

Branching processes and their Applications

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I. del Puerto, R. Martínez, C. Gutiérrez, C. Minuesa



FIRST JOINT MEETING
ÉVORA-EXTREMADURA
ON MATHEMATICS



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Branching processes and their Applications

- Members:
 - Manuel Molina.
 - Miguel González.
 - Manuel Mota.
 - Alfonso Ramos.
 - Inés María del Puerto.
 - Rodrigo Martínez.
 - Cristina Gutiérrez.
 - Carmen Minuesa.
- Main lines of research:
 - Multitype branching processes.
 - Two-sex branching models.
 - Controlled branching processes.
 - Continuous-time branching processes.
- Workshops: On Branching Processes and their Applications, 2009, 2012, 2015 (<http://branching.unex.es/>).
- Website: <http://www.unex.es/investigacion/grupos/gipra>.

Contents

1 Standard branching process

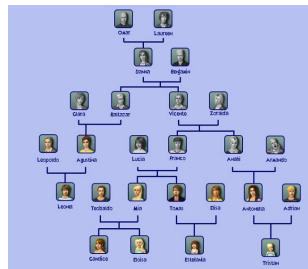
2 Main lines of research

- Multitype branching processes
- Two-sex branching processes
- Continuous-time branching processes in epidemiology
- Controlled branching processes

3 Current and future research

Background

- The probabilistic theory of branching processes was motivated by the study of the extinction of certain family lines of the European aristocracy from a scientist point of view in the 19th century.
- The standard branching process is called Bienaymè-Galton-Watson process and is named after Irénée-Jules Bienaymè, Francis Galton and Henry Watson, who defined it.
- This model has been applied in diverse fields such as Biology, Epidemiology, Genetics, Medicine, Demography, Economy, Algorithms and Data Structures etc.



Standard branching process

$\{Z_n\}_{n \geq 0}$, Z_n = number of individuals in the n -th generation.

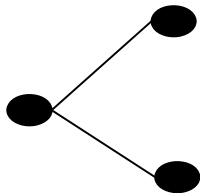
$n = 0$



Standard branching process

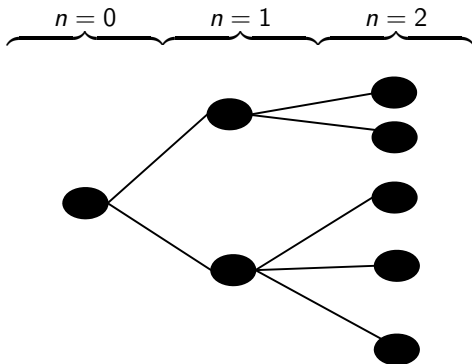
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$\underbrace{\hspace{10em}}_{n=0} \quad \underbrace{\hspace{10em}}_{n=1}$



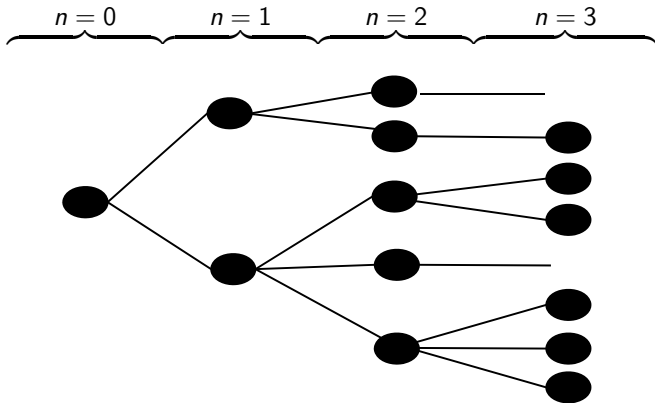
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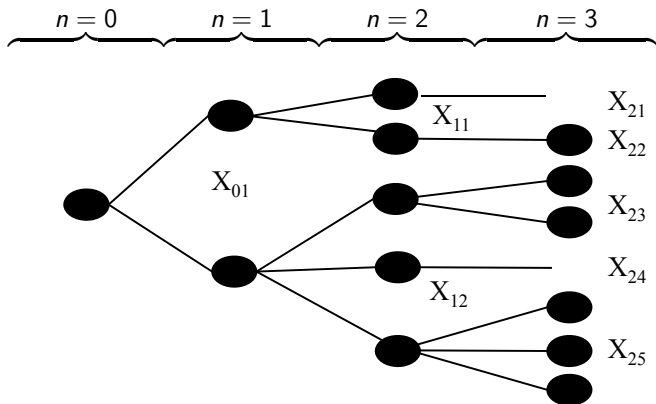
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Standard branching process

