is a functor of algebras and there exists a natural and obvious morphism of functors of monoids $G \hookrightarrow \mathcal{A}^*$. Then every functor of \mathcal{A}^* -modules is a functor of G -modules (see Definition 1.3). In this paper, we prove the following theorem.

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Theorem 1. *The category of dual functors of G -modules is equal to the category of dual functors of \mathcal{A}^* -modules.*

“In the classical situation” observe that if we consider the rational points of G , $G(R)$, and the inclusion $G(R) \hookrightarrow A^*$, the category of (rational) $G(R)$ -modules it is not equivalent to the category of A^* -modules, even if G is a smooth algebraic group over an algebraically closed field (in this last case it is necessary to introduce topologies on A^* and on the modules).

We say that $G = \text{Spec } A$ is invariant exact if taking invariants is an exact functor (see 2.2 and 2.13). If A is a projective R -module and G is invariant exact then it is linearly reductive (3.4)

We prove that an affine R -group scheme $G = \text{Spec } A$ is invariant exact if and only if $\mathcal{A}^* = \mathcal{R} \times \mathcal{B}^*$ as functors of \mathcal{R} -algebras (Corollary 2.6). If A is a projective R -module, $G = \text{Spec } A$ is invariant exact if and only if $A^* = R \times C$ as R -algebras (Remark 2.7). If G is invariant exact there exists an isomorphism $A^* = R \times B^*$ such that the first projection $A^* \rightarrow R$ is the unit of A . Let $w_G := (1, 0) \in R \times B^* = A^*$ be the “invariant integral” on G . We prove the following theorem.

Theorem 2. *Let $G = \text{Spec } A$ be an invariant exact R -group and let $w_G \in A^*$ be the invariant integral on G . Let \mathbb{M} be a dual functor of G -modules. It holds that:*

- (1) $\mathbb{M}^{G'} = w_G \cdot \mathbb{M}$.
- (2) \mathbb{M} splits uniquely as a direct sum of $\mathbb{M}^{G'}$ and another subfunctor of G -modules, explicitly

$$\mathbb{M} = w_G \cdot \mathbb{M} \oplus (1 - w_G) \cdot \mathbb{M}.$$

We call the projection $\mathbb{M} \rightarrow \mathbb{M}^{G'} = w_G \cdot \mathbb{M}$ the Reynolds operator. Taking sections we have $\mathbb{M}(R)^{G'} = w_G \cdot \mathbb{M}(R)$ and the morphism $\mathbb{M}(R) \rightarrow \mathbb{M}(R)^{G'}$, $m \mapsto w_G \cdot m$. The classical Reynolds operator is a particular case of Theorem 2, for $\mathbb{M} = \mathcal{M}$. The previous theorem still holds for any separated functor of \mathcal{A}^* -modules (see Definition 3.1). More generally, for every functor \mathbb{N} of G -modules, we prove that there exists the maximal separated G -invariant quotient of \mathbb{N} and that the dual of this quotient is $\mathbb{N}^{*G'}$ (Theorem 4.1).

In [4] it is proved that a Reynolds operator can be defined on $\text{Hom}_B(M, M')$, where B is a G -algebra and M and M' are two BG -modules. Obviously G' operates on $\mathbb{H}om_B(\mathcal{M}, \mathcal{M}')$ and it is a separated functor by Proposition 3.8. Hence the Reynolds operator can be defined on $\text{Hom}_B(M, M')$ by Theorem 2. This is an example that shows that functorial treatment can clarify some problems.

Let $\chi: G \rightarrow G_m$ be a multiplicative character. Given a functor of G -modules let \mathbb{M}^χ be the subfunctor of the χ -semi-invariant elements of \mathbb{M} (see Definition 5.1). In Section 5, we extend the previous theorems about the invariant integral and the Reynolds operator to the semi-invariant integral and the Reynolds χ -operator. We apply these results to prove some results of [5] about generalized Cayley’s Ω -processes in Example 5.6.

Finally, these results about the invariant integral, the Reynolds operator, etc., can be extended to functors of monoids with a reflexive functor of functions. We will explain it in detail in a next paper.

1. PRELIMINARY RESULTS

[1] is the basic reference for reading this paper.

Let R be a commutative ring (associative with unit). All functors considered in this paper are covariant functors over the category of commutative R -algebras (associative with unit). We will say that \mathbb{X} is a functor of sets (resp. monoids, etc.) if \mathbb{X} is a functor from the category of commutative R -algebras to the category of sets (resp. monoids and so forth).

Let \mathcal{R} be the functor of rings defined by $\mathcal{R}(S) := S$ for every commutative R -algebra S . We will say that a functor of commutative groups \mathbb{M} is a functor of \mathcal{R} -modules if we have a morphism of functors of sets $\mathcal{R} \times \mathbb{M} \rightarrow \mathbb{M}$, so that $\mathbb{M}(S)$ is an S -module for every commutative R -algebra S . We will say that a functor of \mathcal{R} -modules \mathbb{A} is a functor of \mathcal{R} -algebras if $\mathbb{A}(S)$ is a S -algebra with unit and S commutes with all the elements of $\mathbb{A}(S)$. If \mathbb{M} and \mathbb{N} are functors of \mathcal{R} -modules, we will denote by $\mathbb{H}om_{\mathcal{R}}(\mathbb{M}, \mathbb{N})$ the functor of \mathcal{R} -modules

$$\mathbb{H}om_{\mathcal{R}}(\mathbb{M}, \mathbb{N})(S) := \text{Hom}_{\mathcal{S}}(\mathbb{M}|_S, \mathbb{N}|_S)$$

where $\mathbb{M}|_S$ is the functor \mathbb{M} restricted to the category of commutative S -algebras.

The functor $\mathbb{M}^* := \mathbb{H}om_{\mathcal{R}}(\mathbb{M}, \mathcal{R})$ is said to be a dual functor.

Proposition 1.1. *If \mathbb{M}^* is a dual functor of \mathcal{R} -modules and \mathbb{N} is a functor of \mathcal{R} -modules, then $\mathbb{H}om_{\mathcal{R}}(\mathbb{N}, \mathbb{M}^*)$ is a dual functor of \mathcal{R} -modules.*

Proof. Actually, $\mathbb{H}om_{\mathcal{R}}(\mathbb{N}, \mathbb{M}^*) = \mathbb{H}om_{\mathcal{R}}(\mathbb{N} \otimes \mathbb{M}, \mathcal{R})$. □

Given an R -module M , the functor of \mathcal{R} -modules \mathcal{M} defined by $\mathcal{M}(S) := M \otimes_R S$ is called a quasi-coherent \mathcal{R} -module. The functors $M \rightsquigarrow \mathcal{M}$, $\mathcal{M} \rightsquigarrow \mathcal{M}(R) = M$ establish an equivalence between the category of \mathcal{R} -modules and the category of quasi-coherent \mathcal{R} -modules ([1, 1.12]). In particular, $\text{Hom}_{\mathcal{R}}(\mathcal{M}, \mathcal{M}') = \text{Hom}_R(M, M')$. The notion of quasi-coherent \mathcal{R} -module is stable under base change $R \rightarrow S$, that is, $\mathcal{M}|_S$ is equal to the quasi-coherent \mathcal{S} -module associated to the S -module $M \otimes_R S$.

