

Ordered Spaces and Quasi-Uniformities on Spaces of Continuous Order-Preserving Functions

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(Research paper presented by F. Montalvo)

AMS *Subject Class.* (1991): 54A05, 54B30, 54F05

Received November 26, 1998

1. INTRODUCTION

In this paper we introduce and investigate the notions of the point open order topology, compact open order topology, the order topology of quasi-uniform pointwise convergence and the order topology of quasi-uniform convergence on compacta. We consider the functorial correspondence between function spaces in the categories of topological spaces, bitopological spaces and ordered topological spaces. We obtain extensions to the topological ordered case of classical topological results on function spaces. We also investigate the property of strict complete regularity of these function spaces.

A motivation for the study of ordered spaces of continuous order-preserving functions and quasi-uniform spaces is that a typical property on the codomain ordered topological space is characterized in terms of the same property of the ordered function space. The interaction between properties of ordered topological spaces and those of the ordered topological space of continuous order-preserving functions provides the general framework for our study. We will show how, for instance, complete regularity on the ordered topological space of continuous order-preserving functions is characterized by the complete regularity on the corresponding codomain ordered topological space. The celebrated theorem of Arens in spaces of continuous functions is also generalized to ordered topological spaces of continuous order-preserving functions.

2. PRELIMINARIES

In this section we recall some basic definitions and well known results.

A topological space X endowed with a closed partial order \leq will be called an ordered topological space. We will interchangeably use the terms ordered space and ordered topological space. If no confusion can arise we use simply X for the ordered topological space (X, τ, \leq) . A mapping $f : (X, \tau, \leq) \rightarrow (Y, \tau', \leq')$ between two ordered topological spaces (X, τ, \leq) and (Y, τ', \leq') is said to be *order-preserving* (*order-reversing*) if $f(x) \leq' f(y)$ ($f(y) \leq' f(x)$) whenever $x, y \in X$ and $x \leq y$, and is continuous if it is continuous with respect to the given topologies. A one-to-one continuous order preserving function f mapping X onto Y is called an order homeomorphism provided that f^{-1} is also continuous order-preserving. If f is an order homeomorphism of X onto a subspace of Y , then f is called an order embedding of X into Y .

Given a topological ordered space X , we call a subset U of X an upper set if $x \leq y$ and $x \in U$ imply that $y \in U$. Similarly, we say that a subset L of X is a lower set if $y \leq x$ and $x \in L$ imply that $y \in L$. For any subset F of X , $i(F)$ ($d(F)$ respectively) will denote the intersection of all upper (lower respectively) sets of X containing F . An ordered topological space X is called an *I-space* if $i(U)$ and $d(U)$ are open whenever U is an open set in X . Similarly, an ordered topological space X is called a *C-space* if $i(F)$ and $d(F)$ are closed whenever F is a closed set in X . Our ordered topological subspace is as defined in [Fetcher and Lindgren], i.e. for any subset of an ordered topological space (X, τ, \leq) we have that $(S, T|_S, \leq_S)$, where $T|_S$ is the topology induced by T on S and \leq_S is the order induced by \leq on S , is an ordered topological space called an ordered topological subspace of (X, τ, \leq) or simply an ordered subspace of (X, τ, \leq) .

We will use \mathbb{R}_{\leq} to denote the set of real numbers with the usual topology and the usual order. For a subset A of the set of real numbers, we use A_{\leq} as an ordered subspace of \mathbb{R}_{\leq} , e.g. I_{\leq} denotes the closed unit interval with the usual topology and the usual order. An ordered topological space (X, τ, \leq) is a *completely regular ordered space* if it satisfies the following two conditions: (i) Let $a \in X$ and let V be a neighborhood of a in X . Then there exist two continuous functions $f, g : (X, \tau, \leq) \rightarrow I_{\leq}$ such that f is order-preserving, g is order-reversing, $f(a) = 1 = g(a)$, and either $f(x) = 0$ or $g(x) = 0$, whenever $x \in X \setminus V$. (ii) Let $a, b \in X$ such that $a \not\leq b$. Then there exists an order-preserving function f on X such that $f(a) > f(b)$ [13].

If Y is a set and \mathcal{U} is a quasi-uniformity on Y such that $(Y, T(\mathcal{U}))$ is T_0 , then $(Y, T(\mathcal{U} \vee \mathcal{U}^{-1}), \cap \mathcal{U})$ is an ordered space. In fact $(Y, T(\mathcal{U} \vee \mathcal{U}^{-1}), \cap \mathcal{U})$ is a

completely regular ordered space. An ordered space (Y, τ', \leq') is said to admit a quasi-uniformity \mathcal{U} on Y if $\tau' = T(\mathcal{U} \vee \mathcal{U}^{-1})$ and $\leq' = \cap \mathcal{U}$, in this case we say the quasi-uniformity \mathcal{U} is compatible with (τ', \leq') , and (Y, τ', \leq') is also said to be quasi-uniformizable. It is well known ([3] and [13]) that an ordered space is quasi-uniformizable iff it is a completely regular ordered space.

For ordered spaces and quasi-uniform spaces we refer the reader to P. Fletcher and W. Lindgren [3] and L. Nachbin [13]. For bitopological spaces we refer to [4], [8] and [21]. For Function spaces we refer the reader to [5], [12] and [23].

We now recall some definitions on function spaces and state some of the results that will be generalised to the ordered case.

Let (X, τ) and (Y, τ') be two topological spaces. Let $C(X, Y)$ denote the set of all continuous functions from X to Y . Let $\mathcal{F} = \{F \subseteq X : F \text{ is finite}\}$. For each $F \in \mathcal{F}$ and $G \in \tau'$; consider the set $[F, G] = \{f \in C(X, Y) : f(F) \subseteq G\}$. The collection $\{[F, G] : F \in \mathcal{F} \text{ and } G \in \tau'\}$ is a subbase for a topology T_p on $C(X, Y)$. The topology T_p on $C(X, Y)$ is called the pointwise topology. The set $C(X, Y)$ with this topology is denoted by $C_p(X, Y)$.

Let (X, τ) and (Y, τ') be two topological spaces. Let $\mathcal{K} = \{K \subseteq X : K \text{ is compact}\}$. For each $K \in \mathcal{K}$ and $G \in \tau'$ we consider the set $[K, G] = \{f \in C(X, Y) : f(K) \subseteq G\}$. The collection $\{[K, G] : K \in \mathcal{K} \text{ and } G \in \tau'\}$ is a subbase for a topology T_k on $C(X, Y)$. The topology T_k on $C(X, Y)$ is called the compact open topology. The set $C(X, Y)$ with this topology is denoted by $C_k(X, Y)$.