$\{Z_n\}_{n \geq 0}$, $Z_n =$ number of individuals in the n -th generation.



$$Z_0 = N, \quad Z_1 = \sum_{i=1}^{Z_0} X_{0i}, \quad Z_2 = \sum_{i=1}^{Z_1} X_{1i}, \quad Z_3 = \sum_{i=1}^{Z_2} X_{2i}.$$

Standard branching process

Definition

The **standard Branching Process** is the stochastic process $\{Z_n\}_{n \geq 0}$ defined as:

$$Z_0 = N \geq 0, \quad Z_{n+1} = \sum_{i=1}^{Z_n} X_{ni}, \quad n = 0, 1, \dots \quad \left(\sum_1^0 = 0 \right),$$

with $\{X_{ni} : n = 0, 1, \dots; i = 1, 2, \dots\}$ independent and identically distributed random variables (iid).

Standard branching process

- **Properties:**

- $\{Z_n\}_{n \geq 0}$ is an homogeneous Markov Chain with stationary transition probabilities.
- Duality Extinction-Explosion: $P(Z_n \rightarrow 0) + P(Z_n \rightarrow \infty) = 1$.

- **Modified branching processes**

- Age-dependent branching processes.
- Bisexual branching processes.
- Branching processes with immigration.
- Branching processes with varying environment.
- Branching processes with continuous state space.
- Controlled branching processes.
- Multitype branching processes.
- Population-size-dependent branching processes.



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Main contributions

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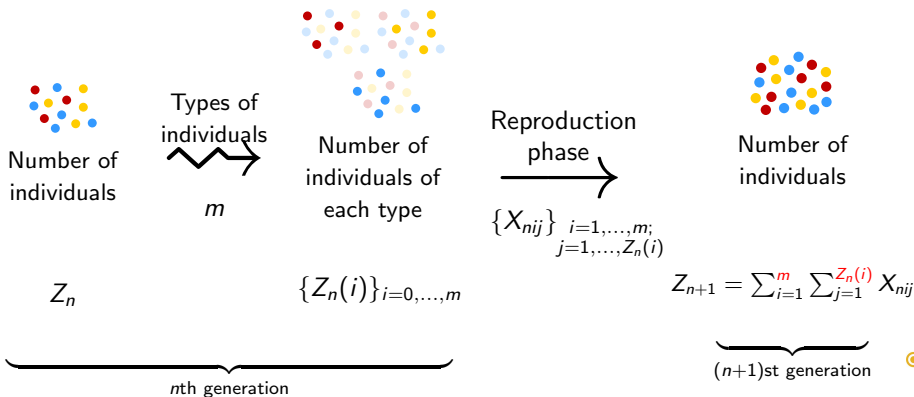
2 Main lines of research

- Multitype branching processes
- Two-sex branching processes
- Continuous-time branching processes in epidemiology
- Controlled branching processes

3 Current and future research

Multitype branching processes

- Discrete-time stochastic model (non-overlapping generations).
- There are **different types of individuals** in the population.



MBP: Contributions

- Growth Markovian Multitype Stochastic Processes:
 - Basic properties.
 - Extinction problem.
 - Asymptotic behaviour.
 - Population-size-dependent controlled multitype branching processes.

Contents

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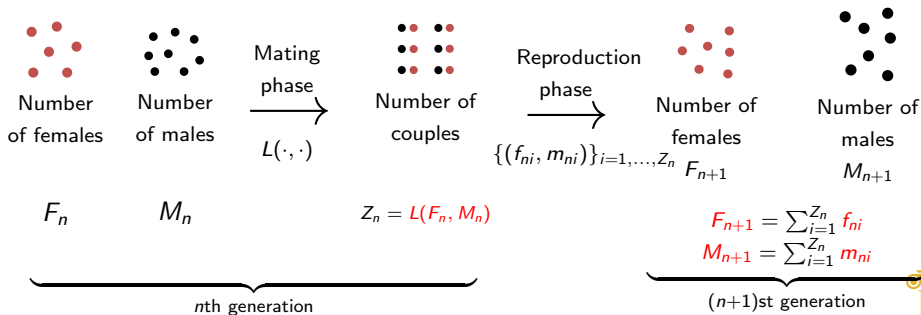
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Two-sex branching processes