The functor $\mathcal{M}^* = \mathbb{H}om_{\mathcal{R}}(\mathcal{M}, \mathcal{R})$ is called an \mathcal{R} -module scheme. Specifically, $\mathcal{M}^*(S) = \text{Hom}_{\mathcal{S}}(M \otimes_R S, S) = \text{Hom}_R(M, S)$. It is easy to check that given two functors of \mathcal{R} -modules \mathbb{M} and \mathbb{M}' , then

$$(\mathbb{H}om_{\mathcal{R}}(\mathbb{M}_1, \mathbb{M}_2))|_S = \mathbb{H}om_{\mathcal{S}}(\mathbb{M}_1|_S, \mathbb{M}'_2|_S).$$

In particular, $(\mathcal{M}^*)|_S$ is an \mathcal{S} -module scheme. An \mathcal{R} -module scheme \mathcal{M}^* is a quasi-coherent \mathcal{R} -module if and only if M is a projective R -module of finite type ([2]). A basic result says that quasi-coherent modules and module schemes are reflexive, that is,

$$\mathcal{M}^{**} = \mathcal{M}$$

([1, 1.10]); thus, the functors $\mathcal{M} \rightsquigarrow \mathcal{M}^*$ and $\mathcal{M}^* \rightsquigarrow \mathcal{M}^{**} = \mathcal{M}$ establish an equivalence between the category of quasi-coherent modules and the category of module schemes. \mathcal{M} and \mathcal{M}^* are examples of dual functors.

Let $X = \text{Spec } A$ be an affine R -scheme and let X^\cdot be the functor of points of X , that is, the functor of sets

$$X^\cdot(S) = \text{Hom}_{R\text{-sch}}(\text{Spec } S, X) = \text{Hom}_{R\text{-alg}}(A, S)$$

“points of X with values in S ”. Given another affine scheme $Y = \text{Spec } B$, by Yoneda’s lemma

$$\text{Hom}_{R\text{-sch}}(X, Y) = \text{Hom}(X^\cdot, Y^\cdot),$$

and $X^\cdot \simeq Y^\cdot$ if and only if $X \simeq Y$. We will sometimes denote $X^\cdot = X$.

Let $R \rightarrow S$ be a morphism of rings and let $X_S = \text{Spec}(A \otimes_R S)$, then $(X^\cdot)_{|S} = (X_S)^\cdot$. Observe that $\text{Hom}(X^\cdot, \mathcal{R}) = \text{Hom}(X^\cdot, (\text{Spec } R[X])^\cdot) = \text{Hom}_{R\text{-alg}}(R[X], A) = A$, then $\mathbb{H}\text{om}(X^\cdot, \mathcal{R}) = \mathcal{A}$. There is a natural morphism $X^\cdot \rightarrow \mathcal{A}^*$, because $X^\cdot(S) = \text{Hom}_{R\text{-alg}}(A, S) \subset \text{Hom}_R(A, S) = \mathcal{A}^*(S)$.

Proposition 1.2. *Let \mathbb{M}^* be a dual functor of \mathcal{R} -modules, any morphism of functors $X^\cdot \rightarrow \mathbb{M}^*$ factorizes via a unique morphism of functors of \mathcal{R} -modules $\mathcal{A}^* \rightarrow \mathbb{M}^*$.*

Proof. It is a consequence of the equalities

$$\begin{aligned} \text{Hom}(X^\cdot, \mathbb{M}^*) &= \text{Hom}_{\mathcal{R}}(\mathbb{M}, \mathbb{H}\text{om}(X^\cdot, \mathcal{R})) = \text{Hom}_{\mathcal{R}}(\mathbb{M}, \mathcal{A}) = \text{Hom}_{\mathcal{R}}(\mathbb{M} \otimes_{\mathcal{R}} \mathcal{A}^*, \mathcal{R}) \\ &= \text{Hom}_{\mathcal{R}}(\mathcal{A}^*, \mathbb{M}^*). \end{aligned}$$

□

Let $G = \text{Spec } A$ be an affine R -monoid scheme, that is, G^\cdot is a functor of monoids. \mathcal{A}^* is an \mathcal{R} -algebra scheme, that is, besides from being an \mathcal{R} -module scheme it is a functor of \mathcal{R} -algebras. The natural morphism $G^\cdot \rightarrow \mathcal{A}^*$ is a morphism of functors monoids. By [1, 5.3], given any dual functor of algebras \mathbb{B}^* (that is, a dual functor of \mathcal{R} -modules which is a functor of \mathcal{R} -algebras), then any morphism of functors of monoids $G^\cdot \rightarrow \mathbb{B}^*$ factorizes via a unique morphism of functors of \mathcal{R} -algebras $\mathcal{A}^* \rightarrow \mathbb{B}^*$.

Definition 1.3. *A functor \mathbb{M} of (left) G -modules is a functor of \mathcal{R} -modules endowed with an action of G , i.e., a morphism of functors of monoids $G^\cdot \rightarrow \text{End}_{\mathcal{R}}(\mathbb{M})$. A functor \mathbb{M} of (left) \mathcal{A}^* -modules is a functor of \mathcal{R} -modules endowed with a morphism of functors of \mathcal{R} -algebras $\mathcal{A}^* \rightarrow \text{End}_{\mathcal{R}}(\mathbb{M})$.*

The functors $M \rightsquigarrow \mathcal{M}$, $\mathcal{M} \rightsquigarrow \mathcal{M}(R) = M$ establish an equivalence between the category of rational G -modules and the category of quasi-coherent G -modules. The category of quasi-coherent G -modules is equal to the category of quasi-coherent \mathcal{A}^* -modules by [1, 5.5].

Notation 1.4. *For abbreviation, we sometimes use $g \in G$ or $m \in \mathbb{M}$ to denote $g \in G(S)$ or $m \in \mathbb{M}(S)$ respectively. Given $m \in \mathbb{M}(S)$ and a morphism of \mathcal{R} -algebras $S \rightarrow T$, we still denote by m its image by the morphism $\mathbb{M}(S) \rightarrow \mathbb{M}(T)$.*

Definition 1.5. *Let \mathbb{M} be a functor of G -modules. We define*

$$\mathbb{M}(S)^G := \{m \in \mathbb{M}(S), \text{ such that } g \cdot m = m \text{ for every } g \in G\}^1$$

and we denote by \mathbb{M}^G the subfunctor of \mathcal{R} -modules of \mathbb{M} defined by $\mathbb{M}^G(S) := \mathbb{M}(S)^G$. We will say that $m \in \mathbb{M}$ is left G -invariant if $m \in \mathbb{M}^G$.

If \mathbb{M} is a functor of G -modules (resp. of right G -modules), then \mathbb{M}^* is a functor of right G -modules: $w * g := w(g \cdot -)$, for every $w \in \mathbb{M}^*$ and $g \in G$ (resp. of left G -modules: $g * w := w(- \cdot g)$). If \mathbb{M}_1 and \mathbb{M}_2 are two functors of G -modules and G is an affine R -group scheme, then $\mathbb{H}\text{om}_{\mathcal{R}}(\mathbb{M}_1, \mathbb{M}_2)$ is a functor of G -modules, with the natural action $g * f := g \cdot f(g^{-1} \cdot -)$, and it holds that

$$\mathbb{H}\text{om}_{\mathcal{R}}(\mathbb{M}_1, \mathbb{M}_2)^G = \mathbb{H}\text{om}_G(\mathbb{M}_1, \mathbb{M}_2).$$

Let \mathbb{M} be a functor of G -modules, then $(\mathbb{M}^G)_{|S} = (\mathbb{M}_{|S})^{G_S}$.