Suppose X is a topological space and Y has a uniformity \mathcal{U} . Then the topology of compact convergence on $C(X, Y)$ is the topology induced by the uniformity which has for a subbase sets of the form $E_{K,U} = \{(f, g) : (f(x), g(x)) \in U, \text{ for each } x \in K\}$, where K is a compact subset of X and $U \in \mathcal{U}$.

We now state the following well known results:

THEOREM 1. *Let X and Y be topological spaces. The following statements are equivalent:*

- (a) Y is uniformizable;
- (b) $C_p(X, Y)$ is uniformizable;
- (c) $C_k(X, Y)$ is uniformizable.

THEOREM 2. (Arens' theorem) *For spaces of continuous functions the topology of compact convergence is the compact-open topology.*

3. SOME REMARKS ON THE POINTWISE ORDER TOPOLOGY AND COMPACT OPEN ORDER TOPOLOGY

Let (X, τ, \leq) and (Y, τ', \leq') be ordered topological spaces. Let $C^\uparrow(X, Y)$ denote the set of all continuous order preserving functions from X to Y . Let $\mathcal{F} = \{F \subseteq X : F \text{ is finite}\}$. For each $F \in \mathcal{F}$ and $G \in \tau'$ we consider the set $[F, G] = \{f \in C^\uparrow(X, Y) : f(F) \subseteq G\}$. The collection $\{[F, G] : F \in \mathcal{F} \text{ and } G \in \tau\}$ is a subbase for a topology T_p on $C^\uparrow(X, Y)$. Define an order on $C^\uparrow(X, Y)$ as follows: for each $f, g \in C^\uparrow(X, Y)$, $f \leq_s g \iff f(x) \le' g(x)$ for all $x \in X$ (i.e., \leq_s is the order on $C^\uparrow(X, Y)$ defined pointwise). We refer to $(C^\uparrow(X, Y), T_p, \leq_s)$ as the ordered space of continuous order-preserving functions with the point open ordered space topology. The ordered space $(C^\uparrow(X, Y), T_p, \leq_s)$ will be denoted by $C_p^\uparrow(X, Y)$. We refer to T_p as the point open order topology.

Let (Y, τ', \leq') be quasi-uniformizable and \mathcal{U} be a quasi-uniformity compatible with (τ', \leq') . For each $F \in \mathcal{F}$ and each $U \in \mathcal{U}$ we consider the set $(F, U) = \{(f, g) \in C^\uparrow(X, Y) \times C^\uparrow(X, Y) : (f(x), g(x)) \in U \text{ for all } x \in F\}$. Then $\{(F, U) : F \in \mathcal{F} \text{ and } U \in \mathcal{U}\}$ is a base for a quasi-uniformity \mathcal{U}_p on $C^\uparrow(X, Y)$ called the quasi-uniformity of quasi-uniform pointwise convergence induced by \mathcal{U} . The topology $T(\mathcal{U}_p^*)$ is said to be the topology of quasi-uniform pointwise convergence $(C^\uparrow(X, Y), \cap \mathcal{U}_p)$, where $\mathcal{U}_p^* = \mathcal{U}_p \vee \mathcal{U}_p^{-1}$.

It is clear that the order topology of pointwise convergence coincides with the order topology of a subspace of the product ordered topological space. It is easy to see that the function $\Phi : (Y, \tau', \leq') \rightarrow C_p^\uparrow(X, Y)$ given by $\Phi(y) = f_y$ (where $f_y(x) = y$ for all $x \in X$) for all $y \in Y$ is order embedding.

From [3] we know that the product ordered space of quasi-uniformizable ordered spaces is a quasi-uniformizable ordered space. Since product ordered space of T_i -ordered ($i = 1, 2$) and regular-ordered are T_i -ordered ($i = 1, 2$) and regular-ordered, respectively, we have that if (X, τ, \leq) and (Y, τ', \leq') are ordered topological spaces then we have the following:

- (a) $C_p^\uparrow(X, Y)$ is T_1 -ordered if and only if (Y, τ', \leq') is T_1 -ordered;
- (b) $C_p^\uparrow(X, Y)$ is T_2 -ordered if and only if (Y, τ', \leq') is T_2 -ordered;
- (c) $C_p^\uparrow(X, Y)$ is regular-ordered if and only if (Y, τ', \leq') is regular-ordered;
- (d) $C_p^\uparrow(X, Y)$ is quasi-uniformizable if and only if (Y, τ', \leq') is quasi-uniformizable.

It is also straightforward to verify that if (X, τ, \leq) is an ordered space and (Y, τ', \leq') a quasi-uniformizable ordered space then for each quasi-uniformity

on Y compatible with (τ', \le') , the order topology of quasi-uniform pointwise convergence coincides with the point-open order topology.

Let (X, τ, \le) and (Y, τ', \le') be two ordered topological spaces. Let $\mathcal{K} = \{K \subseteq X : K \text{ is compact}\}$. For each $K \in \mathcal{K}$ and $G \in \tau'$ we consider the set $[K, G] = \{f \in C^\uparrow(X, Y) : f(K) \subseteq G\}$. The collection $\{[K, G] : K \in \mathcal{K} \text{ and } G \in \tau'\}$ is a subbase for a topology T_k on $C^\uparrow(X, Y)$. We now define an order on $C^\uparrow(X, Y)$ as follows: for each $f, g \in C^\uparrow(X, Y)$, $f \le_s g \iff f(x) \le' g(x)$ for all $x \in X$ (i.e., \le_s is the order on $C^\uparrow(X, Y)$ defined pointwise). We refer to T_k as the compact open order topology. An ordered topological space $(C^\uparrow(X, Y), T_k, \le_s)$ will be denoted by $C_k^\uparrow(X, Y)$.

Analogously, let (Y, τ', \le') be quasi-uniformizable and let \mathcal{U} be a quasi-uniformity compatible with (τ', \le') . For each $K \in \mathcal{K}$ and each $U \in \mathcal{U}$ we consider the set $(K, U) = \{(f, g) \in C^\uparrow(X, Y) \times C^\uparrow(X, Y) : (f(x), g(x)) \in U \text{ for all } x \in K\}$. Then $\{(K, U) : K \in \mathcal{K} \text{ and } U \in \mathcal{U}\}$ is a base for a quasi-uniformity \mathcal{U}_k on $C^\uparrow(X, Y)$ which we refer to as the quasi-uniformity of quasi-uniform convergence on compacta. Then the ordered space $(C^\uparrow(X, Y), T(\mathcal{U}_k^*), \cap \mathcal{U}_k)$, where $\mathcal{U}_k^* = \mathcal{U}_k \vee \mathcal{U}_k^{-1}$, is said to have the order topology of quasi-uniform convergence on compacta.