- 2 types of individuals in the population: **males and females**.
- To model **sexual reproduction** in a population.
- 2 phases: $\left\{ \begin{array}{l} \text{Reproduction phase.} \\ \text{Mating phase.} \end{array} \right.$



2SBP: Contributions

- Basic properties.
- Extinction problem.
- Asymptotic behaviour.
- Frequentist and Bayesian estimation for modified two-sex models:
 - Two-sex branching processes with immigration.
 - Two-sex branching processes in varying environment.
 - Two-sex branching processes with random control on the number of progenitor couples
 - Two-sex branching processes with progenitor couples in a random environment.
 - Multitype two-sex branching model in a genetic context:
 - Y-linked two-sex branching processes.
 - X-linked two-sex branching processes.

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- **Continuous-time branching processes in epidemiology**
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Continuous-time branching processes in epidemiology

- Bellman-Harris (age-dependent) branching processes.
- Sevast'yanov branching processes.
- Crump-Mode-Jagers branching processes.
 - Modeling epidemic processes.
 - Modeling vaccination processes.
 - Studying optimum vaccination policies.

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Controlled branching processes

- Discrete-time stochastic model (non-overlapping generations).
- A **mechanism** establishes the **number of progenitors** in each generation.
- 2 phases: $\left\{ \begin{array}{l} \text{Reproduction phase.} \\ \text{Control phase.} \end{array} \right.$

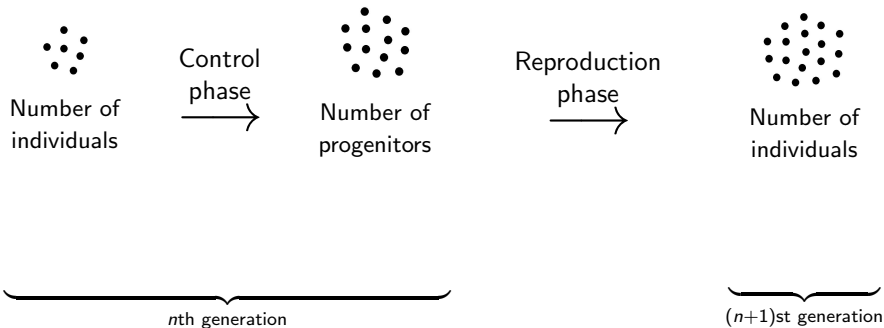
$$\{Z_n\}_{n \in \mathbb{N}_0}$$

Z_n = number of **individuals** in the n -th generation.

$\phi_n(Z_n)$ = number of **progenitors** in the n -th generation.

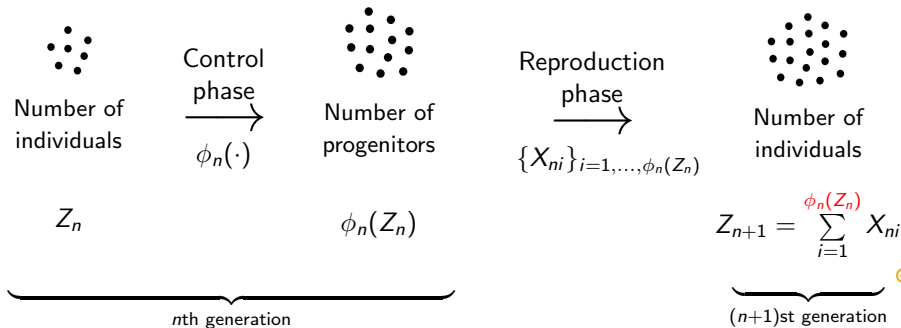
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CBP: Probability model

Main parameters of the model

- $p = \{p_k\}_{k \geq 0}$: **offspring distribution** or **reproduction law**.
- $m = E[X_{01}]$: **offspring mean**.
- $\sigma^2 = \text{Var}[X_{01}]$: **offspring variance**.
- $\varepsilon(k) = E[\phi_0(k)]$, $k = 0, 1, \dots$: **control mean**.
- $\sigma^2(k) = \text{Var}[\phi_0(k)]$, $k = 0, 1, \dots$: **control variance**.
- $\tau_m(k) = k^{-1} m \varepsilon(k) < \infty$: **mean growth rate**.

