Algebraic and Differential Geometry. Computational Algebra.

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University of Extremadura, Badajoz, Spain

January 14th, 2016

FIRST JOINT MEETING ÉVORA-EXTREMADURA ON MATHEMATICS

The research group GADAC

web page http://www.unex.es/investigacion/grupos/gadac

Group Coordinator **IGNACIO OJEDA**

Research Lines of IGNACIO OJEDA

- Semigroup algebras. Toric geometry.
- Classification of projective curves.

Group Members and Research lines

- **Amelia Álvarez** – Schemes algebras and their representations.
- **Teresa Arias-Marco**–Homogeneous spaces with additional geometric structures.
- **Adrián Gordillo**–Natural tensors, differential invariants and Riemannian metrics.
- **José Navarro**–Natural tensors, differential invariants and Riemannian metrics.
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Homogeneous spaces with additional geometric structures

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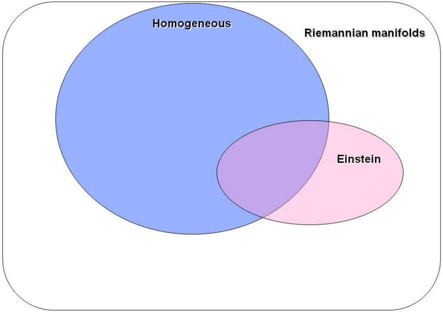
Einstein manifolds and Local homogeneity

Definition

(M, g) is an Einstein manifold if **ric** $= C \cdot g$ *for a certain constant* C *at each point of* M.



Definition.

(M, g) is an *homogeneous Riemannian manifold* if $\mathcal{I}(M)$ acts transitively on M.

Definition

(M, g) is locally homogeneous if the pseudogroup of local isometries acts transitively on M.

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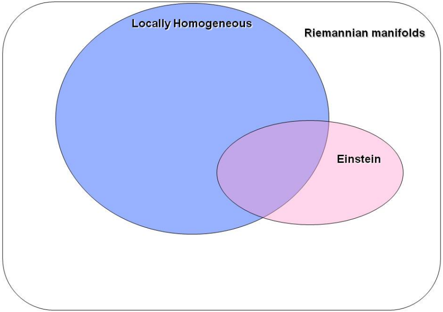

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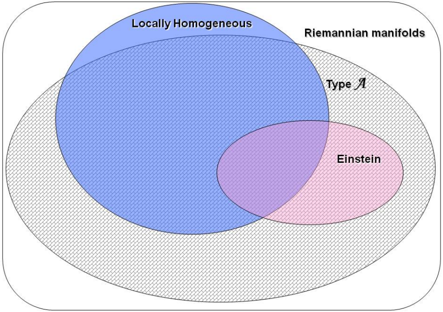
Spaces of type $\mathcal{A}$

Definition (A. Gray, 1978)

(M, g) is of type $\mathcal{A}$ ($T\mathcal{A}m$) if **ric** *is cyclic-parallel; i.e.* $(\nabla_X \mathbf{ric})(X, X) = 0$ *for all $X \in TM$.*



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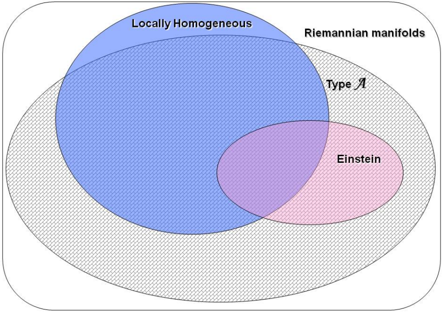

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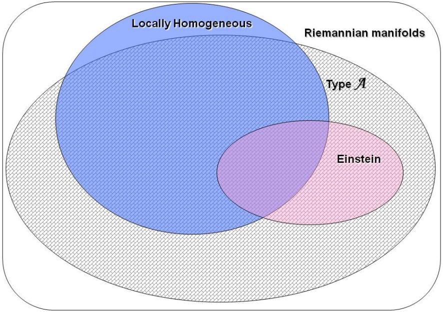

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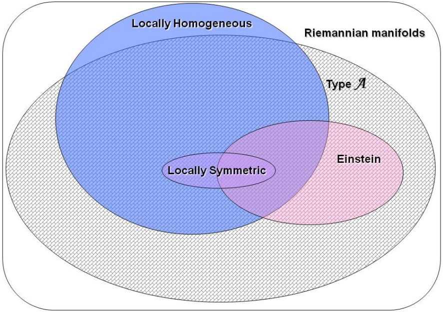
Symmetric spaces

Definition. s_p at $p \in M$ is a *local geodesic symmetry* if for all $X \in N(p)$ in T_pM, $s_p(\exp_p(X)) = \exp_p(-X)$.