4. FUNCTORIAL CORRESPONDENCE BETWEEN FUNCTION SPACES IN THE CATEGORIES OF TOPOLOGICAL SPACES, BITOPOLOGICAL SPACES AND ORDERED SPACES

In this section we look at a unifying categorical approach, where we discuss the correspondence between function spaces in the categories of topological spaces, bitopological spaces and ordered topological spaces.

Let \mathbf{Top} be the category of topological spaces and continuous maps, \mathbf{BiTop} be the category of bitopological spaces and maps which are continuous with respect to the corresponding topologies, \mathbf{TopOrd} be the category of ordered spaces and continuous order-preserving functions. Let \mathbb{R}_u be the set of real numbers with the usual topology, let \mathbb{R}_b be the real numbers with the upper topology and the lower topology, where the upper topology is the topology which has sets of the form $\{(-\infty, a) : a \in \mathbb{R}\}$ as a base and the lower topology has sets of the form $\{(a, \infty) : a \in \mathbb{R}\}$ as a base, and let \mathbb{R}_\le denote the set of real numbers with the usual topology and the usual order.

Put $C((X, \tau), \mathbb{R}_u) = C_1X$, $C((X, \tau_1, \tau_2), \mathbb{R}_b) = C_2X$ and $C((X, \tau, \le), \mathbb{R}_\le) = C_\le X$

Now let $C_{p1}X$ denote C_1X with the point open topology and $C_{k1}X$ de-

note C_1X with the compact open topology. We similarly use the notation $C_{p2}X$ and $C_{k2}X$ in bitopological spaces, and $C_{p\leq}X$ and $C_{k\leq}X$ in ordered spaces. It is known that if $f : X \rightarrow Y$ is continuous then the map $f^* : C_{i1}Y \rightarrow C_{i1}X$ ($i = p, k$) given by $f^*(g) = g \circ f$, for all $g \in C_{i1}X$ is continuous ([12] and [16]). From this result we have the contravariant functors $C_{p1} : \text{Top} \rightarrow \text{Top}$ and $C_{k1} : \text{Top} \rightarrow \text{Top}$ assigning each topological space X , the point open topology $C_{p1}X$ and the compact open topology $C_{k1}X$ respectively.

We also have the following results in bitopological spaces and ordered spaces. The proofs are straightforward.

PROPOSITION 3. (a) *If $f : (X, \tau_1, \tau_2) \rightarrow (Y, \tau'_1, \tau'_2)$ ($i = p, k$) is bicontinuous then $f^* : C_{i2}Y \rightarrow C_{i2}X$ ($i = p, k$) is bicontinuous.*

(b) *If $f : (X, \tau, \leq) \rightarrow (Y, \tau', \leq')$ ($i = p, k$) is a continuous order-preserving function then $f^* : C_{i\leq}Y \rightarrow C_{i\leq}X$ ($i = p, k$) is a continuous order-preserving function.*

From Proposition 3 we have the contravariant functors: $C_{p2} : \text{BiTop} \rightarrow \text{BiTop}$, $C_{k2} : \text{BiTop} \rightarrow \text{BiTop}$, $C_{p\leq} : \text{TopOrd} \rightarrow \text{TopOrd}$ and $C_{k\leq} : \text{TopOrd} \rightarrow \text{TopOrd}$. The functors C_{p2} and C_{k2} assign each bitopological space X the bitopology of pointwise convergence $C_{p2}X$ and the 2compact open bitopology $C_{k2}X$ respectively, as defined by Romaguera and Ruiz-Gómez [18] and [19]. The functors $C_{p\leq}$ and $C_{k\leq}$ assigns each ordered space X , the point open order topology and the compact open order topology respectively, as defined in section 3.

Let $C2Top$ be the category of completely regular bitopological spaces and $CTopOrd$ be the category of completely regular ordered spaces.

Now consider the following well known functors: $D : \text{Top} \rightarrow \text{BiTop}$ given by $D(X, \tau) = (X, \tau, \tau)$ (the doubling functor), $S : \text{BiTop} \rightarrow \text{Top}$ given by $S(X, \tau_1, \tau_2) = (X, \tau_1 \vee \tau_2)$ (D is the left adjoint right inverse of S), $M : C2Top \rightarrow CTopOrd$ given by $M(X, \tau_1, \tau_2) = (X, \tau_1 \vee \tau_2, \leq_{\tau_1})$, where $x \leq_{\tau_1} y \iff x \in \overline{\{y\}}^{\tau_1}$.

Let $A = \{x \in X : f(x) \leq 0\}$ where $f : (X, \tau, \leq) \rightarrow \mathbb{R}_{\leq}$ is continuous order-preserving function. In [10] sets of this form are called the decreasing zero sets and sets of the form $B = \{x \in X : f(x) \geq 0\}$ where $f : (X, \tau, \leq) \rightarrow \mathbb{R}_{\leq}$ is continuous order-preserving function are called the increasing zero sets. Let \mathcal{A}_0 be a collection of the decreasing zero sets and \mathcal{B}_0 be the collection of the increasing zero sets. In [22] the functor $I : CTopOrd \rightarrow C2Top$ was defined using initiality. It was shown in [14] that $I(X, \tau, \leq) = (X, \tau_{\mathcal{A}_0}, \tau_{\mathcal{B}_0})$, where $\tau_{\mathcal{A}_0}$ is the topology having \mathcal{A}_0 as a base for closed sets and $\tau_{\mathcal{B}_0}$ is the topology

having \mathcal{B}_0 as a base for closed sets.

PROPOSITION 4. $SC_{p2}D = C_{p1}$ and $SC_{k2}D = C_{k1}$.

EXAMPLE 5. $DC_{p1}S \neq C_{p2}$ and $DC_{k1}S \neq C_{k2}$. Let (X, T) be a cofinite topology on an infinite set X . Let T^* be the topology which has a base the closed sets of T . Then (X, T, T^*) is pairwise completely regular ([21] and [22]). Since (X, T) is a T_1 -space we have that T^* is a discrete topology. Then $S(X, T_1, T_2)$ is a discrete space. Then $C_{p1}S(X, T, T^*)$ is the topological product \mathbb{R}^X (see [1], Corollary II.8.7). Clearly $DC_{p1}S(X, T, T^*)$ is the product in BiTop. But $C_{p2}(X, T, T^*)$ is a proper bitopological subspace of the product. Hence $DC_{p1}S(X, T, T^*) \neq C_{p2}(X, T, T^*)$.

THEOREM 6. $MC_{p2}I = C_{p\leq}$ and $MC_{k2}I = C_{k\leq}$.