Classification of CBPs

- The process is **subcritical** if $\limsup_{k \rightarrow \infty} \tau_m(k) < 1$.
- The process is **critical** if $\liminf_{k \rightarrow \infty} \tau_m(k) \leq 1 \leq \limsup_{k \rightarrow \infty} \tau_m(k)$.
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CBP: Contributions to the main investigated topics

- **Extinction Problem:**
 - González, Molina, and del Puerto (2002, 2005a).
- **Asymptotic Behaviour. Growth rates:**
 - González, Molina, and del Puerto (2002, 2003, 2005a,b).
- **Statistical Inference:**
 - **Frequentist viewpoint: Method of moment, Likelihood method.**
 - González, Martínez, del Puerto (2004, 2005).
 - González, Minuesa, del Puerto, (2015).
 - **Frequentist viewpoint: WCLS estimator, bootstrap approximation.**
 - Sriram, Bhattacharya, González, Martínez, del Puerto (2007).
 - González, del Puerto (2012).
 - **Frequentist viewpoint: robust estimation.**
 - González, Minuesa, del Puerto, (2016-*submitted*).
 - **Bayesian viewpoint: Gibbs sampler, ABC methodology.**
 - González, Gutiérrez, Martínez, del Puerto (2013).
 - González, Gutiérrez, Martínez, Minuesa, del Puerto, (2016-*accepted*).
 - Martínez, Mota, del Puerto (2009).

CBP: current research

- **Assumption:** the offspring distribution belongs to a **parametric** family

$$\mathcal{F}_\theta = \{p(\theta) : \theta \in \Theta\}, \quad \Theta \subseteq \mathbb{R},$$

that is, $p = p(\theta_0)$, with $\theta_0 \in \Theta$.

- **Sample:** $\mathcal{Z}_n^* = \left\{ Z_l(k) = \sum_{i=1}^{\phi_l(Z_l)} I_{\{X_{li}=k\}} : k \geq 0; l = 0, \dots, n-1 \right\}$.

Aim

In a Bayesian framework, to obtain robust estimators of θ_0 given the **entire family tree**.

CBP: current research

Disparity measure

A **disparity measure** between $q \in \Gamma$ and $p(\theta) \in \mathcal{F}_\theta$ is defined by:

$$D(q, \theta) = \sum_{k=0}^{\infty} G(\delta(q, \theta, k)) p_k(\theta),$$

with $G(\cdot)$ a three times differentiable and strictly convex function on $[-1, \infty)$ with $G(0) = 0$ and

$$\delta(q, \theta, k) = \frac{q_k}{p_k(\theta)} - 1 \quad (\text{Pearson residual}).$$

Examples of disparity measures

Disparity measure	MDE	Notation	$G(\delta)$	$A(\delta)$
Likelihood disparity	MLDE	$LD(\tilde{p}_n, \theta)$	$(\delta + 1) \log(\delta + 1)$	δ
Squared Hellinger distance	MHDE	$HD(\tilde{p}_n, \theta)$	$[(\delta + 1)^{1/2} - 1]^2$	$2[(\delta + 1)^{1/2} - 1]$
Negative exponential disparity	MNEDE	$NED(\tilde{p}_n, \theta)$	$\exp(-\delta) - 1$	$1 - (2 + \delta) \exp(-\delta)$

CBP: current research

- Posterior density:

$$\pi(\theta|\mathcal{Z}_n^*) \propto \exp(-\Delta_{n-1} \mathbf{KL}(\hat{\mathbf{p}}_n, \theta)) \pi(\theta).$$

- Expectation a posteriori (EAP):

$$\theta_n^* = \int_{\Theta} \theta \pi(\theta|\mathcal{Z}_n^*) d\theta.$$

- Maximum a posteriori (MAP):

$$\theta_n^+ = \arg \max_{\theta \in \Theta} \pi(\theta|\mathcal{Z}_n^*).$$

- D-Posterior density:

$$\pi_D(\theta|\hat{\mathbf{p}}_n) \propto \exp(-\Delta_{n-1} \mathbf{D}(\hat{\mathbf{p}}_n, \theta)) \pi(\theta).$$

