EXTRACTA MATHEMATICAE Vol. **15**, Núm. 3, 547 – 561 (2000)

Two Remarks on the Structure of Sets of Exposed and Extreme Points[†]

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(Research paper presented by D. Yost)

AMS *Subject Class.* (1991): 28A05, 46B20, 46B26 *Received March* 6, 2000

In Section 1 we are showing that the set $\exp K$ of exposed points of a convex subset K of a Banach space, where K is separable and compact with respect to the weak topology, is Borel in the weak topology and $\mathbf{F}_{\sigma\delta}$ in the norm topology. If K is norm compact and convex, then we get that $\exp K$ is even a norm $\mathbf{K}_{\sigma\delta}$-set. This solves Problem 1.14 of [2] asking whether the set of exposed points $\exp K$ of a norm compact convex subset K of a Banach space is analytic or even Borel (see also Remark 1 below). Let us remark that a partial solution saying that $\exp K$ is analytic in such a case, and in fact even in the case of a norm $\mathbf{K}_\sigma$ and convex set K, was given in [4, p.255]. Moreover, our result obviously implies that $\exp K$ is (weakly) Borel for every closed bounded convex K in a separable reflexive Banach space. The latter fact was conjectured in [4, p.254]. We get the results of Section 1 using the methods of [2] with minor modifications.

Section 2 concerns some "counterexamples" in nonseparable Banach spaces. We are showing that in many nonseparable Banach spaces, e.g. in those that admit a Markushevich basis or in $\ell_\infty(\mathbb{N})$, there is a bounded closed convex set K (even a ball with respect to some equivalent norm) such that the sets $\exp K$ and $\operatorname{ext} K$ of all exposed or extreme points of K, respectively, are not Borel (see Theorem 3 below for the exact, and in fact stronger, formulation). Theorem 2 and its Corollary are devoted to the case of weakly compactly generated spaces, where weakly compact convex sets K and the corresponding sets $\operatorname{ext} K$ and $\exp K$ are discussed.

[†]Our investigation was supported by GAUK 160/1999, GAČR 201/97/0216, GAČR 201/97/1161, and MSM 113200007.

Let us first recall some necessary notions. Let K be a convex subset of a Banach space X. The element $x \in K$ is an *exposed point* of K if there is a continuous linear form $f \in X^*$ such that $\{x\} = \{z \in K\colon f(z) = \sup_K f\}$. We use $\exp K$ to denote the set of all exposed points of K. The element x of K is an *extreme point* of K if x does not belong to any open segment contained in K. The set of all extreme points of K is denoted by $\operatorname{ext} K$ here.

1. Exposed points of weakly compact convex sets

First, we formulate the main result of this section in the part (b) of the following theorem. We state the claim (a) for completeness. We shall see that it is an immediate corollary of well-known facts from [4].

Theorem 1. *Let K be a convex subset of a Banach space X such that K is compact and separable with respect to the weak topology.*

(a) *Then $\operatorname{ext} K$ is a $\mathbf{K}_{\sigma\delta}$ subset of X endowed with the weak topology.*

(b) *Then $\exp K$ is an $\mathbf{F}_{\sigma\delta}$ subset of X endowed with the norm topology. It is also Borel in the weak topology of X. Moreover, it is $\mathbf{G}_{\delta\sigma\delta}$ in the compact metrizable weak topology of K.*

Before proving the theorem we shall do few auxiliary observations.

We introduce first a notation that plays a crucial role in what follows. Let K^* be the set of restrictions of elements of the closed unit ball B_{X^*} of the dual space X^* to K. We use w^*_K to denote the topology of pointwise convergence in K^* with respect to points from K, i.e. the trace of the product topology from $\mathbb{R}^K$ to K^*. Notice that the space (K^*, w^*_K) is compact as the continuous image of (B_{X^*}, w^*).

Now we point out the metrizability of (K^*, w^*_K) and of K, two easy and well-known facts, with their short and straightforward proofs for the convenience of the reader.