Proof. Consider the second equality. We want to show that $MC_{k2}I = C_{k\leq}$. Let (X, τ, \leq) be a completely regular ordered space. Then $I(X, \tau, \leq) = (X, \tau_{\mathcal{A}_0}, \tau_{\mathcal{B}_0})$. We now consider the bitopological space $C_{k2}(X, \tau_{\mathcal{A}_0}, \tau_{\mathcal{B}_0})$. The first topology $T_{\tau_{\mathcal{A}_0}}$ of $C_{k2}(X, \tau_{\mathcal{A}_0}, \tau_{\mathcal{B}_0})$ has subbase $\{[K, (a, \infty)] : K \text{ is } \tau_{\mathcal{A}_0} \vee \tau_{\mathcal{B}_0} = \tau\text{-compact}\}$ and the second topology $T_{\tau_{\mathcal{B}_0}}$ of $C_{k2}(X, \tau_{\mathcal{A}_0}, \tau_{\mathcal{B}_0})$ has subbase $\{[K, (-\infty, b)] : K \text{ is } \tau_{\mathcal{A}_0} \vee \tau_{\mathcal{B}_0} = \tau\text{-compact}\}$ where $a, b \in \mathbb{R}$. Without loss of generality we can assume that $a < b$. We now consider $M(X, T_{\tau_{\mathcal{A}_0}}, T_{\tau_{\mathcal{B}_0}})$. Now $[K, (a, \infty)] \cap [K, (-\infty, b)] = [K, (a, b)]$. Therefore $T_{\tau_{\mathcal{A}_0}} \vee T_{\tau_{\mathcal{B}_0}}$ is the topology of the ordered space $C_{k\leq}(X, \tau, \leq)$. We now show that the order $\leq_{T_{\tau_{\mathcal{B}_0}}}$ is the pointwise order. Let $f \leq_{T_{\tau_{\mathcal{B}_0}}} g$. Then $f \in cl_{T_{\tau_{\mathcal{B}_0}}} \{g\}$. Thus $f \in [x, (a, \infty)]$ implies $g \in [x, (a, \infty)]$ for all $x \in X$ and for all $a \in \mathbb{R}$. Hence $f(x) \in (a, \infty)$ implies $g(x) \in (a, \infty)$ for all $x \in X$ and for all $a \in \mathbb{R}$. Then $f(x) > a$ implies $g(x) > a$. Then $f(x) \leq_u g(x)$ for all $x \in X$.

On the other hand suppose $f \not\leq_{T_{\tau_{\mathcal{B}_0}}} g$ implies that there exists $x_0 \in X$ and $a \in \mathbb{R}$ such that $f \in [x_0, (a, \infty)]$ and $g \notin [x_0, (a, \infty)]$. Then there exists $x_0 \in X$ and $a \in \mathbb{R}$ such that $f(x_0) > a$ and $g(x_0) < a$. Thus there exists $x_0 \in X$ and $a \in \mathbb{R}$ such that $g(x_0) < a < f(x_0)$. Hence $f(x_0) \not\leq_u g(x_0)$. Therefore $f \leq_{T_{\tau_{\mathcal{B}_0}}} g \iff f(x) \leq_u g(x)$ for all $x \in X$.

This completes the proof. ■

EXAMPLE 7. $IC_{p\leq}M \neq C_{p2}$ and $IC_{k\leq}M \neq C_{k2}$. Consider the case $i = p$. Let (\mathbb{Q}, us, ls) be the rational with the upper Sorgenfrey topology and the lower Sorgenfrey topology (i.e., us has basic open sets of the form $(a, b]$ where $a, b \in \mathbb{Q}$, $a < b$, and ls has basic open sets of the form $[a, b)$ where $a, b \in \mathbb{Q}$,

$a < b$). Then $C_{p2}(\mathbb{Q}, us, ls)$ has the bitopology of a subspace of the product $P = (\prod_{q \in \mathbb{Q}} \mathbb{R}_q, \prod_{q \in \mathbb{Q}} (i)_q, \prod_{q \in \mathbb{Q}} (d)_q)$. Now $M(\mathbb{Q}, us, sl) = (\mathbb{Q}, \text{discrete topology, discrete order})$. Then every function $f : \mathbb{Q} \rightarrow \mathbb{R}$ is in $C_{p\leq}(\mathbb{Q}, \mathbb{R}_{\leq})$. Therefore $C_{p\leq}(\mathbb{Q}, \mathbb{R}_{\leq})$ is the product topological space $\mathbb{R}^{\mathbb{Q}}$. Applying the functor I and using the fact that I preserves initial sources, we have that $IC_{p\leq}(\mathbb{Q}, \mathbb{R}_{\leq})$ is the product bitopology $P = (\prod_{q \in \mathbb{Q}} \mathbb{R}_q, \prod_{q \in \mathbb{Q}} (i)_q, \prod_{q \in \mathbb{Q}} (d)_q)$. Since $C_{p2}(\mathbb{Q}, us, ls)$ is a proper bitopological subspace of P we have $IC_{p\leq}M \neq C_{p2}$.

Example 7 is also valid for the case of compact open bitopology and compact open order topology since in a discrete topology compact sets are finite sets and thus the bitopology of pointwise convergence coincides with the 2compact open bitopology.

If we replace \mathbb{R}_b and \mathbb{R}_{\leq} by a completely regular bitopological space (Y, τ'_1, τ'_2) and respectively by a completely regular ordered space (Y, τ', \leq') then we get the following proposition which is analogously to Proposition 3.

PROPOSITION 8. (a) *If $f : (X, \tau_1, \tau_2) \rightarrow (Y, \tau'_1, \tau'_2)$ ($i = p, k$) is bicontinuous and $Z \in C2Top$ then $f^* : C_{i2}(Y, Z) \rightarrow C_{i2}(X, Z)$ ($i = p, k$) is bicontinuous.*

(b) *If $f : (X, \tau, \leq) \rightarrow (Y, \tau', \leq')$ ($i = p, k$) is a continuous order-preserving function and $Z \in CTopOrd$ then $f^* : C_{i\leq}(Y, Z) \rightarrow C_{i\leq}(X, Z)$ ($i = p, k$) is a continuous order-preserving function.*

From Proposition 8 we have the following contravariant functors $C_{i2} : C2Top \rightarrow C2Top$ ($i = p, k$) and $C_{i\leq} : CTopOrd \rightarrow CTopOrd$ ($i = p, k$) which assign for each $X \in C2Top$ the bitopological space $C_{i2}(X, Z)$ ($i = p, k$) and for each $X \in CTopOrd$ the ordered topological space $C_{i\leq}(X, Z)$ ($i = p, k$) where $Z \in CTopOrd$ is fixed.