¹More precisely, $g \cdot m = m$ for every $g \in G(T)$ and every morphism of \mathcal{R} -algebras $S \rightarrow T$.

2. INVARIANT EXACT MONOIDS

From now on, through out this paper $G = \text{Spec } A$ is an affine R -monoid scheme.

Theorem 2.1. *The category of dual functors of G -modules is equivalent to the category of dual functors of \mathcal{A}^* -modules.*

Proof. Let \mathbb{M} be a dual functor of \mathcal{R} -modules. By Proposition 1.1, $\mathbb{E}nd_{\mathcal{R}}(\mathbb{M})$ is a dual functor of \mathcal{R} -algebras. Hence $\text{Hom}_{\text{mon}}(G, \mathbb{E}nd_{\mathcal{R}}(\mathbb{M})) = \text{Hom}_{\mathcal{R}\text{-alg}}(\mathcal{A}^*, \mathbb{E}nd_{\mathcal{R}}(\mathbb{M}))$, and giving a structure of functor of G -modules on \mathbb{M} is equivalent to giving a structure of functor of \mathcal{A}^* -modules on \mathbb{M} .

Given two dual functors of G -modules \mathbb{M} and \mathbb{M}' , $\text{Hom}_G(\mathbb{M}, \mathbb{M}') = \text{Hom}_{\mathcal{A}^*}(\mathbb{M}, \mathbb{M}')$: observe that given a morphism of functors of \mathcal{R} -modules $L : \mathbb{M} \rightarrow \mathbb{M}'$ and $m \in \mathbb{M}$, the morphism $L_1 : G \rightarrow \mathbb{M}'$, $L_1(g) := L(gm) - gL(m)$ is null if and only if the morphism $L_2 : \mathcal{A}^* \rightarrow \mathbb{M}'$, $L_2(a) := L(am) - aL(m)$ is null. \square

Definition 2.2. *An affine R -monoid scheme $G = \text{Spec } A$ is said to be left invariant exact if for any exact sequence (in the category of functors of \mathcal{R} -modules) of dual functors of left G -modules*

$$0 \rightarrow \mathbb{M}_1 \rightarrow \mathbb{M}_2 \rightarrow \mathbb{M}_3 \rightarrow 0$$

the sequence

$$0 \rightarrow \mathbb{M}_1^G \rightarrow \mathbb{M}_2^G \rightarrow \mathbb{M}_3^G \rightarrow 0$$

is exact. G is said to be invariant exact if it is left and right invariant exact.

If G is an affine R -group scheme and it is left invariant exact, then it is right invariant exact since every functor of right G -modules \mathbb{M} can be regarded as a functor of left G -modules: $g \cdot m := m \cdot g^{-1}$.

Let $\Theta : G \rightarrow \mathcal{R}$, $g \mapsto 1$ be the trivial character, which induces the trivial representation $\Theta : \mathcal{A}^* \rightarrow \mathcal{R}$. Observe that $\Theta = 1 \in A$.

Theorem 2.3. *An affine R -monoid scheme $G = \text{Spec } A$ is invariant exact if and only if $\mathcal{A}^* = \mathcal{R} \times \mathcal{B}^*$ as \mathcal{R} -algebra schemes, where the projection $\mathcal{A}^* \rightarrow \mathcal{R}$ is Θ .*

Proof. Let us assume that G is invariant exact. The projection $\Theta : \mathcal{A}^* \rightarrow \mathcal{R}$ is a morphism of left and right G -modules (or \mathcal{A}^* -modules). Taking left invariants one obtains an epimorphism $\Theta : \mathcal{A}^{*G} \rightarrow \mathcal{R}$. Let $w_l \in \mathcal{A}^*$ be left G -invariant such that $\Theta(w_l) = 1$. Likewise, taking right invariants let $w_r \in \mathcal{A}^*$ be right G -invariant such that $\Theta(w_r) = 1$. Then $w = w_l \cdot w_r \in \mathcal{A}^*$ is left and right G -invariant and $\Theta(w) = 1$. Then, $w' \cdot w = w'(1) \cdot w = w \cdot w'$, because $g \cdot w = w = w \cdot g$. Hence, w is idempotent. Therefore, one finds a decomposition as a product of \mathcal{R} -algebra schemes $\mathcal{A}^* = w \cdot \mathcal{A}^* \oplus (1 - w) \cdot \mathcal{A}^*$; moreover, the morphism $\mathcal{R} \rightarrow \mathcal{A}^*$, $\lambda \mapsto \lambda \cdot w$, is a section of Θ , $w \cdot \mathcal{A}^* = \mathcal{R} \cdot w$ and Θ vanishes on $(1 - w) \cdot \mathcal{A}^*$.

Let us assume now that $\mathcal{A}^* = \mathcal{R} \times \mathcal{B}^*$ and $\pi_1 = \Theta$. Let $w = (1, 0) \in \mathcal{R} \times \mathcal{B}^* = \mathcal{A}^*$ and let us prove that G is invariant exact.

For any dual functor of G -modules \mathbb{M} , let us see that $w \cdot \mathbb{M} = \mathbb{M}^G$. One sees that $w \cdot \mathbb{M} \subseteq \mathbb{M}^G$, because $g \cdot (w \cdot m) = (g \cdot w) \cdot m = w \cdot m$, for every $g \in G$ and every $m \in \mathbb{M}$. Conversely, $\mathbb{M}^G \subseteq w \cdot \mathbb{M}$: Let $m \in \mathbb{M}$ be G -invariant. The morphism $G \rightarrow \mathbb{M}$, $g \mapsto g \cdot m = m$, extends to a unique morphism $\mathcal{A}^* \rightarrow \mathbb{M}$. The uniqueness implies that $w' \cdot m = w'(1) \cdot m$ and then $m = w \cdot m \in w \cdot \mathbb{M}$.

Taking invariants is a left exact functor. If $\mathbb{M}_2 \rightarrow \mathbb{M}_3$ is a surjective morphism, then the morphism $\mathbb{M}_2^G \rightarrow \mathbb{M}_3^G$ is surjective because so is the morphism $\mathbb{M}_2^G = w \cdot \mathbb{M}_2 \rightarrow w \cdot \mathbb{M}_3 = \mathbb{M}_3^G$. \square

Remark 2.4. *If a quasi-coherent \mathcal{R} -module \mathcal{M} is isomorphic to a direct product $\mathbb{M} \times \mathbb{N}$ of functors of \mathcal{R} -modules, then \mathbb{M} and \mathbb{N} are quasi-coherent (specifically, they are the quasi-coherent modules associated to the modules $\mathbb{M}(R)$ and $\mathbb{N}(R)$). Dually, if \mathcal{M}^* is isomorphic to a direct product $\mathbb{M} \times \mathbb{N}$ of functors of \mathcal{R} -modules, then \mathbb{M} and \mathbb{N} are \mathcal{R} -module schemes. If $\mathcal{A}^* = \mathbb{B} \times \mathbb{C}$ as functors of \mathcal{R} -algebras, then \mathbb{B} and \mathbb{C} are \mathcal{R} -algebra schemes.*

Let $\chi : G \rightarrow G_m$ be a multiplicative character and let $\chi : \mathcal{A}^* \rightarrow \mathcal{R}$ be the induced morphism of functors of \mathcal{R} -algebras.