Definition.

(M, g) is locally symmetric if for all $p \in M$, s_p is an isometry. $\equiv \nabla R = 0$.



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(M, g) is a Riemannian symmetric space if for all $p \in M$, s_p is globally defined and it is an isometry.

$\Rightarrow$ Riemannian symmetric spaces are spaces with metric - preserving geodesic symmetries.

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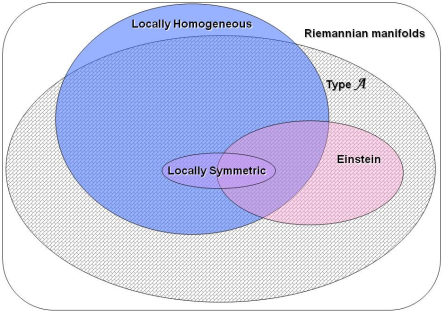

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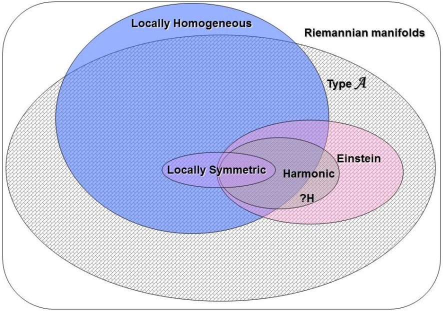
Harmonic Spaces

Definition

(M, g) *is an harmonic space iff*

- *the volume density function of the geodesic exponential map is radial around each point.*
- *at each $m \in M$ there exists a normal neighborhood of M on which $\Delta u = 0$ admits a real solution depending only upon the distance r to m and being analytic for $r \neq 0$.*



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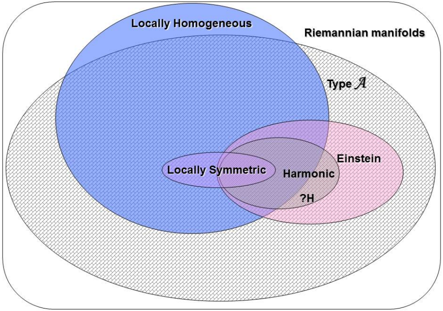

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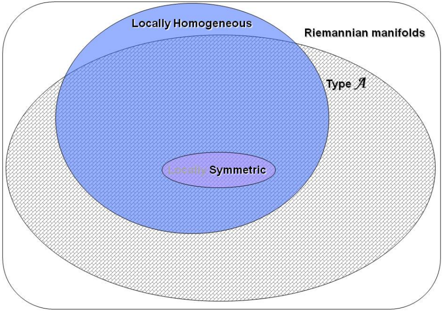
Weakly symmetric spaces

Definition (Selberg, 56, Szabó, 93)

(M, g) is called weakly symmetric if each $p \in M$ and each nontrivial γ starting in p there exists an isometry of M which fixes p and reverses γ.



- Berndt,Vanhecke,96. Weakly sym.examples non-symmetric.
- **Classifications**
  3, 4-dim by Berndt,Vanhecke,96.
  5-dim by Kowalski,Marinosci,97.
- **Weakly symm. spaces are:**
  Homogeneous by Selberg,56.
  Type $\mathcal{A}$ by Kowalski, Prüfer,89.
- **Remark.**

If a Riemannian covering of (M, g) is weakly symmetric

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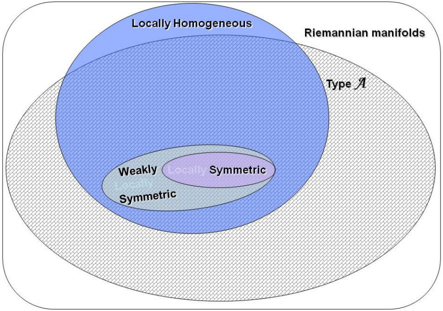