The proof of the following proposition follows as in the proof of Theorem 5.

PROPOSITION 9. *Let X be any ordered topological space and Y be a fixed completely regular ordered space. Then $MC_{i2}(IX, IY) = C_{i\leq}(X, Y)$ where $i = p, k$.*

5. THE COMPACT OPEN ORDER TOPOLOGY AND THE ORDER TOPOLOGY OF QUASI-UNIFORM CONVERGENCE ON O-COMPACTA

In this section we study the quasi-uniformizability of the ordered topological space of continuous order-preserving functions. We also generalize Arens' Theorem.

THEOREM 10. *Let (X, τ, \leq) and (Y, τ', \leq') be two ordered topological spaces. Then (Y, τ', \leq') is quasi-uniformizable if and only if $C_k^\uparrow(X, Y)$ is quasi-uniformizable.*

Proof. Suppose (Y, τ', \leq') is quasi-uniformizable. Then IY is a quasi-uniformizable bitopological space. By theorem 1 in [18], $C_{k2}(IX, IY)$ is quasi-uniformizable. By applying the functor M and using proposition 9, we have that $MC_{k2}(IX, IY) = C_{i\leq}(X, Y)$ is quasi-uniformizable.

Conversely suppose $C_k^\uparrow(X, Y)$ is quasi-uniformizable. Let \mathcal{U} be a quasi-uniformity on $C^\uparrow(X, Y)$ compatible with (T_k, \leq_s) , i.e., $T_k = T(\mathcal{U}^*)$ and $\leq_s = \cap \mathcal{U}$. Now for each $y \in Y$ define $f_y : X \rightarrow Y$ by $f_y(x) = y$ for all $x \in X$. For each $U \in \mathcal{U}$ put

$$\tilde{U} = \{(y, z) \in Y \times Y : (f_y, f_z) \in U\}.$$

We claim that $\{\tilde{U} : U \in \mathcal{U}\}$ is a base for a quasi-uniformity $\tilde{\mathcal{U}}$ on Y compatible with (τ', \leq') , i.e., $T(\tilde{\mathcal{U}}^*) = \tau'$ and $\leq' = \cap \tilde{\mathcal{U}}$. Let $y \in H \in T(\tilde{\mathcal{U}}^*)$. There is $U \in \mathcal{U}$ such that $(\tilde{U} \cap \tilde{U}^{-1})(y) \subseteq H$. Since $(U \cap U^{-1})(f_y) \in T_k$, there are sets $K_1, \dots, K_n \in \mathcal{K}$ and $G_1, \dots, G_n \in \tau'$ such that

$$f_y \in \cap_{i=1}^n [K_i, G_i] \subseteq (U \cap U^{-1})(f_y).$$

Put $G = \cap_{i=1}^n G_i$. We have $f_y(K_i) \subseteq G_i$ for $i = 1, \dots, n$. Thus $y \in G_i$ for all i . Then $y \in G$. We now show that $G \subseteq H$. Let $z \in G$. Then $z \in G_i$ for $i = 1, \dots, n$. Then $f_z \in \cap_{i=1}^n [K_i, G_i] \subseteq (U \cap U^{-1})(f_y)$. This implies $(f_y, f_z) \in U \cap U^{-1}$. Thus $(y, z) \in \tilde{U} \cap \tilde{U}^{-1}$ and hence $z \in (\tilde{U} \cap \tilde{U}^{-1})(y) \subseteq H$. Therefore H is open in τ' and thus $T(\tilde{\mathcal{U}}^*) \subseteq \tau'$. On the other hand if $y \in G \in \tau'$. Then for each $x \in X$, $f_y \in [x, G] \in T_k = T(\mathcal{U}^*)$. There exists $U \in \mathcal{U}$ such that $(U \cap U^{-1})(f_y) \subseteq [x, G]$. We now show that $(\tilde{U} \cap \tilde{U}^{-1})(y) \subseteq G$. We have that

$$\begin{aligned} z \in (\tilde{U} \cap \tilde{U}^{-1})(y) &\implies (y, z) \in \tilde{U} \cap \tilde{U}^{-1} \\ &\implies (f_y, f_z) \in U \cap U^{-1} \end{aligned}$$

$$\begin{aligned}
&\implies f_z \in (U \cap U^{-1})(f_y) \subseteq [x, G] \\
&\implies f_z(x) \in G \\
&\implies z \in G.
\end{aligned}$$

Therefore $G \in T(\tilde{\mathcal{U}}^*)$. Hence $\tau' = T(\tilde{\mathcal{U}}^*)$. We now show that $\leq' = \cap \tilde{\mathcal{U}}$:

$$\begin{aligned}
(y, z) \in \cap \tilde{\mathcal{U}} &\iff (y, z) \in \tilde{U} \text{ for all } U \in \mathcal{U} \\
&\iff (f_y, f_z) \in U \text{ for all } U \in \mathcal{U} \\
&\iff (f_y, f_z) \in \cap \mathcal{U} \\
&\iff f_y \leq_s f_z \\
&\iff f_y(x) \leq' f_z(x) \text{ for all } x \in X \\
&\iff y \leq' z.
\end{aligned}$$

Therefore (Y, τ', \leq') is quasi-uniformizable. ■

COROLLARY 11. *Let (X, τ, \leq) be an ordered space. Then the following statements are equivalent:*

- (i) (Y, τ', \leq') is quasi-uniformizable;
- (ii) $C_k^\uparrow(X, Y)$ is quasi-uniformizable;
- (iii) $C_p^\uparrow(X, Y)$ is quasi-uniformizable.

Proof. Follows from Section 3(d) and Theorem 10. ■

We now generalize Arens' theorem.

THEOREM 12. *Let (X, τ, \leq) be an ordered topological space and (Y, τ', \leq') a quasi-uniformizable ordered space. Then for each quasi-uniformity on Y compatible with (τ', \leq') the order topology of quasi-uniform convergence on compacta coincides with the compact open order topology.*

Proof. Let \mathcal{U} be a quasi-uniformity on Y compatible with (τ', \leq') , i.e., $T(\mathcal{U}^*) = \tau'$ and $\leq' = \cap \mathcal{U}$. The topology T_k is the restriction to $C^\uparrow(X, Y)$ of the compact open topology on the space of continuous functions. Therefore the topology of uniform convergence on compacta (of \mathcal{U}^*) coincides with the compact open topology on the spaces of continuous functions. Hence $T_k =$

$T(\mathcal{U}_k^*)$ on $C^\uparrow(X, Y)$. It remains to show that $\leq_s = \cap \mathcal{U}_k$:

$$\begin{aligned} f \leq_s g &\iff f(x) \leq' g(x) \text{ for all } x \in X \\ &\iff (f(x), g(x)) \in \cap \mathcal{U} \text{ for all } x \in X \\ &\iff (f(x), g(x)) \in U \text{ for all } U \in \mathcal{U}, \text{ for all } x \in X \\ &\iff (f, g) \in (K, U) \text{ for all } K \in \mathcal{K}, U \in \mathcal{U} \\ &\iff (f, g) \in \cap \{(K, U) : K \in \mathcal{K}, U \in \mathcal{U}\} \\ &\iff (f, g) \in \cap \mathcal{U}_k. \end{aligned}$$

■

Let $C^{b\uparrow}(X, \mathbb{R}_{\leq})$ denotes the set of bounded continuous order-preserving functions. Then we have the following:

PROPOSITION 13. $C^{b\uparrow}(X, \mathbb{R}_{\leq})$ is dense in $C_k^\uparrow(X, \mathbb{R}_{\leq})$.