Corollary 2.5. *An affine R -monoid scheme $G = \text{Spec } A$ is invariant exact if and only if $\mathcal{A}^* = \mathcal{R} \times \mathcal{B}^*$ as \mathcal{R} -algebra schemes, where the projection $\mathcal{A}^* \rightarrow \mathcal{R}$ is χ .*

Proof. The character χ induces the morphism $G \rightarrow \mathcal{A}^*$, $g \mapsto \chi(g) \cdot g$, which induces a morphism of \mathcal{R} -algebra schemes $\varphi : \mathcal{A}^* \rightarrow \mathcal{A}^*$. This last morphism is an isomorphism because its inverse morphism is the morphism induced by χ^{-1} .

The diagram

$$\begin{array}{ccc} \mathcal{A}^* & \xrightarrow{\varphi} & \mathcal{A}^* \\ & \searrow \chi & \downarrow \Theta \\ & & \mathcal{R} \end{array}$$

is commutative. Hence, via φ , “ $\mathcal{A}^* = \mathcal{R} \times \mathcal{B}^*$ as \mathcal{R} -algebra schemes, where the projection $\mathcal{A}^* \rightarrow \mathcal{R}$ is Θ ” if and only if “ $\mathcal{A}^* = \mathcal{R} \times \mathcal{B}^*$ as \mathcal{R} -algebra schemes, where the projection $\mathcal{A}^* \rightarrow \mathcal{R}$ is χ ”. Then, Theorem 2.3 proves this corollary. \square

Corollary 2.6. *An affine R -group scheme $G = \text{Spec } A$ is invariant exact if and only if $\mathcal{A}^* = \mathcal{R} \times \mathcal{B}^*$ as \mathcal{R} -algebra schemes.*

Proof. Assume that $\mathcal{A}^* = \mathcal{R} \times \mathcal{B}^*$ and let $G \hookrightarrow \mathcal{A}^*$, $g \mapsto g$ be the natural morphism. The composite morphism

$$G \hookrightarrow \mathcal{A}^* = \mathcal{R} \times \mathcal{B}^* \xrightarrow{\pi_1} \mathcal{R}$$

is a multiplicative character and π_1 is the morphism induced by this character. Now it is easy to prove that this corollary is a consequence of Corollary 2.5. \square

If M is an R -module and the natural morphism $M \rightarrow M^{**}$ is injective, for example if M is a projective module, then

$$\text{Hom}_{\mathcal{R}}(\mathcal{M}^*, \mathcal{M}'^*) = \text{Hom}_{\mathcal{R}}(\mathcal{M}', \mathcal{M}) = \text{Hom}_R(M', M) \subseteq \text{Hom}_R(M^*, M'^*).$$

Remark 2.7. *Assume that A is a projective R -module. If $A^* = C_1 \times C_2$ as R -algebras then the morphisms $\mathcal{A}^* \rightarrow \mathcal{A}^*$, $w \mapsto (1, 0) \cdot w - w \cdot (1, 0)$, $(0, 1) \cdot w - w \cdot (0, 1)$ are null and $\mathcal{A}^* = \mathcal{A}^* \cdot (1, 0) \times \mathcal{A}^* \cdot (0, 1)$ as functors of \mathcal{R} -algebras. Then, an affine R -group scheme $G = \text{Spec } A$ is invariant exact if and only if $A^* = R \times C$ as R -algebras.*

Theorem 2.8. *An affine R -group scheme $G = \text{Spec } A$ is invariant exact if and only if there exists a left G -invariant 1-form $w \in A^*$ such that $w(1) = 1$. Moreover, w is unique, it is right G -invariant and $*(w) = w$ (where $*$ is the morphism induced on \mathcal{A}^* by the morphism $G \rightarrow G$, $g \mapsto g^{-1}$).*

Proof. If w_l is left invariant and $w_l(1) = 1$, then $*w =: w_r$ is right invariant, $w := w_l \cdot w_r$ is left and right invariant and $w(1) = 1$. Now we can proceed as in Theorem 2.3 in order to prove that G is invariant exact.

Let us only prove the last statement. We follow the notation used in the proof of the last theorem. We know that $\mathcal{A}^{*G} = (1, 0) \cdot \mathcal{A}^* = \mathcal{R} \times 0$, then $(1, 0) : A \rightarrow R$ is the only left G -invariant linear map $w : A \rightarrow R$ such that $(1, 0)(1) = 1$. As well, $(1, 0)$ is right invariant. Finally, $*(1, 0)$ is left invariant and $*(1, 0)(1) = (1, 0)(1) = 1$, then $*(1, 0) = (1, 0)$. \square

Remark 2.9. *This result can be found, in [3] and [7], when R is a field and G is a linearly reductive algebraic group (that is, every rational G -module, M , is direct sum of irreducible G -modules). If R is an algebraically closed field of characteristic zero, then G is a reductive group if and only if G is linearly reductive, by a theorem of H. Weyl. If R is a field of positive characteristic, then the monoid of matrices $M_n(R)$ is not a linearly reductive monoid (there exists rational representations of $M_n(R)$ no completely reducible), however $M_n(R)$ is invariant exact (observe that given $0 \in M_n(R)$ and a $M_n(R)$ -module \mathbb{M} then $0 \cdot \mathbb{M} = \mathbb{M}^{M_n(R)}$).*

In the proof of Theorem 2.3 we have also proved Theorem 2.10 and Theorem 2.12.

Theorem 2.10. *An affine R -monoid scheme $G = \text{Spec } A$ is invariant exact if and only if there exists a left and right G -invariant 1-form $w \in A^*$ such that $w(1) = 1$.*

Definition 2.11. *Let $G = \text{Spec } A$ be an invariant exact affine R -monoid scheme. The only 1-form $w_G \in A^*$ that is left and right G -invariant and such that $w_G(1) = 1$ is called the invariant integral on G (influenced by the theory of compact Lie groups).*

Theorem 2.12. *An affine R -group scheme $G = \text{Spec } A$ is invariant exact if and only if for every exact sequence (in the category of functors of \mathcal{R} -modules) of G -module schemes*

$$0 \rightarrow \mathcal{M}_1^* \rightarrow \mathcal{M}_2^* \rightarrow \mathcal{M}_3^* \rightarrow 0$$

the sequence

$$0 \rightarrow \mathcal{M}_1^{*G} \rightarrow \mathcal{M}_2^{*G} \rightarrow \mathcal{M}_3^{*G} \rightarrow 0$$

is exact.

