Logic, Algebra, Geometry group of CIMA at Évora

Imme van den Berg, Coordinator

January 13, 2016

1 Overview of the group

1. Logic

   (a) (José Carmo, currently rector of University of Madeira: Model Logic).

   (b) Imme van den Berg, Júlia Justino (Politécnico Setúbal) with Tran Van Nam (Kantum, Vietnam, Erasmusmundus Ph-student): Nonstandard Analysis.

2. Algebra

   Manuel Branco: Numerical Semigroups.

3. Geometry

   (a) Rui Albuquerque: Differential Geometry.

   (b) Pedro Marques: Algebraic Geometry.

1.1 Numerical Semigroups, Manuel Branco.

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Thesis supervisor: J.C. Rosales, Granada, collaboration continues.

Frobenius coin problem,

A numerical semigroup is a subset S of $\mathbb{N}$ closed under addition, it contains the zero element and has finite complement in $\mathbb{N}$. Given a nonempty subset A of $\mathbb{N}$ we will denote by $\langle A \rangle$ the submonoid of $(\mathbb{N}, +)$ generated by A, that is,

$$\langle A \rangle = \left\{ \begin{array}{c} \lambda_1 a_1 + \cdots + \lambda_n a_n \mid n \in \mathbb{N} \backslash \{0\}, \\ a_i \in A, \lambda_i \in \mathbb{N} \text{ for all } i \in \{1, \ldots, n\} \end{array} \right\}.$$

If $S = \langle A \rangle$ and there exists no proper subset of A that generates S we say that A is a minimal system of generators for S.

Two invariants have special relevance to a numerical semigroups: the greatest integer that does not belong to S, called the Frobenius number of S denoted by $\mathsf{F}(S)$, and the cardinality of $\mathbb{N}\backslash S$, called the genus of S denoted by $\mathsf{g}(S)$.

The *Frobenius coin problem*, or the *linear Diophantine problem of Frobenius*, consists in finding a formula, in terms of the elements in a minimal system of generators of S, for computing $\mathsf{F}(S)$ and $\mathsf{g}(S)$; this is solved in the case of $|S| = 2$.

Rosales and Branco study the Frobenius number, the type and the genus for some classes of numerical semigroups with three or more generators: MED-semigroups, Mersenne numerical semigroups, Thabit numerical semigroups, Repunit numerical semigroups and numerical semigroups with minimal set of generators $\{4 < n_2 < n_3\}$.

Some references:

1. Rosales, J.C., Branco M.B.: *Numerical semigroups that can be expressed as an intersection of symmetric numerical semigroups*, J. Pure and Applied Algebra, 171, 303-314 (2002)

2. Rosales, J.C., Branco M.B.: *Irreducible numerical semigroups*, Pacific J. Math. 209, 131–143, (2003)

3. Rosales, J.C. and Branco, M.B.: *The Frobenius problem for numerical semigroups with multiplicity four*, Semigroup Forum 83, no3, 468-478, (2011)

4. Rosales, J.C., Branco, M.B.: *Irreducible numerical semigroups*, Pacific J. Math. 209, no. 1, 131-143. 20M14, (2003)

5. Rosales, J.C., Branco, M.B., Torrão, D.: *The Frobenius problem for Thabit numerical semigroups*, J. of Number Theory 155, 85-99, (2015)

6. Rosales, J.C., Branco. M.B., Torrão, D.: *The Frobenius problem for Repunit numerical semigroups*, The Ramanujan Journal, 1-12, (2015)

7. J. C. Rosales, J.C., Branco, M.B., Torrão, D,: *The Frobenius problem for Mersernne numerical semigroups*, submitted

2 Differential Geometry, Rui Albuquerque.

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On a fundamental differential system of Riemannian geometry.

Riemannian manifold, sphere bundle, calibration, exterior differential system, Euler-Lagrange system, hypersurface theory.

The discovery of an exterior differential system (EDS) naturally associated with every oriented Riemannian $n+1$-manifold M is being recognized. Somehow it completes the well-known contact tangent structure. I.e. that which is given by a natural 1-form θ, such that $\theta \wedge d\theta^n \neq 0$ on the unit tangent sphere bundle SM (The manifold SM inherits a Riemannian structure.)

The EDS consists of $n+1$ global invariant n-forms $\alpha_0, \ldots, \alpha_n \in \Omega^n$. Their definition requires merely the orientation, volume and a splitting property.