Proof. Let B be a basic open set in $C_k^\uparrow(X, \mathbb{R}_{\leq})$ and let $f \in B$. Then there exist $K_1, \dots, K_n \in \mathcal{K}$ and V_1, \dots, V_n open in \mathbb{R}_{\leq} such that $f \in B = \cap_{i=1}^n [K_i, V_i]$. Put $K = \cup_{i=1}^n K_i$. Then K is compact. Since $f \in C_k^\uparrow(X, \mathbb{R}_{\leq})$, we have that the restriction $f_K : K \rightarrow \mathbb{R}_{\leq}$ is a bounded (because K is compact) continuous order preserving function. Let ε be such that $|f(x)| \leq \varepsilon$ for all $x \in K$. Then $g = f \wedge \varepsilon$ is in $C^{b\uparrow}(X, \mathbb{R}_{\leq})$ and $g[K_i] = f[K_i] \subseteq V_i, 1 \leq i \leq n$, as required. ■

COROLLARY 14. $C^{b\uparrow}(X, \mathbb{R}_{\leq})$ is dense in $C_p^\uparrow(X, \mathbb{R}_{\leq})$.

Proof. Follows from $C_p^\uparrow(X, \mathbb{R}_{\leq}) \subset C_k^\uparrow(X, \mathbb{R}_{\leq})$. ■

6. MORE PROPERTIES OF $C_p^\uparrow(X, Y)$, $C_k^\uparrow(X, Y)$ INHERITED FROM THE CODOMAIN ORDERED SPACE Y

The notion of a strictly completely ordered space was introduced by J. D. Lawson [9]. We investigate how this notion, and the notions of an I-space and a C-space interact between the codomain ordered space (Y, τ', \leq') and the ordered function spaces $C_p^\uparrow(X, Y)$ and $C_k^\uparrow(X, Y)$. In [9] Lawson gave the following definition:

DEFINITION 1. Let (X, τ, \leq) be an ordered topological space. Then X is said to be *strictly completely regular ordered space* if: (a) the order on X is semiclosed; (b) X is strongly order convex, i.e., the open upper sets

and open lower sets form a subbasis for the topology; (c) given a closed lower (upper, respectively) set A and a point $x \notin A$, there exists a continuous order-preserving function $f : (X, \tau, \leq) \rightarrow I_{\leq}$ such that $f(A) = 0$ and $f(x) = 1$ ($f(A) = 1$ and $f(x) = 0$, respectively).

Remark 1. Condition (c) in Definition 1 is equivalent to: given an open upper set U and $x \in U$ there exists a continuous order-preserving function $f : (X, \tau, \leq) \rightarrow I_{\leq}$ such that $f(X \setminus U) = 0$ and $f(x) = 1$.

The straightforward proof of the following lemma will be omitted.

LEMMA 15. Let (X, τ, \leq) and (Y, τ', \leq') be ordered topological spaces. Then for any subset A of X and any subset B of Y we have $i([A, B]) = [A, i(B)]$ and $d([A, B]) = [A, d(B)]$

THEOREM 16. Let (X, τ, \leq) and (Y, τ', \leq') be ordered topological spaces.

(a) The following are equivalent:

- (i) (Y, τ', \leq') is an I-space;
- (ii) $C_p^\uparrow(X, Y)$ is an I-space;
- (iii) $C_k^\uparrow(X, Y)$ is an I-space.

(b) The following are equivalent:

- (i) (Y, τ', \leq') is a C-space;
- (ii) $C_p^\uparrow(X, Y)$ is a C-space;
- (iii) $C_k^\uparrow(X, Y)$ is a C-space.

Proof. We will only prove (a), the proof of (b) is similar.

(i) \implies (iii) Let (Y, τ', \leq') be an I-space. Let F be an open subset of $C_k^\uparrow(X, Y)$ and $f \in F$. Then there are compact subsets K_1, \dots, K_n of X and open subsets V_1, \dots, V_n of Y such that $f \in \bigcap_{j=1}^n [K_j, V_j] \subset F$. Then $F = \bigcup_{f \in F} (\bigcap_{j=1}^n [K_j, V_j])_f$. Then

$$i(F) = i(\bigcup_{f \in F} (\bigcap_{j=1}^n [K_j, V_j])_f) = \bigcup_{f \in F} (\bigcap_{j=1}^n [K_j, i(V_j)])_f$$

since i preserves unions and intersections. Since (Y, τ', \leq') is an I-space each $[K_j, i(V_j)]$ is open in $C_k^\uparrow(X, Y)$ and thus $i(F)$ is open. The proof that $d(F)$ is open is analogous. Therefore $C^\uparrow(X, Y)$ is an I-space.

(iii) \implies (i) Suppose $C_k^\uparrow(X, Y)$ is an I-space. Let U be open in (Y, τ', \leq') . Let $x \in X$. Then $[x, U]$ is a subbasic open set in $C_k^\uparrow(X, Y)$. Therefore $i([x, U]) = [x, i(U)]$ is open in $C_k^\uparrow(X, Y)$. We now show that for any subset B of Y , if $[x, B]$ is open in $C_k^\uparrow(X, Y)$ then B is open in Y . Let $y \in B$. Then $c_y : X \rightarrow Y$ defined by $c_y(x) = y$ for all $x \in X$, is in $C_k^\uparrow(X, Y)$. Since $c_y \in [x, B]$ and $[x, B]$ is open in $C_k^\uparrow(X, Y)$, there exist compact subsets K_1, \dots, K_n of X and open subsets V_1, \dots, V_n of Y such that $c_y \in \bigcap_{j=1}^n [K_j, V_j] \subset [x, B]$. Put $V = \bigcap_{j=1}^n V_j$. Then $y \in V$. We now show that $V \subset B$. Let $z \in V$. Then $z \in V_j$ for all j . Then $c_z \in \bigcap_{j=1}^n [K_j, V_j]$ and thus $z \in B$. Therefore B is open. From this we have that $i(U)$ is open. Similarly $d(U)$ is open. Therefore Y is an I-space.