Theorem 2.13. *Let $G = \text{Spec } A$ be an affine R -monoid scheme. Assume that A is a projective R -module. G is invariant exact if and only if the functor “take invariants” is (left and right) exact on the category of coherent G -modules (or equivalently, the category of rational G -modules).*

Proof. Assume that the functor “take invariants” is (left and right) exact on the category of quasi-coherent G -modules. \mathcal{A}^* is an inverse limit of quotients \mathcal{B}_i , which are coherent \mathcal{R} -algebras by [1, 4.12]. It can be assumed that the morphism $\Theta : \mathcal{A}^* \rightarrow \mathcal{R}$ factorizes via \mathcal{B}_i for all i . Observe that \mathcal{B}_i are (left and right) \mathcal{A}^* -modules, then they are G -modules. Now, as in Theorem 2.10, we can prove that $\mathcal{B}_i = \mathcal{R} \times \mathcal{B}'_i$ as coherent \mathcal{R} -algebras, (where the projection onto the first factor is Θ). Then, taking inverse limit $\mathcal{A}^* = \mathcal{R} \times \mathcal{B}^*$ and, by Theorem 2.3, G is invariant exact. \square

3. REYNOLDS OPERATOR ON SEPARATED FUNCTORS

Let \mathbb{M} be a functor of \mathcal{R} -modules and let \mathbb{K} be the kernel of the natural morphism $\mathbb{M} \rightarrow \mathbb{M}^{**}$. One has that $\mathbb{K}(S) = \{m \in \mathbb{M}(S) : w(m) = 0, \text{ for every } w \in \mathbb{M}^*(T) \text{ and every morphism of } R\text{-algebras } S \rightarrow T\}$. Moreover, $(\mathbb{M}/\mathbb{K})^* = \mathbb{M}^*$ (then $(\mathbb{M}/\mathbb{K})^{**} = \mathbb{M}^{**}$) and the morphism $\mathbb{M}/\mathbb{K} \rightarrow (\mathbb{M}/\mathbb{K})^{**}$ is injective.

Definition 3.1. *We will say that \mathbb{M} is a separated functor of \mathcal{R} -modules if the morphism $\mathbb{M} \rightarrow \mathbb{M}^{**}$ is injective, that is, $m \in \mathbb{M}$ is null if and only if $w(m) = 0$ for every $w \in \mathbb{M}^*$.*

Dual functors of \mathcal{R} -modules are separated: Given $0 \neq m \in \mathbb{M} = \mathbb{N}^*$ there exists $n \in \mathbb{N}$ such that $m(n) \neq 0$; if \tilde{n} is the image of n by the morphism $\mathbb{N} \rightarrow \mathbb{N}^{**} = \mathbb{M}^*$, then $\tilde{n}(m) = m(n) \neq 0$. Every subfunctor of \mathcal{R} -modules of a separated functor of \mathcal{R} -modules is separated.

Proposition 3.2. *Let $G = \text{Spec } A$ be an invariant exact R -monoid and let $w_G \in \mathcal{A}^*$ be the invariant integral on G . Let \mathbb{M} be a separated functor of \mathcal{A}^* -modules. It holds that:*

- (1) $\mathbb{M}^G = w_G \cdot \mathbb{M}$.
- (2) \mathbb{M} splits uniquely as a direct sum of \mathbb{M}^G and another subfunctor of G -modules, explicitly

$$\mathbb{M} = w_G \cdot \mathbb{M} \oplus (1 - w_G) \cdot \mathbb{M}.$$

The morphism $\mathbb{M} \rightarrow \mathbb{M}^G$, $m \mapsto w_G \cdot m$ will be called the Reynolds operator of \mathbb{M} .

Proof.

- (1) One deduces that $w_G \cdot \mathbb{M} \subseteq \mathbb{M}^G$, because $g \cdot (w_G \cdot m) = (g \cdot w_G) \cdot m = w_G \cdot m$ for every $g \in G$ and every $m \in \mathbb{M}$. Conversely, let us see that $\mathbb{M}^G \subseteq w_G \cdot \mathbb{M}$. Let $m \in \mathbb{M}^G$. The morphism $G \rightarrow \mathbb{M} \hookrightarrow \mathbb{M}^{**}$, $g \mapsto g \cdot m = m$, extends to a unique morphism $\mathcal{A}^* \rightarrow \mathbb{M}^{**}$. The uniqueness implies that $w' \cdot m = w'(1) \cdot m$ and then $m = w_G \cdot m \in w_G \cdot \mathbb{M}$.
- (2) Since $\mathcal{A}^* = w_G \cdot \mathcal{A}^* \oplus (1 - w_G) \cdot \mathcal{A}^*$, then

$$\mathbb{M} = \mathcal{A}^* \otimes_{\mathcal{A}^*} \mathbb{M} = w_G \cdot \mathbb{M} \oplus (1 - w_G) \cdot \mathbb{M}.$$

Let $\mathbb{M} = \mathbb{M}^G \oplus \mathbb{N}$ be an isomorphism of G -modules. The G -module structure of \mathbb{N} extends to an \mathcal{A}^* -module structure, because $\mathbb{N} = \mathbb{M}/\mathbb{M}^G$. Moreover, \mathbb{N} is separated because it is a subfunctor of \mathcal{R} -modules of \mathbb{M} . Now, every morphism of G -modules between separated \mathcal{A}^* -modules is a morphism of \mathcal{A}^* -modules, because the morphism between the double duals is of \mathcal{A}^* -modules by Theorem 2.1. Thus, multiplying by w_G one concludes that $w_G \cdot \mathbb{M} = \mathbb{M}^G \oplus w_G \cdot \mathbb{N}$; hence, $w_G \cdot \mathbb{N} = 0$ and $(1 - w_G) \cdot \mathbb{M} = (1 - w_G) \cdot \mathbb{N} = \mathbb{N}$. \square

Proposition 3.3. *Let $G = \text{Spec } A$ be an invariant exact affine R -group scheme and let \mathbb{M} and \mathbb{N} be dual functors of G -modules. If $\pi : \mathbb{M} \rightarrow \mathbb{N}$ is an epimorphism of functors of G -modules and $s : \mathbb{N} \rightarrow \mathbb{M}$ is a section of functors of \mathcal{R} -modules of π , then $w_G \cdot s$ is a section of functors of G -modules of π .*

Proof. Let us consider the epimorphism of functors of G -modules (then of \mathcal{A}^* -modules)

$$\pi_* : \mathbb{H}om_{\mathcal{R}}(\mathbb{N}, \mathbb{M}) \rightarrow \mathbb{H}om_{\mathcal{R}}(\mathbb{N}, \mathbb{N}), f \mapsto \pi \circ f.$$

Then, $\pi \circ (w_G \cdot s) = \pi_*(w_G \cdot s) = w_G \cdot \pi_*(s) = w_G \cdot \text{Id} = \text{Id}$. \square

Likewise, it can be proved that if \mathbb{M} and \mathbb{N} are functors of G -modules, \mathbb{M} is a dual functor, $i : \mathbb{M} \rightarrow \mathbb{N}$ is an injective morphism of G -modules and r is a retract of functors R -modules of i , then $w_G \cdot r$ is a retract of functors of G -modules of i .