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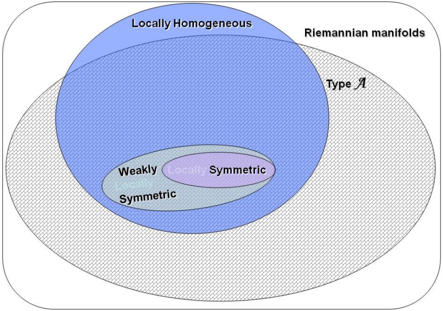

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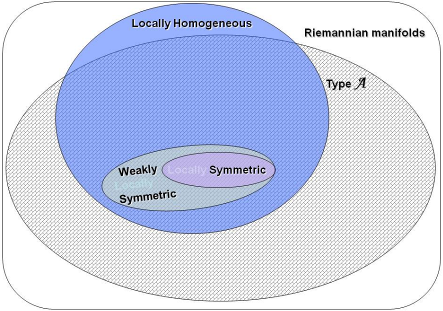
(M, g) is weakly locally symmetric if $\forall p \in M$, $\exists \varepsilon > 0$ such that $\forall \gamma$ in M, $\gamma(0) = p$, $\exists$ an isometry of $B_\varepsilon(p)$ which fixes p and reverses $\gamma|_{(-\varepsilon,\varepsilon)}$.



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- **Lemma.** Let (M, g) be complete, simply connected and, weakly locally symmetric $\Rightarrow$ M is weakly sym.
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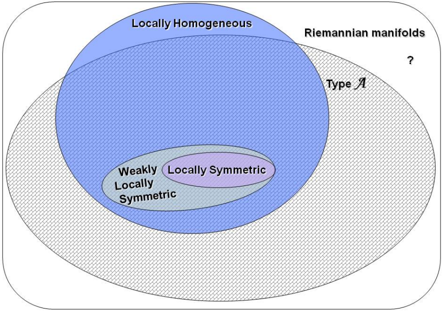

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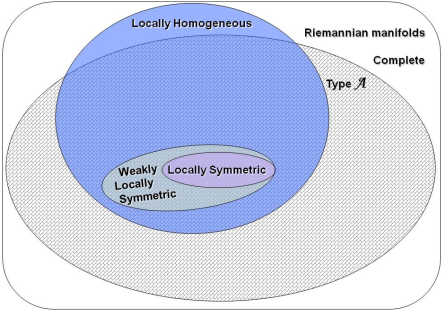

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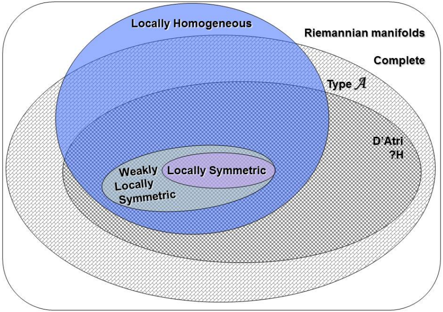
Others symmetric-like manifolds

Definition (D'Atri,Nickerson,69,74)

(M, g) is called D'Atri space if the local geodesic symmetries are Riemannian volume-preserving.



On complete Riemannian manifolds

Weak local symmetry

$\Downarrow$

- D'Atri property,
- the $\mathfrak{C}$ property,
- probabilistic commutativity,
- the $\mathfrak{TC}$ property,
- the $\mathfrak{GC}$ property,

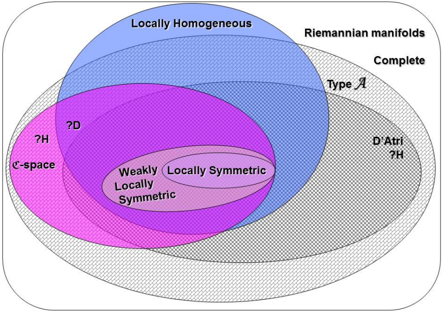
$\Downarrow$

The type $\mathcal{A}$ property.

Others symmetric-like manifolds

Definition (Berndt,Vanhecke,92)

(M, g) is called $\mathfrak{C}$-space if for each γ the eigenvalues of the associated Jacobi operator are constant along γ.



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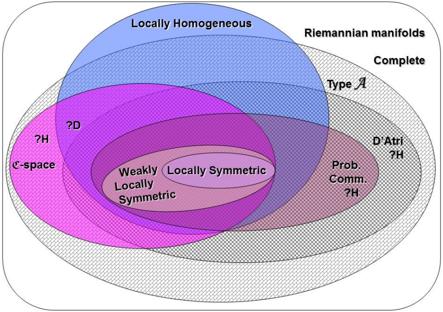
Others symmetric-like manifolds

Definition (Kowalski, Prüfer, 82, 89)

(M, g) is probabilistic commutative if all Euclidean Laplacian $\tilde{\Delta}^{(k)}$, $k \in \mathbb{N}$, commute.