Lemma 1. *If K is as in Theorem 1, the space (K^*, w^*_K) is compact and metrizable and the weak topology of K is metrizable.*

Proof. Indeed, let S be a countable dense subset of K in the weak topology. Now, the compact topology w^*_K coincides on K^* with the (obviously weaker) metrizable topology of pointwise convergence on S.

Similarly, since the elements (that are restrictions of elements of X^*) of any countable dense subset S^* of (K^*, w^*_K) separate points of the weakly compact set K and they are weakly continuous, the weak topology on K coincides with the metrizable topology $\sigma(K, S^*)$ of pointwise convergence on S^*. ∎

So K endowed with the weak topology is analytic (a continuous image of a separable complete metric space) and $\mathbf{K}_\sigma$ (in fact it is even a compact metric space), and we may use the straightforward argument of [4, pp. 251-252] to see that $\operatorname{ext} K$ is a weakly $\mathbf{K}_{\sigma\delta}$ subset of X. Thus the statement (a) is established and it remains to prove our main statement (b).

Let ρ denote a metric on K that induces its weak topology. So (K, ρ) is a compact metric space. Let us denote by $\mathcal{B}_n$ a finite cover of K by open balls having ρ-diameter at most $\frac{1}{n}$. The next lemma expresses one of the main ideas of the method used in [2] to prove their Theorem 1.9 and its corollaries adapted to our special case. We formulate it explicitly with the proof for the convenience of the reader.

LEMMA 2. *Let K, K^*, ρ, and $\mathcal{B}_n$ be as above. Then $\exp K$ is the set of all $x \in K$ such that, for every $n \in \mathbb{N}$, there are $f_n \in K^*$ and $B_n \in \mathcal{B}_n$ such that*

$$f_n(x) = \max_K f_n \quad \text{and} \quad f_n(x) > \max_{K \setminus B_n} f_n.$$

Proof. If $x \in \exp K$, then there is an $f \in X^*$ with $f(x) > f(y)$ for every $y \in K \setminus \{x\}$. Since this holds for each positive multiple of f, we may achieve that f belongs to the unit ball B_{X^*}. For every $n \in \mathbb{N}$ we find an element B_n of $\mathcal{B}_n$ containing x. By the weak compactness of $K \setminus B_n$ we have that $(f|K)(x) > \max_{K \setminus B_n} f$. So we may put $f_n = f|K$ and one inclusion is proved.

Let $x \in K$ be such that there are $B_n \in \mathcal{B}_n$ and $f_n \in K^*$ with $f_n(x) = \max_K f_n$ and $f_n(x) > \max_{K \setminus B_n} f_n$ for every $n \in \mathbb{N}$. Let $f_n^* \in B_{X^*}$ be some extension of $f_n \in K^*$. Put $f^* = \sum_{n=1}^{\infty} \frac{1}{2^n} f_n^*$. Obviously, $f^* \in B_{X^*}$. Also, if $y \in K \setminus \{x\}$, then $f^*(x) > f^*(y)$ because $f_n(x) \geq f_n(y)$ for every $n \in \mathbb{N}$ and $f_n(x) > f_n(y)$ for n's for which $y \notin B_n$. ∎

Proof of Theorem 1(*b*). Let us define

$$F = \{(x, f) \in K \times K^* \colon f(x) = \max_K f\}$$

and

$$F_B = \{(x, f) \in K \times K^* \colon f(x) > \max_{K \setminus B} f\},$$

for every $B \in \bigcup_{n \in \mathbb{N}} \mathcal{B}_n$.

It is almost obvious that the set F is closed in $(K, \|\cdot\|) \times (K^*, w_K^*)$. If $(x, f) \notin F$ then there is an $x_0 \in K$ such that $f(x) + \epsilon < \max_K f = f(x_0)$ for some positive ϵ. Now every pair $(y, g) \in K \times K^*$ such that $\|y - x\| < \epsilon/3$, $|g(x) - f(x)| < \epsilon/3$, and $|g(x_0) - f(x_0)| < \epsilon/3$ also does not belong to F as g is Lipschitz with constant one. Hence F is closed.