(i) \iff (ii) The proof is similar to the above. \blacksquare

EXAMPLE 17. (a) Since \mathbb{R}_\leq is both an I-space and a C-space, we have that $C_k^\uparrow(X, \mathbb{R}_\leq)$ and $C_p^\uparrow(X, \mathbb{R}_\leq)$ are both an I-space and a C-space.

(b) Consider the following ordered space. Let $Y = \{a, b, c\}$, $\tau' = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, Y\}$, and order Y by $a < b < c$. We have that $\{b\}$ is closed but neither $i(b)$ nor $d(b)$ is closed. Therefore Y is not a C-space. By Proposition 13, $C_p^\uparrow(X, Y)$ and $C_k^\uparrow(X, Y)$ are not C-spaces.

The proof of the following lemma is straightforward.

LEMMA 18. Let (X, τ, \leq) and (Y, τ', \leq') be ordered topological spaces. Then the set $[K, V]$ in $C^\uparrow(X, Y)$ is an upper set if and only if V is an upper set.

LEMMA 19. Let (X, τ, \leq) be a strictly completely regular ordered space and $A \subseteq X$ be compact. If V an upper (lower) neighborhood of A then there is a continuous order-preserving function $f : (X, \tau, \leq) \rightarrow I_\leq$ such that $f(A) = 1$ and $f(X \setminus V) = 0$ ($f(A) = 0$ and $f(X \setminus V) = 1$).

Proof. Let $a \in A$ and let V be an increasing neighborhood of A . By remark 1, there is a continuous order-preserving function $f_a : (X, \tau, \leq) \rightarrow I_\leq$ such that $f(X \setminus V) = 0$ and $f_a(a) = 1$. Then $\{f_a^{-1}(\frac{1}{2}, 1] : a \in A\}$ is an open cover of A . Since A is compact there is a finite subset $A' \subseteq A$ such that $A \subseteq \{f_a^{-1}(\frac{1}{2}, 1] : a \in A'\}$. Let $f = \min\{f_a : a \in A'\}$. Since $f^{-1}([0, \frac{1}{2}) = \bigcup_{a \in A'} f_a^{-1}([0, \frac{1}{2}))$, $f^{-1}((\frac{1}{2}, 1]) = \bigcap_{a \in A'} f_a^{-1}((\frac{1}{2}, 1])$, we have that f is continuous. Since each of the functions f_a is order-preserving, we have that f is order-preserving. It is easy to see that $f(A) = 1$ and $f(X \setminus V) = 0$. The dual case is analogous. \blacksquare

THEOREM 20. Let (X, τ, \leq) and (Y, τ', \leq') be two ordered topological spaces. Then

- (i) (Y, τ', \leq') is a strictly completely regular ordered space if and only if $C_p^\uparrow(X, Y)$ is a strictly completely regular ordered space;
- (ii) (Y, τ', \leq') is a strictly completely regular ordered space if and only if $C_k^\uparrow(X, Y)$ is a strictly completely regular ordered space.

Proof. We will prove (ii), the proof of (i) is similar. Suppose (Y, τ', \leq') is a strictly completely regular ordered space. We verify that $C_k^\uparrow(X, Y)$ satisfies the three conditions in Definition 1.

(a) To show that \leq_s is semiclosed is equivalent to showing that for all $f \in C_k^\uparrow(X, Y)$, the sets $i(f)$ and $d(f)$ are closed. Let $g \in C_k^\uparrow(X, Y) \setminus i(f)$. Then

$$\begin{aligned} g \notin i(f) &\implies f \not\leq_k g \\ &\implies f(x_o) \not\leq' g(x_o) \text{ for some } x_o \in X \\ &\implies g(x_o) \notin i(f(x_o)) \\ &\implies g(x_o) \notin Y \setminus i(f(x_o)). \end{aligned}$$

Since \leq' is semiclosed, the set $Y \setminus i(f(x_o))$ is open in τ' . We now show that $g \in [x_o, Y \setminus i(f(x_o))] \subseteq C_k^\uparrow(X, Y) \setminus i(f)$, which will show that $C_k^\uparrow(X, Y) \setminus i(f)$ is open. The function g is obviously in $[x_o, Y \setminus i(f(x_o))]$. Now

$$\begin{aligned} h \in [x_o, Y \setminus i(f(x_o))] &\implies h(x_o) \in Y \setminus i(f(x_o)) \\ &\implies h(x_o) \notin i(f(x_o)) \\ &\implies f(x_o) \not\leq' h(x_o) \\ &\implies f \not\leq_s h \\ &\implies h \notin i(f) \\ &\implies h \in C_k^\uparrow(X, Y) \setminus i(f). \end{aligned}$$

Therefore $C_k^\uparrow(X, Y) \setminus i(f)$ is open and hence $i(f)$ is closed. Similarly $d(f)$ is closed. Therefore \leq_s is semiclosed.

(b) Let B be open in $C_k^\uparrow(X, Y)$ and $f \in B$. Then there exist $K_1, \dots, K_n \in \mathcal{K}$ and $V_1, \dots, V_n \in \tau'$ such that $f \in \bigcap_{i=1}^n [K_i, V_i] \subseteq B$. Then $f(K_i) \subseteq V_i$ for all $i = 1, \dots, n$. For each $x_i \in K_i$ there exists an open upper set O_i and an open lower set U_i such that $f(x_i) \in O_i \cap U_i \subseteq V_i$, since Y is strongly order convex.

We claim that $[K_i, O_i]$ is an open upper set in $C_k^\uparrow(X, Y)$ and $[K_i, U_i]$ is an open lower set such that $f \in (\cap_{i=1}^n [K_i, O_i]) \cap (\cap_{i=1}^n [K_i, U_i]) \subseteq B$. Since $f(K_i) \subseteq O_i$ and $f(K_i) \subseteq U_i$ for all i , we have that $f \in (\cap_{i=1}^n [K_i, O_i]) \cap (\cap_{i=1}^n [K_i, U_i])$. By Remark 1 it follows that each $[K_i, O_i]$ is an upper set and so is $\cap_{i=1}^n [K_i, O_i]$. Similarly $\cap_{i=1}^n [K_i, U_i]$ is a lower set. Now if $g \in (\cap_{i=1}^n [K_i, O_i]) \cap (\cap_{i=1}^n [K_i, U_i])$ then $g(x_i) \in O_i \cap U_i \subseteq V_i$ for all i . Therefore $g \in B$. Hence $C_k^\uparrow(X, Y)$ is strongly order convex.