In case $n = 1$, so that M is a surface, we have 1-forms θ, α_0 and α_1. The following was known to Cartan, where K denotes the Gauss curvature of M:

$$d\theta = \alpha_0 \wedge \alpha_1, \quad d\alpha_1 = K\alpha_0 \wedge \theta, \quad d\alpha_0 = \theta \wedge \alpha_0.$$

For 3-manifolds we have now four pairwise orthogonal 2-forms $\alpha_0, \alpha_1, \alpha_2$ and $d\theta$, satisfying:

$$\begin{aligned} {}^*\theta &= \alpha_0 \wedge \alpha_2 = -\frac{1}{2}\alpha_1 \wedge \alpha_1 = -\frac{1}{2}d\theta \wedge d\theta \\ d\alpha_0 &= \theta \wedge \alpha_1, \quad d\alpha_1 = 2\theta \wedge \alpha_2 - r\theta \wedge \alpha_0, \\ d\alpha_2 &= \mathcal{R}(\alpha_2), \quad dd\theta = 0. \end{aligned}$$

$r = ric(u,u)$, $u \in SM$, is a function and the 3-form $\mathcal{R}(\alpha_2)$ is also a curvature tensor.

The $n+1$ natural n-forms on SM have more complex $d\alpha_i$.

Applications:

So far: the study of pure differential structures of Riemannian geometry, and a new approach to hypersurface theory.

E.g. in dim 4: a natural G_2 structure on SM is associated to any given oriented Riemannian 4-manifold M! The space is called G_2-twistor space. Cocalibrated if and only if M is Einstein. Its fundamental 3-form is

$$\phi = \theta \wedge d\theta + \alpha_1 - \alpha_3,$$

and there are many open questions in this little field of research.

Extensions to dim $n + 1$.

This work was supported by Pierre and Marie Curie-grant and took the author among others to Turin and IHES, Paris.

References:

R. Albuquerque "Curvatures of weighted metrics on tangent sphere bundles". Rivista di Matematica della Università di Parma. Vol 2, Num 2, 229-313 (2011)

R. Albuquerque "Weighted metrics on tangent sphere bundles". Quarterly Journal of Mathematics, Vol 63, Issue 2 (2012)

R. Albuquerque "On the characteristic connection of gwistor space", Cent. Eur. J. Math., 2013, 11(1), pp.149-160 (2013)

R. Albuquerque "Variations of gwistor space", Portugaliae Mathematica, Volume:70, Issue:2, pp. 145-160 (2013)

3 Algebraic Geometry, Pedro Marques

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Thesis supervisor: R. M. Miró-Roíg, Barcelona, collaboration continues.

Syzygy bundles: The study of vector bundles over the projective space known as syzygy bundles, which are constructed from the relations among a set of homogeneous polynomials. They are defined as a kernel of a morphism of sums of line bundles. One can get a good deal of information on syzygy bundles form what we know about line bundles. The study of vector bundles over an algebraic variety provides information on the variety itself. The goal is to obtain classification results in some settings.

Monads: A more general construction from sums of line bundles yields monads over algebraic varieties, which have proved to be a powerful way of obtaining new vector bundles. When studying monads, one is interested in establishing their existence and on questions on the stability of the vector bundles they may provide, a property that plays an essential role when it comes to constructing moduli spaces. Given a smooth projective variety X over an algebraically closed field K of characteristic 0, a *monad* on X is a complex

$$M_\bullet \colon 0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

of coherent sheaves on X, with α an injective map and β surjective. The coherent sheaf $E := \ker \beta / im\alpha$ is called the *cohomology (sheaf)* of the monad $M_\bullet$. Monadshave proved to be very useful objects for constructing vector bundles and studying their properties.

Artinian Gorenstein local rings: The rank of a homogeneous polynomial extends what we know from linear algebra to be the rank of a quadratic form. Artinian Gorenstein local rings are a central object of study when tackling these problems. A local ring (R, m, k) is Artinian if it is finitely generated as a $k-$module, and is Gorenstein if it has a one-dimensional socle (the annihilator of the maximal ideal m). The study is closely related to the problem of representing a homogeneous polynomial as a sum of powers of linear forms. Extension of the well-known study of decomposition of quadratic froms in a sum of powers.

Marques is closely related to a group of young researchers in Algebraic Geometry, and spends each year half of his time abroad (Seoul, Boston, Campinas - Brasil, Leuven).