For every $B \in \bigcup_{n\in\mathbb{N}} \mathcal{B}_n$ we have $F_B = \bigcup\{F_B(a,b)\colon a, b \in \mathbb{Q},\ a < b\}$, where

$$F_B(a,b) = \{(x, f) \in (K, \|\cdot\|) \times (K^*, w_K^*)\colon f(x) \geq b \text{ and } \max_{K\setminus B} f \leq a\}.$$

The set $\{(x, f) \in K \times K^*\colon \max_{K\setminus B} f \leq a\}$ is clearly closed in $(K, \|\cdot\|) \times (K^*, w_K^*)$.

To prove that $\{(x, f) \in (K, \|\cdot\|) \times (K^*, w_K^*)\colon f(x) \geq b\}$ is closed we may proceed similarly as for F.

Due to Lemma 2 we have that

$$\exp K = \bigcap_{n\in\mathbb{N}} \bigcup_{B\in\mathcal{B}_n} \bigcup_{\substack{a,b\in\mathbb{Q}\\ a<b}} \pi(F \cap F_B(a,b)),$$

where π is the projection of $(K, \|\cdot\|) \times (K^*, w_K^*)$ to $(K, \|\cdot\|)$.

It is easy to check that the projections $\pi(F \cap F_B(a,b))$ are closed in $(K, \|\cdot\|)$ as (K^*, w_K^*) is compact and $F \cap F_B(a,b)$ are closed subsets of $(K, \|\cdot\|) \times (K^*, w_K^*)$. So $\exp K$ is an $\mathbf{F}_{\sigma\delta}$ subset of $(K, \|\cdot\|)$ and thus also of X endowed with the norm topology.

So $\exp K$ is of the form $\bigcap_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} F_{nk} \cap K$, where F_{nk} are norm closed. Since K is convex and weakly separable, it is also norm separable. One may notice that the norm closure of rational convex combinations of elements of a countable weakly dense subset is a closed convex set that has to coincide with K. Hence each norm open subset G of K is covered by countably many closed balls intersected with K and so G is $\mathbf{F}_\sigma$ in the weak topology. Thus each norm closed subset of K is weakly $\mathbf{G}_\delta$ in K. Now each set $F_{nk} \cap K$ is norm closed in K and so it is also weakly $\mathbf{G}_\delta$ in K by the preceding argument. Finally, $\exp K$ is $\mathbf{G}_{\delta\sigma\delta}$ in the weak topology of K.

This finishes the proof of the fact that $\exp K$ is weakly Borel (in fact weakly $\mathbf{G}_{\delta\sigma\delta}$ in K and thus also weakly $\mathbf{K}_{\sigma\delta\sigma\delta}$ in X) due to its above description.

If moreover K is compact in the norm topology, then the sets F and $F_B(a,b)$ above are compact in $(K, \|\cdot\|) \times (K^*, w_K^*)$. Thus their projections to $(K, \|\cdot\|)$ are also compact and so $\exp K$ is $\mathbf{K}_{\sigma\delta}$ in the norm topology due to the above description of $\exp K$. ∎

Remark 1. If $(K, \|\cdot\|)$ is compact, then w_K^* on K^* coincides with the topology of uniform convergence on K and Theorem 1.6 of [2] also applies. Indeed, $K^* \subset C(K)$ is a closed subset in $C(K)$ endowed with the topology of pointwise convergence as it is compact. So it is also closed in the topology of uniform convergence. Also K^* is totally bounded in the uniform convergence topology since all its elements are Lipschitz with constant one as restrictions of elements of B_{X^*}.