(c) Consider a fixed subbase $\{[K, V] : K \in \mathcal{K} \text{ and } V \in \tau'\}$ of T_k . Let $f \in [K, V]$, where $[K, V]$ is an open upper set. We want to show that there is a continuous order preserving function $\Psi : C^\uparrow(X, Y) \rightarrow I_{\leq}$ such that $\Psi(C^\uparrow(X, Y) \setminus [K, V]) = 0$ and $\Psi(f) = 1$. Then $f(K) \subseteq V$, and by Remark 1, V is an open upper set. Since $f(K)$ is compact, there is a continuous order-preserving function $g : (Y, \tau', \leq') \rightarrow I_{\leq}$ such that $g(f(K)) = 1$ and $g(Y \setminus V) = 0$. Define $\Psi : C^\uparrow(X, Y) \rightarrow I_{\leq}$ by $\Psi(h) = \sup g(h(K))$ for all $h \in C^\uparrow(X, Y)$. The function Ψ is continuous and order-preserving. (see Theorem 7). Furthermore $\Psi(C^\uparrow(X, Y) \setminus [K, V]) = 0$ and $\Psi(f) = 1$ as required. Therefore $C_k^\uparrow(X, Y)$ is a strictly completely regular ordered space.

Conversely suppose $C_k^\uparrow(X, Y)$ is a strictly completely regular ordered space. Consider the function $\Phi : Y \rightarrow C^\uparrow(X, Y)$ given by $\Phi(y) = f_y$ where $f_y(z) = y$ for all $z \in Y$. Then Φ is an order embedding. Therefore Y is an order subspace of $C^\uparrow(X, Y)$. Hence Y is a strictly completely regular ordered space. ■

COROLLARY 21. *Let (X, τ, \leq) and (Y, τ', \leq') be two ordered topological spaces. The following are equivalent:*

- (i) (Y, τ', \leq') is a strictly completely regular ordered space;
- (ii) $C_p^\uparrow(X, Y)$ is a strictly completely regular ordered space;
- (iii) $C_k^\uparrow(X, Y)$ is a strictly completely regular ordered space.

EXAMPLE 22. (a) Since \mathbb{R}_{\leq} is a strictly completely regular ordered space, $C_k^\uparrow(X, \mathbb{R}_{\leq})$ and $C_p^\uparrow(X, \mathbb{R}_{\leq})$ are strictly completely regular ordered spaces.

(b) ([6], Example 6) On the set $Y = [0, \omega_1] \times [0, \omega_0]$ equipped with the product topology of the interval topologies on the factor sets, define a partial order by $(a, b) \leq (c, d)$ if and only if $a \geq c$ and $b \leq d$ whenever $(a, b), (c, d) \in Y$. Then (Y, T, \leq) is a compact T_2 -ordered space. Let \mathcal{V} be a unique quasi-uniformity on X compatible with (Y, T, \leq) . For each $V \in \mathcal{V}$ set $H_V = V \setminus \{(x, y) \in Y \times Y : x = (\omega_1, \omega_0) \text{ and } y \neq (\omega_1, \omega_0)\}$.

Then $\{H_V : V \in \mathcal{V}\}$ generates a quasi-uniformity \mathcal{U} . The completely regular ordered space $(Y, T(\mathcal{U}^*), \cap \mathcal{U})$ is not a strictly completely regular ordered space. Then by Theorem 20, the ordered spaces $C_k^\uparrow(X, (Y, T(\mathcal{U}^*), \cap \mathcal{U}))$ and $C_p^\uparrow(X, (Y, T(\mathcal{U}^*), \cap \mathcal{U}))$ are not strictly completely regular ordered spaces.

For every ordered space (X, τ, \leq) there is a bitopological space $(X, \tau^\#, \tau^b)$ associated with it where $\tau^\#$ denotes the collection of open upper sets of τ and τ^b denotes the collection of open lower sets of τ .

PROPOSITION 23. ([6], Proposition 1) *Let (Y, τ', \leq') be a completely regular ordered space. Then the bitopological space $(Y, \tau'^\#, \tau'^b)$ is a pairwise completely regular if and only if (Y, τ', \leq') is a strictly completely regular ordered space.*

COROLLARY 24. *Let (X, τ, \leq) be an ordered space and (Y, τ', \leq') be a quasi-uniformizable ordered space. Then the following are equivalent:*

- (a) (Y, τ', \leq') is a strictly completely regular ordered space;
- (b) $C_p^\uparrow(X, Y)$ is a strictly completely regular ordered space;
- (c) $C_k^\uparrow(X, Y)$ is a strictly completely regular ordered space;
- (d) $(Y, \tau'^\#, \tau'^b)$ is quasi-uniformizable;
- (e) $(C^\uparrow(X, Y), T_p^\#, T_p^b)$ is quasi-uniformizable;
- (f) $(C^\uparrow(X, Y), T_k^\#, T_k^b)$ is quasi-uniformizable.

Proof. (a) \iff (b) \iff (c) follows from Corollary 21.

(a) \iff (d) follows from Proposition 23.

(d) \iff (e) \iff (f) follows from [18] and [19]. ■

EXAMPLE 25. Let us consider Example 22(b). From Proposition 23 we have that $(Y, T(\mathcal{U}^*)^\#, T(\mathcal{U}^*)^b)$, $(C^\uparrow(X, Y), T_p^\#, T_p^b)$ and $(C^\uparrow(X, Y), T_k^\#, T_k^b)$ are not quasi-uniformizable.

Remark 2. In section 3, a general collection \mathcal{A} could replace \mathcal{F} and \mathcal{K} . This will mean that for this collection of subsets of X we can generate the topology and the order in the same way we did with \mathcal{F} and \mathcal{K} . In this way we can study the interaction between the properties of an ordered space (Y, τ', \leq') and the properties of such an ordered space $(C^\uparrow(X, Y), T_{\mathcal{A}}, \leq_s)$. Work is in progress in this direction. We also conjecture that this could also be done in

the context of bitopological spaces as in [18] and [19], where in the place of a collection consisting of finite subsets or a collection consisting of compact subsets, a general collection \mathcal{A} , or the collection \mathcal{A} with a specific property could be considered and generate the bitopology on the function space. This approach will be in line with the one used in [12] where the notion of networks was used.

ACKNOWLEDGEMENTS

The author wishes to thank Professor Sergio Salbany, for his continuous encouragement during the preparation of this manuscript. This work was supported by Foundation for Research and Development via the Unisa Research Group.

The author further thanks the referees for their comments which helped to improve the paper.

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