Remark 3.4. *We shall say a rational G -module M is simple if it does not contain any G -submodule $M' \subsetneq M$, such that M' is a direct summand of M as an R -module (this last condition is equivalent to the morphism of functors of \mathcal{R} -modules $\mathcal{M}^* \rightarrow \mathcal{M}'^*$ being surjective, see the previous paragraph to [1, 1.14]). If G is an invariant exact affine R -group scheme, M is a rational G -module and it is a noetherian R -module, then it is easy to prove, using the previous proposition, that M is a direct sum of simple G -modules.*

Example 3.5. *Let us give the proof of the famous Finiteness Theorem of Hilbert, [6], for its simplicity: "Let k be a field, let G be a linearly reductive affine k -group scheme and let us consider an operation of G over an algebraic variety $X = \text{Spec } A$. Then $X/\sim := \text{Spec } A^G$ is an algebraic variety".*

Proof. Let ξ_1, \dots, ξ_m be a system of generators of the k -algebra A . Let V be a finite dimensional G -submodule of A which contains ξ_1, \dots, ξ_m . The natural morphism $S \cdot V \rightarrow A$ is surjective. We have to prove that A^G is an algebra of finite type. As G is invariant exact, it is sufficient to prove that $(S \cdot V)^G = (k[x_1, \dots, x_n])^G$ is a k -algebra of finite type.

Let $I \subset k[x_1, \dots, x_n]$ be the ideal generated by $(x_1, \dots, x_n)^G$. Let $f_1, \dots, f_r \in (x_1, \dots, x_n)^G$ be a finite system of generators of I . We can assume f_i are homogeneous. Let us prove that $k[x_1, \dots, x_n]^G = k[f_1, \dots, f_r]$. Given a homogeneous $h \in k[x_1, \dots, x_n]^G$, we have to prove that $h \in k[f_1, \dots, f_r]$. We are going to proceed by induction on the degree of h . If $\text{dg } h = 0$ then $h \in k \subseteq k[f_1, \dots, f_r]$. Let $\text{dg } h = d > 0$. We can write $h = \sum_{i=1}^r a_i \cdot f_i$, where $a_i \in k[x_1, \dots, x_n]$ are homogeneous of degree $d - \text{dg}(f_i)$ (which are less than d). Then

$$h = w_G \cdot h = \sum_{i=1}^r w_G \cdot (a_i \cdot f_i) \stackrel{*}{=} \sum_{i=1}^r (w_G \cdot a_i) \cdot f_i$$

(Observe in $\stackrel{*}{=}$ that $g \cdot (a_i \cdot f_i) = (g \cdot a_i) \cdot (g \cdot f_i) = (g \cdot a_i) \cdot f_i$, then $w \cdot (a_i \cdot f_i) = (w \cdot a_i) \cdot f_i$ for all $w \in A^*$). By the induction hypothesis $w_G \cdot a_i \in k[f_1, \dots, f_r]$ and therefore $h \in k[f_1, \dots, f_r]$. \square

Let us see more examples where this theory can be applied.

Let $G = \text{Spec } A$ be an affine R -group scheme and let B be an R -algebra.

Definition 3.6. *We say that B is a G -algebra if G acts on B by endomorphisms of R -algebras, that is, there exists a morphism of monoids $G \rightarrow \text{End}_{\mathcal{R}\text{-alg}}(B)$.*

We will say that a functor of \mathcal{R} -modules \mathbb{M} is a functor of \mathcal{B} -modules if there exists a morphism of functors of \mathcal{R} -algebras $\mathcal{B} \rightarrow \text{End}_{\mathcal{R}}(\mathbb{M})$.

Definition 3.7. *Let B be a G -algebra and \mathbb{M} a functor of \mathcal{B} -modules. We say that \mathbb{M} is a $\mathcal{B}G$ -module if it has a G -module structure which is compatible with the \mathcal{B} -module structure, that is,*

$$g(b \cdot m) = g(b) \cdot g(m)$$

for every $g \in G$, $b \in \mathcal{B}$ and $m \in \mathbb{M}$.

If \mathbb{M} and \mathbb{N} are $\mathcal{B}G$ -modules, then it is easy to check that $\mathbb{H}om_{\mathcal{B}}(\mathbb{M}, \mathbb{N})$ is a subfunctor of G -modules of $\mathbb{H}om_{\mathcal{R}}(\mathbb{M}, \mathbb{N})$ and it coincides with the kernel of the morphism of G -modules

$$\begin{array}{ccc} \mathbb{H}om_{\mathcal{R}}(\mathbb{M}, \mathbb{N}) & \xrightarrow{\varphi} & \mathbb{H}om_{\mathcal{R}}(\mathcal{B} \otimes_{\mathcal{R}} \mathbb{M}, \mathbb{N}) \\ L & \mapsto & L_1 - L_2 \end{array}$$

where $L_1(b \otimes m) := L(b \cdot m)$ and $L_2(b \otimes m) := b \cdot L(m)$. Therefore, if \mathbb{N} is a dual functor as well, then $\mathbb{H}om_{\mathcal{B}}(\mathbb{M}, \mathbb{N})$ is an \mathcal{A}^* -module. Moreover, $\mathbb{H}om_{\mathcal{B}}(\mathbb{M}, \mathbb{N})$ is separated because it is an R -submodule of $\mathbb{H}om_{\mathcal{R}}(\mathbb{M}, \mathbb{N})$, and this latter is separated because it is a dual functor. Hence, if $G = \text{Spec } A$ is an invariant exact R -group, $\mathbb{H}om_{\mathcal{B}}(\mathbb{M}, \mathbb{N})^G = w_G \cdot \mathbb{H}om_{\mathcal{B}}(\mathbb{M}, \mathbb{N})$. We have proved the following proposition.

Proposition 3.8. *Let \mathbb{N} be a dual functor of $\mathcal{B}G$ -modules and let \mathbb{M} be a functor of $\mathcal{B}G$ -modules. Then:*

- (1) $\mathbb{H}om_{\mathcal{B}}(\mathbb{M}, \mathbb{N})$ is a separated functor of \mathcal{A}^* -modules.
- (2) If $G = \text{Spec } A$ is an invariant exact affine R -group scheme, then

$$\mathbb{H}om_{\mathcal{B}}(\mathbb{M}, \mathbb{N})^G = w_G \cdot \mathbb{H}om_{\mathcal{B}}(\mathbb{M}, \mathbb{N})$$

and $w_G \cdot : \mathbb{H}om_{\mathcal{B}}(\mathbb{M}, \mathbb{N}) \rightarrow \mathbb{H}om_{\mathcal{B}}(\mathbb{M}, \mathbb{N})^G$ is the Reynolds operator.

4. REYNOLDS OPERATOR ON FUNCTORS

Let us generalize the Reynolds operator to all functors of G -modules.

Let us assume that $G = \text{Spec } A$ is an invariant exact monoid.

Given a dual functor of G -modules \mathbb{M} , the dual morphism of $\mathbb{M}^G \hookrightarrow \mathbb{M}$ is the Reynolds operator of \mathbb{M}^* .