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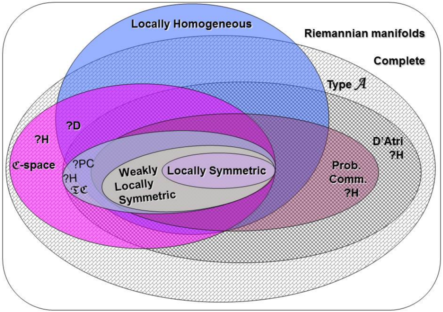
Definition (Berndt, Vanhecke, 1993)

(M,g) *is a $\mathfrak{TC}$-space if* $\forall m \in M$ *and* $\forall p \in M$ *suff. close to m,* $T_p(m)$ *and* $T_{s_m(p)}(m)$ *have the same eigenvalues*



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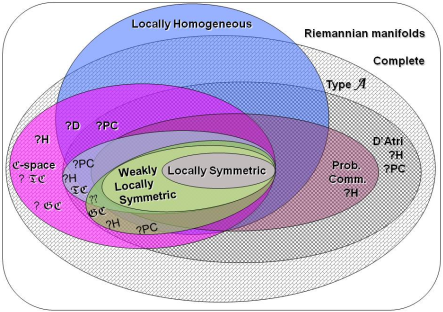
(M, g) is a $\mathfrak{G}\mathfrak{C}$-space if $\forall m \in M$ and $\forall p \in M$ suff. close to m, $T_m(p)$ and $s_{m^*}^{-1} \circ T_m(s_m(p)) \circ s_{m^*}$ have the same eigenvalues



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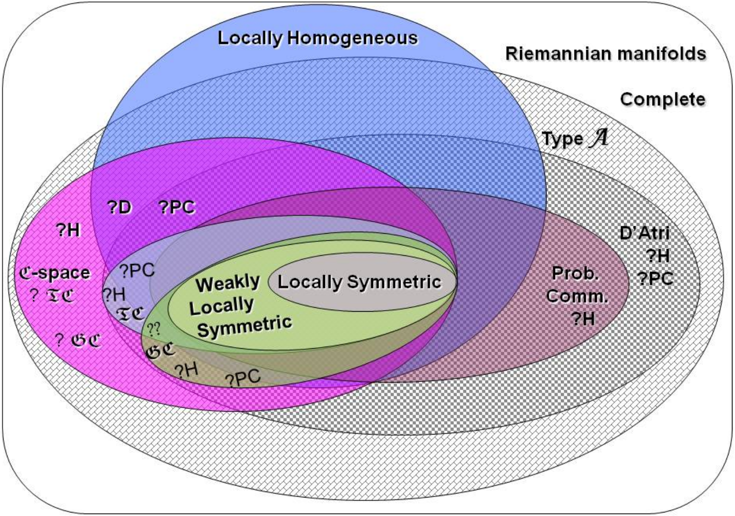
Open problems on symmetric-like manifolds



Eigenvalues problems for the Laplace-Beltrami operator

Let (M, g) be a compact, connected, n-dimensional smooth Riemannian manifold, possibly with boundary.

Let $\Delta = -\mathbf{div}\,\mathbf{grad}$ be the Laplace operator associated with g, acting on functions.

Eigenvalues Problems: Find all real numbers λ for which there exists a nontrivial solution $f \in \mathcal{C}^2(M)$ to $\Delta f + \lambda f = 0$ when

- $\partial M = \emptyset$. Namely, closed eigenvalue problem.
- $f = 0$ *on* ∂M. Namely, Dirichlet eigenvalue problem.
- $\partial_\nu f = 0$ *on* ∂M where ν is the inward unit normal vector field on ∂M. Namely, Neumann eigenvalue problem.

Throughout our work, (M, g) will be a closed manifold.

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- $\partial_\nu f = 0$ *on* ∂M where ν is the inward unit normal vector field on ∂M. Namely, Neumann eigenvalue problem.

Throughout our work, (M, g) will be a closed manifold.