References:

P. M. Marques, L. Oeding “Splitting criteria for vector bundles on the symplectic isotropic Grassmannian”, Le Matematiche (Catania), 64, no. 2, 155-176 (2009)

P.M. Marques, R.M. Miro-Roig “Stability of syzygy bundles”, Proceedings of the American Mathematical Society, V.139, Issue: 9, Pages: 3155-3170 (2011)

P.M. Marques, H. Soares “Monads on Segre varieties”, Boletim da Sociedade Portuguesa de Matemática, Special Issue 2013, 83-86 (2013)

P.M. Marques, H. Soares "Cohomological characterization of monads", Mathematische Nachrichten (2014)

3.1 Nonstandard Analysis, Imme van den Berg, Júlia Justino, with Bruno Dinis, Tran Van Nam, (ex-)PhD students.

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Infinitesimal discretisations: The continuum $\mathbb{R}$ may be imitated by a discrete set of infinitesimally spaced points, say $\mathbb{T} = \{k\delta t \,|k \in \mathbb{N}\}$. Here $\delta t > 0$, $\delta t \simeq 0$ (existence in $\mathbb{R}$ guaranteed by axiom). So analysis or continuous-time stochastics may be imitated by discrete, even finite mathematics.

Examples: Higher-order discrete chain rule (analogue to Faà di Bruno Theorem), I. van den Berg, Discretisations of higher order and the theorems of Faà di Bruno and DeMoivreLaplace, Journal of Logic and Analysis, Vol 5:6 (2013) 1–35 (2013)

Brownian Motion on $[0, T]$ imitated by the finite Wiener Walk W_t on $[0, T] \cap \mathbb{T}$, given by

$$\delta W_t = \begin{cases} \sqrt{\delta t} & \text{probability } \frac{1}{2} \\ -\sqrt{\delta t} & \text{probability } \frac{1}{2}, \end{cases}$$

Lead among others to a simplified presentation of Financial Mathematics, for Black-Scholes theory can be treated entirely within the finite Cox-Ross-Rubinstein model.

Book: I.P. van den Berg, *Principles of infinitesimal stochastic and financial analysis*, World Scientific, Singapore, 146+xii p. (2000); presently serves in the masters-course on Stochastics and Financial Mathematics at the University of Évora, audience usually includes students from Business and Economy.

Orders of magnitude: Classically orders of magnitude are treated functionally: $O(\epsilon), o(\epsilon)$, with $\epsilon \to 0$. With $\epsilon \in \mathbb{R}$, $\epsilon > 0$ infinitesimal we can define orders of magnitude within $\mathbb{R}$:

$$\begin{aligned} \pounds &= \{x \in \mathbb{R} \,||x| \text{ less than some standard number}\} \\ \oslash &= \{x \in \mathbb{R} \,|\; |x| \text{ infinitesimal}\} \\ & \pounds\epsilon, \oslash\sqrt{\epsilon}, \pounds\epsilon^2 \text{ etc., } \epsilon \text{ some fixed infinitesimal} \end{aligned}$$

These "external" sets are convex subgroups of $\mathbb{R}$, called *neutrices*. An *external number* is the sum of a real number and a neutrix. Rules of calculus are almost those of a nonarchimedean ordered field, called *solid*, and include a form of Dedekind Completeness.

Characterized in:

I.P. van den Berg, *Nonstandard Asymptotic Analysis*, Springer Lecture Notes in Mathematics 1249, 187+ xi p. (1987)

F. Koudjeti, I.P. van den Berg, *Neutrices, external numbers and external calculus*, in: Nonstandard Analysis in Practice, F. and M. Diener (eds.), Springer Universitext (1995) 145-170

On the quotient class of non-archimedean fields, Bruno Dinis (Lisboa), I.P. van den Berg, submitted

Characterization of distributivity in a solid, Bruno Dinis, I.P. van den Berg, submitted

Axiomatics for the external numbers of nonstandard analysis, Bruno Dinis, I.P. van den Berg, in preparation

Applications of external numbers:

1. Propagation of errors in matrix calculus, including Gauss-Jordan solution of linear systems.

J. Justino, I.P. van den Berg, *Cramer's Rule applied to flexible systems of linear equations*, Electronic Journal of Linear Algebra, 24, p. 126-152 (2012).

Júlia Justino, *Nonstandard Linear algebra with error analysis*, PhD thesis, Universidade de Évora, 2013

Nam Tran Van, PhD thesis, in preparation

2. Optimization with numerical uncertainties, Nam Tran Van, PhD thesis, in preparation.