Let \mathbb{N} be a separated functor of G -modules. Let $\mathbb{N}_1 = \mathbb{N} \cap (1 - w_G) \cdot \mathbb{N}^{**}$. Then $\mathbb{N}_1 = \{n \in \mathbb{N} : w_G \cdot n = 0\}$, since $(1 - w_G) \cdot \mathbb{N}^{**} = \{n' \in \mathbb{N}^{**} : w_G \cdot n' = 0\}$. One deduces that $\mathbb{N}_1^G = \mathbb{N}_1 \cap \mathbb{N}^{**G} = 0$ and $(\mathbb{N}/\mathbb{N}_1)^G = \mathbb{N}/\mathbb{N}_1$, because \mathbb{N}/\mathbb{N}_1 injects into $\mathbb{N}^{**}/(1 - w_G) \cdot \mathbb{N}^{**} = \mathbb{N}^{**G}$. Moreover, $(\mathbb{N}/\mathbb{N}_1)^* = \mathbb{N}^{*G} : \mathbb{N}^{*G} = \mathbb{N}^* \cdot w_G$ vanishes on \mathbb{N}_1 , then $\mathbb{N}^{*G} \subseteq (\mathbb{N}/\mathbb{N}_1)^*$, and $(\mathbb{N}/\mathbb{N}_1)^* \subseteq \mathbb{N}^{*G}$. Therefore, $\mathbb{N}^{**} \rightarrow (\mathbb{N}/\mathbb{N}_1)^{**}$ is the Reynolds operator of \mathbb{N}^{**} .

Theorem 4.1. *Let $G = \text{Spec } A$ be an invariant exact R -monoid, let \mathbb{N} be a functor of G -modules and let $\mathbb{N}_1 \subset \mathbb{N}$ be the subfunctor of G -modules defined by $\mathbb{N}_1 := \{n \in \mathbb{N} : w_G \cdot \tilde{n} = 0\}$, where \tilde{n} denotes the image of n by the morphism $\mathbb{N} \rightarrow \mathbb{N}^{**}$. It holds that:*

- (1) \mathbb{N}/\mathbb{N}_1 is the maximal separated G -invariant quotient of \mathbb{N} .
- (2) The double dual of the morphism $\mathbb{N} \rightarrow \mathbb{N}/\mathbb{N}_1$ is the Reynolds operator $\mathbb{N}^{**} \rightarrow \mathbb{N}^{**G}$ and one has the commutative diagram

$$\begin{array}{ccc} \mathbb{N} & \longrightarrow & \mathbb{N}/\mathbb{N}_1 \\ \downarrow & & \downarrow \\ \mathbb{N}^{**} & \longrightarrow & \mathbb{N}^{**G} = (\mathbb{N}/\mathbb{N}_1)^{**}. \end{array} \quad (1)$$

- (3) If \mathbb{N} is a dual functor, then $\mathbb{N}/\mathbb{N}_1 = \mathbb{N}^G$ and the morphism $\mathbb{N} \rightarrow \mathbb{N}/\mathbb{N}_1$ is the Reynolds operator of \mathbb{N} .

Proof.

- (2) \mathbb{N}_1 is the kernel of the composite morphism $\mathbb{N} \rightarrow \mathbb{N}^{**} \rightarrow \mathbb{N}^{**G}$, then \mathbb{N}_1 contains the kernel \mathbb{K} of the morphism $\mathbb{N} \rightarrow \mathbb{N}^{**}$. Let $\mathbb{N}' = \mathbb{N}/\mathbb{K}$. Observe that $\mathbb{N}'^* = \mathbb{N}^*$, that \mathbb{N}' is separated, $\mathbb{N}'_1 = \mathbb{N}_1/\mathbb{K}$ and $\mathbb{N}'/\mathbb{N}'_1 = \mathbb{N}/\mathbb{N}_1$. Therefore the diagram (1) is commutative because is commutative for $\mathbb{N} = \mathbb{N}'$. In particular, \mathbb{N}/\mathbb{N}_1 is separated.
- (1) We must prove that if $\mathbb{P} \subseteq \mathbb{N}$ is a subfunctor of G -modules such that \mathbb{N}/\mathbb{P} is separated and G -invariant, then $\mathbb{N}_1 \subseteq \mathbb{P}$, i.e., \mathbb{N}/\mathbb{P} is a quotient of \mathbb{N}/\mathbb{N}_1 . $\mathbb{N}/(\mathbb{P} \cap \mathbb{N}_1)$ is G -invariant and separated functor, because the morphism $\mathbb{N}/(\mathbb{P} \cap \mathbb{N}_1) \hookrightarrow \mathbb{N}/\mathbb{N}_1 \oplus \mathbb{N}/\mathbb{P}$, $\bar{h} \mapsto (\bar{h}, \bar{h})$ is injective. It is enough to prove that $\mathbb{N}_1 = \mathbb{P} \cap \mathbb{N}_1$. Let us denote $\mathbb{P}' = \mathbb{P} \cap \mathbb{N}_1$. From the composition of injections $\mathbb{N}^{*G} = (\mathbb{N}/\mathbb{N}_1)^* \hookrightarrow (\mathbb{N}/\mathbb{P}')^* \hookrightarrow \mathbb{N}^{*G}$ one concludes that $(\mathbb{N}/\mathbb{N}_1)^* = (\mathbb{N}/\mathbb{P}')^*$. Now, the commutative diagram

$$\begin{array}{ccc} \mathbb{N}/\mathbb{P}' & \longrightarrow & \mathbb{N}/\mathbb{N}_1 \\ \downarrow & & \downarrow \\ (\mathbb{N}/\mathbb{P}')^{**} & \xlongequal{\quad} & (\mathbb{N}/\mathbb{N}_1)^{**} \end{array}$$

implies that the morphism $\mathbb{N}/\mathbb{P}' \rightarrow \mathbb{N}/\mathbb{N}_1$ is injective; hence, $\mathbb{P}' = \mathbb{N}_1$.

- (3) Recall that $\mathbb{N} = w_G \cdot \mathbb{N} \oplus (1 - w_G) \cdot \mathbb{N}$, $\mathbb{N}_1 = (1 - w_G) \cdot \mathbb{N}$ and $\mathbb{N}/\mathbb{N}_1 = w_G \cdot \mathbb{N} = \mathbb{N}^G$.

□

Let us observe that $\mathbb{N}_1^0 := \{w \in \mathbb{N}^* : w(\mathbb{N}_1) = 0\} = (\mathbb{N}/\mathbb{N}_1)^* = \mathbb{N}^{*G}$. On the other hand, $(\mathbb{N}^{*G})^0 := \{n \in \mathbb{N} : \mathbb{N}^{*G}(n) = 0\} = \{n \in \mathbb{N} : \tilde{n}(\mathbb{N}^{*G} = \mathbb{N}^* \cdot w_G) = 0\} = \{n \in \mathbb{N} : (w_G \cdot \tilde{n})(\mathbb{N}^*) = 0\} = \{n \in \mathbb{N} : w_G \cdot \tilde{n} = 0\} = \mathbb{N}_1$.

5. SEMI-INVARIANTS

Let $\chi: G = \text{Spec } A \rightarrow \mathcal{R}$ be a multiplicative character and let $\chi: \mathcal{A}^* \rightarrow \mathcal{R}$ be the induced morphism.

Definition 5.1. Let \mathbb{M} be a functor of G -modules. An element $m \in \mathbb{M}$ is said to be (left) χ -semi-invariant if $g \cdot m = \chi(g) \cdot m$ for every $g \in G$.

Definition 5.2. Let $G = \text{Spec } A$ be an affine R -monoid scheme and let $\chi: G \rightarrow \mathcal{R}$ be a multiplicative character. We will call the 1-form $w_\chi \in A^*$ which is left and right χ -semi-invariant and such that $w_\chi(\chi) = 1$, if it exists, a χ -semi-invariant integral on G .