Added in proof. It seems to be of some interest that it is not difficult to modify the proof of Theorem 1(b) to get that $\exp K$ is $\mathbf{F}_{\sigma\delta}$ in the norm topology if $K = \bigcup_{m\in\mathbb{N}} K(m)$, K convex and $K(m)$ compact and separable with respect to the weak topology of a Banach space X.

Roughly speaking, it suffices to use $\mathcal{B}_n(m)$ related to $K(m)$ as $\mathcal{B}_n$ were related to K in the above proof. Further, if K^* is again the set of all restrictions of elements of B_{X^*} to K, then $\exp K$ is the set of all $x \in K$ such that, for every $m, n \in \mathbb{N}$, there are $f_n^m \in K^*$ and $B_n(m) \in \mathcal{B}_n(m)$ such that

$$f_n^m(x) = \sup_K f_n^m \quad \text{and} \quad f_n^m(x) > \max_{K(m)\setminus B_n(m)} f_n^m,$$

and thus it can be shown that $\exp K$ is $\mathbf{F}_{\sigma\delta}$ in the norm topology similarly as in the proof of Theorem 1(b).

2. Extreme and exposed points in nonseparable spaces

We shall give examples of nonseparable Banach spaces X in which we are able to find a bounded closed convex set K with the sets $\exp K$ and $\operatorname{ext} K$ being “nonmeasurable” in various senses, e.g. non-Borel, nonanalytic, or even without the Baire property in some closed set.

Remark 2. The idea of our following examples goes back to Example 2 of [4]. In fact, we obtain a different, perhaps simpler, example in the space of signed Radon measures than that of Example 2 from [4]. See the discussion preceding Theorem 3 below.

Our construction is also inspired by the examples of convex continuous functions with “nonmeasurable” set of points of Gâteaux differentiablity from [3]. In fact, L. Zajíček noticed first that e.g. in the case of nonseparable Hilbert spaces we may use Fenchel duality to get examples of balls (with respect to equivalent norms) with “bad” sets of exposed points from the examples of continuous convex functions in [3]. He also noticed that, since we are able to modify the examples from [3] so that the functions constructed there are equivalent norms, we may use the correspondence between the norm in X with the set of its Gâteaux smooth points, and the dual ball in X^* with its exposed points. This idea works only in reflexive spaces having a strictly convex norm. We do not see how we could get our more general results in this way. Therefore we use the more straightforward construction below.

We need for our construction few elementary and almost evident facts concerning the unit disc $D_0 = \{x \in \mathbb{R}^2 \colon \langle x, x\rangle \leq 1\}$. Since we did not find appropriate reference, we prove here the following lemma. We need it just in the case when the measures represent finite convex combinations, i.e. they are probabilities with finite supports. However it seems that the more general formulation is more transparent. One may notice that we use in the proof that elements of the boundary $T_0 = \{x \in \mathbb{R}^2 \colon \langle x, x\rangle = 1\}$ are strongly exposed. By the weak convergence of measures we mean the convergence on (bounded) continuous functions on D_0.

Lemma 3. *If μ_n are Borel probabilities on D_0 such that the sequence of their barycenters $b\mu_n = \int\limits_{D_0} x d\mu_n(x)$ converges to $x_0 \in T_0$, then μ_n converge weakly to the Dirac measure δ_{x_0} at x_0, i.e. for every open $U \subset \mathbb{R}^2$ containing x_0, we have that $\lim\limits_{n\to\infty} \mu_n(D_0 \setminus U) = 0$.*

Proof. For an arbitrary open U containing x_0 we find $\varepsilon > 0$ such that

$$K_\varepsilon = \{x \in D_0 \colon \langle x, x_0\rangle > 1 - \varepsilon\} \subset U \cap D_0.$$

We put $L_\varepsilon = D_0 \setminus K_\varepsilon$.