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$$\theta_n^{D*} = \int_{\Theta} \theta \pi_D(\theta|\hat{\mathbf{p}}_n) d\theta.$$

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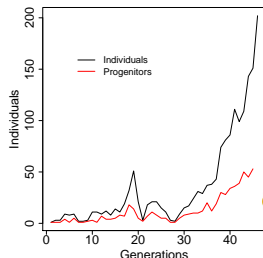
CBP: current research

Behaviour of the posterior density

- Parametric family:** $\mathcal{F}_\theta = \{NB(r, \theta) : \theta \in (0, 1), r = 3\}$,
 where $NB(r, \theta)$ denotes a negative binomial distribution with parameters r and θ .
- Mixture model for gross errors:** $p(\theta, \alpha, L) = (1 - \alpha)p(\theta) + \alpha\delta_L$,
 where $p(\theta)$ is the probability mass function of $NB(3, 0.65)$, $\alpha = 0.05$ and $L = 9$.

We have simulated the first 45 generations of a CBP which verifies:

- It starts with $Z_0 = 1$ individual. 
- The distribution of the variables X_{ij} is $p(\theta, \alpha, L)$, for $i = 0, 1, \dots, j = 1, \dots$
- $\phi_n(k) \sim B(k, q)$, with $q = 0.35, k \geq 0$.
- $m = 1.5154$ and $\sigma^2 = 2.4852$.



CBP: current research

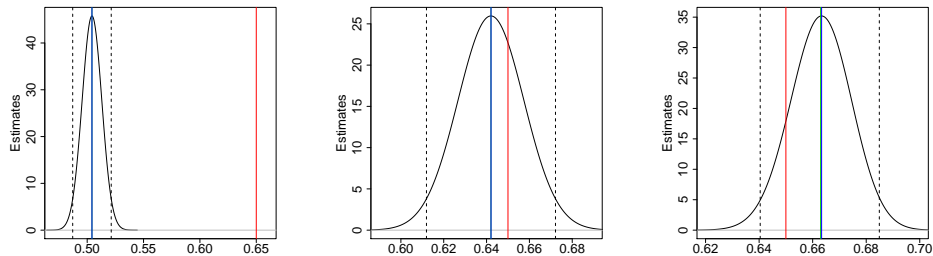


Fig: Posterior density (left), HD-Posterior density (centre) and NED-Posterior density (right) of θ at the generation 45. Red lines represent the true value of the parameters and dashed lines represent the 95% HPD interval.

CBP: current research

- Asymptotic normality of posterior distribution

$$\pi(\theta | \mathcal{Z}_n^*) - \varphi_n(\hat{\rho}_n) \xrightarrow[n \rightarrow \infty]{L_1} 0, \text{ a.s. on } \{Z_n \rightarrow \infty\},$$

with $\varphi_n(\hat{\rho}_n)$ the density function of $N(\hat{\theta}_n, (\Delta_{n-1} I(\theta_0))^{-1})$.

- Asymptotic normality of D-posterior distribution

$$\pi_D(\theta | \hat{\rho}_n) - \varphi_n(\hat{\rho}_n) \xrightarrow[n \rightarrow \infty]{L_1} 0, \text{ a.s. on } \{Z_n \rightarrow \infty\},$$

with $\varphi_n(\hat{\rho}_n)$ the density function of $N(\tilde{\theta}_n^D(\hat{\rho}_n), (\Delta_{n-1} I(\theta_0))^{-1})$.

- Strongly consistency of EAP:

$$\theta_n^* \xrightarrow[n \rightarrow \infty]{a.s.} \theta_0, \text{ on } \{Z_n \rightarrow \infty\}.$$

- Strongly consistency of EDAP:

$$\theta_n^{D*} \xrightarrow[n \rightarrow \infty]{a.s.} \theta_0, \text{ on } \{Z_n \rightarrow \infty\}.$$

- Efficiency of EAP:

$$\Delta_{n-1}^{1/2}(\theta_n^* - \theta_0) \xrightarrow[n \rightarrow \infty]{d} N(0, I(\theta_0)^{-1}),$$

on $\{Z_n \rightarrow \infty\}$.