If a χ -semi-invariant integral exists, then it is unique: Observe that $w \cdot w_\chi = w(\chi) \cdot w_\chi$, because $g \cdot w_\chi = \chi(g) \cdot w_\chi$ for every $g \in G$, since w_χ is left χ -semi-invariant. Likewise, $w_\chi \cdot w = w(\chi) \cdot w_\chi$. Given a left χ -semi-invariant $w \in A^*$ such that $w(\chi) = 1$, one concludes that $w = w_\chi(\chi) \cdot w = w_\chi \cdot w = w(\chi) \cdot w_\chi = w_\chi$.

Given a functor \mathbb{M} of G -modules we will define \mathbb{M}^χ to be the functor $\mathbb{M}^\chi(S) := \{m \in \mathbb{M}(S) : g \cdot m = \chi(g) \cdot m, \text{ for every } g \in G(T), \text{ and every } S\text{-algebra } T\}$.

Proposition 5.3. Let $G = \text{Spec } A$ be an affine R -monoid scheme and let $\chi: G \rightarrow \mathcal{R}$ be a multiplicative character. $\mathcal{A}^* = \mathcal{R} \times \mathcal{B}^*$ as functors of \mathcal{R} -algebras, where the projection onto the first factor is χ if and only if there exists the χ -semi-invariant integral on G .

Proof. If there exists the χ -semi-invariant integral on G , w_χ , then $\mathcal{A}^* = w_\chi \cdot \mathcal{A}^* \times (1 - w_\chi) \cdot \mathcal{A}^* = \mathcal{R} \times \mathcal{B}^*$ as functors of \mathcal{R} -algebras, where the projection onto the first factor is χ . Conversely, if $\mathcal{A}^* = \mathcal{R} \times \mathcal{B}^*$, where the first projection $\mathcal{A}^* \rightarrow \mathcal{R}$ is χ , then $w_\chi = (1, 0) \in \mathcal{R} \times \mathcal{B}^*$. \square

By Corollary 2.5 we obtain the following theorem.

Theorem 5.4. *Let G be an affine monoid scheme and let $\chi: G \rightarrow G_m$ be a multiplicative character. G is invariant exact if and only if there exists the χ -semi-invariant integral on G .*

Likewise as in Proposition 3.2, we obtain the following result.

Proposition 5.5. *Let $G = \text{Spec } A$ be an affine R -monoid scheme and assume there exists the χ -semi-invariant integral on G , $w_\chi \in \mathcal{A}^*$. Let \mathbb{M} be a separated functor of \mathcal{A}^* -modules. It holds that:*

- (1) $\mathbb{M}^\chi = w_\chi \cdot \mathbb{M}$.
- (2) \mathbb{M} splits uniquely as a direct sum of \mathbb{M}^χ and another subfunctor of G -modules, explicitly

$$\mathbb{M} = w_\chi \cdot \mathbb{M} \oplus (1 - w_\chi) \cdot \mathbb{M}.$$

We call the morphism $\mathbb{M} \rightarrow \mathbb{M}^\chi$, $m \mapsto w_\chi \cdot m$, the Reynolds χ -operator.

Example 5.6. *Let $G = \text{Spec } A$ be an affine R -monoid scheme and let $\chi: G \rightarrow \mathcal{R}$ be a multiplicative character. An Ω -process associated to χ (see [5, 3.1]) is a nonzero linear operator $\Omega: A \rightarrow A$ such that*

$$\Omega(a \cdot g) = \chi(g) \cdot (\Omega(a) \cdot g); \quad \Omega(g \cdot a) = \chi(g) \cdot (g \cdot \Omega(a))$$

for all $a \in A$ and $g \in G$. The composite morphism

$$A \xrightarrow{\Omega} A \xrightarrow{\chi} A$$

is a morphism of left and right G -modules: $(\chi \cdot \circ \Omega)(g \cdot a) = \chi \cdot \chi(g) \cdot (g \cdot \Omega(a)) = (g \cdot \chi) \cdot (g \cdot \Omega(a)) = g \cdot (\chi \cdot \Omega(a)) = g \cdot ((\chi \cdot \circ \Omega)(a))$, likewise we prove that $\chi \cdot \circ \Omega$ is a morphism of right G -modules. Since

$$\text{Hom}_{\text{left-right } G\text{-modules}}(A, A) = \text{Hom}_{\text{left-right } \mathcal{A}^*\text{-modules}}(\mathcal{A}^*, \mathcal{A}^*) = Z(\mathcal{A}^*)$$

($Z(\mathcal{A}^*)$ is the center of \mathcal{A}^*), then $\chi \cdot \circ \Omega = z \cdot$ for some $z \in Z(\mathcal{A}^*)$. If G is a linearly reductive monoid and R is an algebraically closed field, then \mathcal{A}^* is a semisimple algebra scheme and $\mathcal{A}^* = \prod_{E_i \in I} \text{End}_R(E_i)$ where I is the set of irreducible representations of G (up to isomorphism), by [1, 6.2, 6.8], hence $\chi \cdot \circ \Omega \in Z(\mathcal{A}^*) = \prod_{i \in I} R$ (on the other hand, see [5, 4.4]).

Assume now that $0 \in G$ (that is an element such that $0 \cdot g = g \cdot 0 = 0$ for all $g \in G$) and that $\Omega(\chi) = 1$ (generally $\chi \cdot \Omega(\chi) = z \cdot \chi = \chi(z) \cdot \chi = \chi \cdot \chi(z)$, $\chi(z) \in R$). The projection $w: A \rightarrow R$, $a \mapsto a(0)$ is left and right invariant and $w(1) = 1$, then G is an invariant exact R -monoid and $w = w_G$. The composite morphism $w' = w_G \circ \Omega$ is left and right χ -semi-invariant and $w'(\chi) = 1$, then $w' = w_\chi$. Given a rational G -module M , let us calculate the Reynolds χ -operator of M , that is, the morphism $M \rightarrow M$, $m \mapsto w_\chi \cdot m$ (on the other hand, see [5, 5.1]). The dual morphism of the multiplication morphism $\mathcal{M}^* \otimes \mathcal{A}^* \rightarrow \mathcal{M}^*$ is the comultiplication morphism $\mu: M \rightarrow M \otimes A$. If $\mu(m) = \sum_l m_l \otimes a_l$, then $g \cdot m = \sum_l a_l(g) \cdot m_l$, for all $g \in G$. Hence,

$$w_\chi \cdot m = \sum_l a_l(w_\chi) \cdot m_l = \sum_l w_\chi(a_l) \cdot m_l = \sum_l \Omega(a_l)(0) \cdot m_l.$$

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DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD DE EXTREMADURA, AVENIDA DE ELVAS S/N,
06071 BADAJOZ, SPAIN
E-mail address: `aalarma@unex.es`

DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD DE SALAMANCA, PLAZA DE LA MERCED 1-4,
37008 SALAMANCA, SPAIN
E-mail address: `mplu@usal.es`

DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD DE EXTREMADURA, AVENIDA DE ELVAS S/N,
06071 BADAJOZ, SPAIN
E-mail address: `sancho@unex.es`