The spectrum of Δ and the geometry of M

Definition. The **heat kernel** $K(t,x,y)$ is the fundamental solution of $(\frac{\partial}{\partial t}+\Delta)f=0$. $K(t,x,y)$ is analytic in $t>0$ and $\mathcal{C}^\infty$ in $(x,y)\in M\times M$.

The Sturm-Liouville decomposition:
$\exists$ a complete orthonormal basis $\{\varphi_0,\varphi_1,....\}$ of $\mathcal{C}^\infty(M)$, with φ_j having eigenvalue λ_j of Δ satisfying $0\leq\lambda_0\leq\lambda_1\leq\lambda_2\leq\cdots\to\infty$. Each eigenvalue has finite multiplicity. Finally,

$$K(t,x,y)=\sum_{j=0}^{\infty}e^{-\lambda_j t}\varphi_j(x)\varphi_j(y)$$

with convergence absolute, and uniform, for each $t>0$. In particular,

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Minakshisundaram-Pleijel Asymptotic Expansion, 1949:

$$K(t,x,x) \backsim (4\pi t)^{\frac{-n}{2}} \sum_{k=0}^{\infty} u_k(x) t^k$$

where $u_k(x)$ are $\mathcal{C}^\infty(M)$ and polynomials in the components of the curvature tensor $\mathbf{R}$ and its covariant derivatives. Moreover,

$$\sum_{j=0}^{\infty} e^{-\lambda_j t} \backsim (4\pi t)^{\frac{-n}{2}} \sum_{k=0}^{\infty} a_k t^k,$$

where the coefficients a_k are called classical heat invariants. In particular,

$$a_0 = \int_M \omega = Vol(M,g), \quad a_1 = \frac{1}{6}\int_M \tau\omega,$$

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The spectrum of Δ and the geometry of M

$\Rightarrow$ **The spectrum of Δ depends only on the Riemannian structure of (M, g).**

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Direct and inverse problems on spectral theory.

Direct: To use geometrical properties of M to determine information about the eigenvalues and eigenfunctions of Δ.

Theorem (Lichnerowicz)

(M, g) closed and **ric** $\geq (n-1)k > 0$*, for* $k \in \mathbb{N}$*.*
Then, the first positive eigenvalue λ *satisfies* $\lambda \geq nk$*.*

Inverse: one assumes knowledge about $Spec(\Delta, M)$ and attempts to determine information about the geometry of (M, g).

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Inverse problems in spectral geometry

To which extent does the spectrum of Δ on (M, g) determine its geometry?

Important fact:

If (M, g) is isometric to $(M', g') \Rightarrow Spec(\Delta, M) = Spec(\Delta', M')$.

Natural question:

Does $Spec(\Delta, M) = Spec(\Delta', M')$ imply M isometric to M'**?**

Negative answer, 1964: Milnor's example(a pair of 16-dim flat tori)

Famous question by M. Kac, 1966:

Can one hear the shape of a drum**?**

Partial positive result: $Spec(\Delta, M) = Spec(\Delta', a\ ball)$ implies M is a ball of the same radius so isometric to that ball.

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Basic definitions

Definition

Two compact Riemannian manifolds (M, g) and (M', g') are said to be isospectral if $Spec(\Delta, M) = Spec(\Delta', M')$.

Definition

A geometric property of a compact Riemannian manifold M can be heard if it can be determined from $Spec(\Delta, M)$.

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A geometric property is inaudible, i.e. not determined by the spectrum, if there exist pairs of isospectral manifolds which differ with respect to this property.

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Some Positive and Negative Results

Some positive results:

- Heat invariants can be heard:

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$\Rightarrow$ Therefore, in dim 2, the total curvature and the Euler characteristic of the manifold can be heard.

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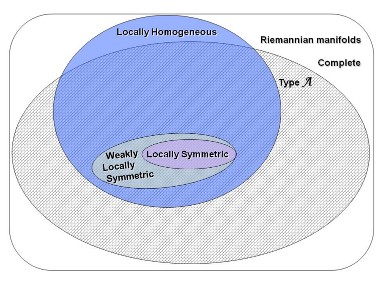
Our Results

Main Theorem (*Ann. Glob. Anal. Geom.* **37***(2010), 339–349.*)

Each of the following properties is an inaudible property of Riemannian manifolds:



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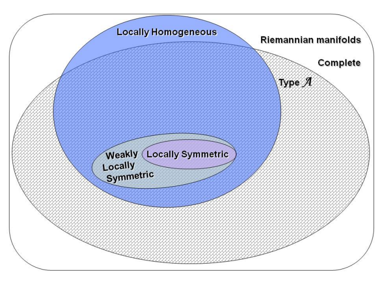

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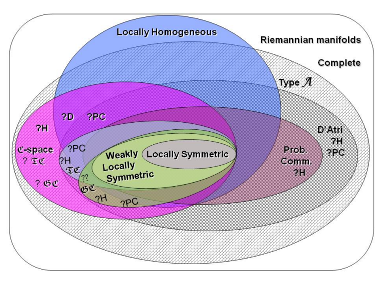

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An interesting question

Problem by B.Y. Chen, L. Vanhecke in 1981:

To what extent do the properties of sufficiently small geodesic spheres determine the Riemannian geometry of the ambient space?

$T_p(m)$:Shape op.(at m)of the geodesic sph. with center p, radius r

Characterization (Vanhecke, Willmore, 1983)

M is locally symmetric space iff $\forall m \in M$ and $\forall p \in M$ sufficiently close to m, $T_p(m) = T_{s_m(p)}(m)$.

Characterization (Ledger, Vanhecke, 1987)

M is locally symmetric space iff $\forall m \in M$ and $\forall p \in M$ sufficiently close to m, $s_{m*} \circ T_m(p) = T_m(s_m(p)) \circ s_m$.

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Question 1

Are the properties of being locally symmetric, Einstein or harmonic spectrally determined on closed manifold?

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To what extent do the spectra of small geodesic spheres in a (possibly noncompact) Riemannian manifold M determine the geometry of M?

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Our Question

Do the spectra of geodesic spheres distinguish symmetric $(\nabla R = 0)$ harmonic spaces from nonsymmetric $(\nabla R \neq 0)$ harmonic spaces?

Main Theorem (*Geom. Funct. Anal. (GAFA)* **22***(2012), 1–21.*)

Let M_1 and M_2 be harmonic spaces, and let $p_1 \in M_1$, $p_2 \in M_2$. If there exists $\varepsilon > 0$ such that for each $r \in (0, \varepsilon)$ the geodesic spheres $S_r(p_1)$ and $S_r(p_2)$ are isospectral, then

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Consequently,

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The Steklov problem

In 1895 V. A. Steklov posed the following problem:

Let M be a smooth manifold of dimension ≥ 2 with smooth boundary $bd(M)$. Find functions u on M and scalars $\sigma \in \mathbb{R}$ that satisfy,

$$\begin{aligned} \Delta u = 0 &\quad \text{in } M \\ \partial_\nu u = \sigma u &\quad \text{on } bd(M) \end{aligned}$$

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The Steklov problem, rephrased

The Steklov spectrum is the spectrum of the "Dirichlet-to-Neumann" operator,

$$\mathcal{D} : C^\infty(bd(M)) \to C^\infty(bd(M)),$$

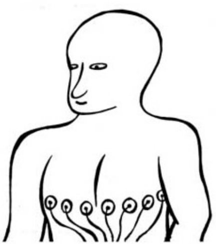
defined as follows.

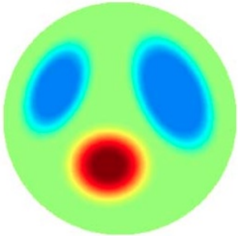
- Take $u \in C^\infty(bd(M))$.
- Let $\tilde{u}$ be the harmonic extension of u to M.
- $\mathcal{D}(u) = (\partial_\nu \tilde{u})|_{bd(M)}$.

Operator $\mathcal{D}$ is an elliptic pseudodifferential operator of order one with the same principal symbol as $\sqrt{\Delta^{bd(M)}}$.

Applied context: Electrical impedance tomography

In electrical impedance tomography electrical stimuli are placed around a body. Resulting surface voltages are recorded and used to infer the structure of objects inside the body.





This applied (numerical) technique relies on use of the Dirichlet-to-Neumann operator.

Algebraic and Differential Geometry. Computational Algebra.

Teresa Arias-Marco

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University of Extremadura, Badajoz, Spain

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FONDO EUROPEO DE DESARROLLO REGIONAL: UNA MANERA DE HACER EUROPA

January 14th, 2016

FIRST JOINT MEETING ÉVORA-EXTREMADURA ON MATHEMATICS