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CBP: current research

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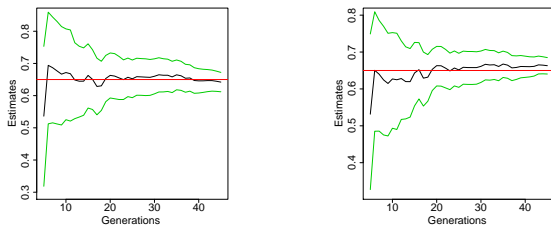


Fig: EDAP estimates (black lines) for the HD (left) and NED (right), with the 95% HPD intervals (green lines) and true value of θ (red lines).

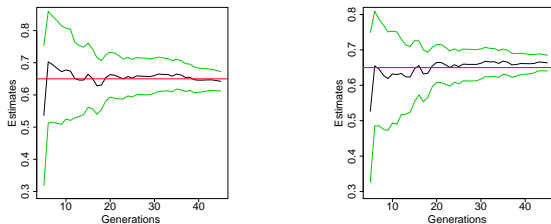


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Current and future research

● **Controlled Branching Processes.**

- To develop the **inferential theory** in controlled process from both frequentist and Bayesian points of view.
- To introduce and study the probabilistic theory of **continuous time** controlled processes and of **modified** controlled processes.

● **Two-sex Branching Processes.**

- To investigate **modified** two-sex branching processes.
- To introduce and study new two-sex branching models with **random mating**.
- To introduce and study new **two-sex multitype** branching processes depending on the population density.

● **Epidemic Stochastic Models.**

- To introduce **generalizations of the SIR general epidemic model** and study their basic probabilistic and inferential properties.

● **Stochastic Models in System Biology.**

- To introduce new population **stochastic models to model complex biological systems** and develop their probabilistic and inferential theory.



References I



González, M., Gutiérrez, C., Martínez, R., del Puerto, I. (2013). Bayesian inference for controlled branching processes through MCMC and ABC methodologies. *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*, **107(2)**, 459-473.



González, M., Gutiérrez, C., Martínez, R., Minuesa, C., del Puerto, I. (2016). Bayesian analysis for controlled branching processes. *Branching Processes and their Applications. Lecture Notes in Statistics-Proceedings* (in press).



González, M., Martínez, R., del Puerto, I. (2004). Nonparametric estimation of the offspring distribution and the mean for a controlled branching process. *Test*, **13(2)**, 465-479.



González, M., Minuesa, C., del Puerto, I. (2015). Maximum likelihood estimation and Expectation-Maximization algorithm for controlled branching processes. *Computational Statistics and Data Analysis*, DOI: 10.1016/j.csda.2015.01.015.



González, M., Minuesa, C., del Puerto, I. (2016) Minimum disparity estimation in controlled branching process. arXiv:1511.06400



González, M., Molina, M., del Puerto, I. (2002). On the class of controlled branching process with random control functions. *Journal of Applied Probability*, **39(4)**, 804-815.



González, M., Molina, M., del Puerto, I. (2003). On the geometric growth in controlled branching processes with random control function. *Journal of Applied Probability*, **40(4)**, 995-1006.



González, M., Molina, M., del Puerto, I. (2005a). Asymptotic behaviour of critical controlled branching process with random control function. *Journal of Applied Probability*, **42(2)**, 463-477.

References II



González, M., Molina, M., del Puerto, I. (2005b). On the L^2 -convergence of controlled branching processes with random control function. *Bernoulli*, **11(1)**, 37-46.



González, M. and del Puerto, I. (2012). Diffusion Approximation of an Array of Controlled Branching Processes. *Methodology and Computing in Applied Probability*, **14**, 843-861.



Martínez, R., del Puerto, I., Mota, M. (2009). On asymptotic posterior normality for controlled branching processes. *Statistics*, **43**, 367-378.



Sriram, T.N., Bhattacharya, A., González, M., Martínez, R., del Puerto, I. (2007). Estimation of the offspring mean in a controlled branching process with a random control function. *Stochastic Processes and their Applications*, **117**, 928-946.



Yanev, N.M. (1975). Conditions for degeneracy of φ -branching processes with random φ . *Theory of Probability and its Applications*, **20**, 421-428.

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