Now

$$\langle x_0 - b\mu_n, x_0\rangle = \int_{L_\varepsilon} \langle x_0 - x, x_0\rangle d\mu_n + \int_{K_\varepsilon} \langle x_0 - x, x_0\rangle d\mu_n \geq$$

$$\geq \varepsilon\mu_n(L_\varepsilon) \geq \varepsilon\mu_n(D_0 \setminus U).$$

Hence $\lim\limits_{n\to\infty} \mu_n(D_0 \setminus U) = 0$. ∎

The main construction concerning our examples of sets K with "nonmeasurable" $\exp K$ and $\operatorname{ext} K$ follows.

MAIN LEMMA 4. *Let X be a Banach space with* $\dim X \geq 3$. *Let* $S_0 \subset T_0$, $E(s) = [-e_s, e_s] = \operatorname{conv}\{-e_s, e_s\}$ *with* $e_s \in X$, $0 < \|e_s\| \leq 1$ *for* $s \in S_0$, *and* $E(t) = \{0\}$ *for* $t \in T_0 \setminus S_0$ *be such that*

$$\bigcap \{\overline{\operatorname{conv}}\ E(U \cap T_0) \colon t \in U,\ U \text{ open}\} = \{0\} \tag{1}$$

for every $t \in T_0 \setminus S_0$. *Let P be an arbitrary two-dimensional subspace of X and* $S, T \subset P$ *be the corresponding images of* $S_0, T_0 \subset \mathbb{R}^2$ *under a linear isometry of* $\mathbb{R}^2$ *onto P endowed with a scalar product* $\langle \cdot, \cdot \rangle$. *Then,*

(i) *there is a closed convex bounded set* $K \subset X$ *such that*

$$P \cap \exp K = P \cap \operatorname{ext} K = T \setminus S. \tag{2}$$

(ii) *If moreover $E(T)$ is relatively weakly compact, then there is a weakly compact convex set K such that (2) holds.*

(iii) *If* $S \subset T$ *is symmetric, i.e. if* $S = -S$, *then there is an equivalent norm* $|\cdot|$ *on X such that*

$$P \cap \exp B = P \cap \operatorname{ext} B = T \setminus S,$$

where B is the closed unit ball of $|\cdot|$.

Proof. Let P with $\langle \cdot, \cdot \rangle$ from the statement be chosen arbitrarily. We may and shall identify S_0 with S and T_0 with T. Let Y be a closed linear subspace of X that is a complement to P. As $\dim X \geq 3$, Y is nontrivial. Let $p_Y \colon X \to Y$ be the continuous linear projection of X onto Y with $p_Y^{-1}(0) = P$ and p be the continuous linear projection of X onto P defined by $p(x) = x - p_Y(x)$.

We are going to show that we may suppose that $E(s) \subset Y \cap B_X$ for every $s \in T$.

We prove first that we may suppose without loss of generality that $e_s \notin P$ for all $s \in S$ in the definition of E. Notice that if $s_n \in S$ tend to $t \in T \setminus S$ are such that $e_{s_n} \in P$, then by the compactness of the closed unit ball in P and by the condition (1), the norms $\|e_{s_n}\|$ tend to zero. Therefore it suffices to replace each $e_s \in P$ in the definition of E by any $e_s^* \in Y$ for which $\|e_s^*\| = \|e_s\|$.

To verify it let us write e_s^* for e_s if $e_s \notin P$. Let U_n be a sequence of open neighbourhoods of $t \in T \setminus S$ with $\bigcap_{n \in \mathbb{N}} U_n = \{t\}$. Considering now any sequence of absolutely convex combinations $x_n = \sum_{i=1}^{k(n)} a_i(n) e^*_{s_i(n)}$ with $s_i(n) \in S \cap U_n$, the corresponding absolutely convex combinations x_n^* that arise by omitting those summands for which $e^*_{s_i(n)} = e_{s_i(n)} \notin P$ converge to zero.

So if x_n converge to some $x \in \bigcap\{\overline{\text{conv}}\ E(U \cap T) : t \in U, U \text{ open}\}$, then also $x_n - x_n^*$ converge to x. However $x_n - x_n^*$ are absolutely convex combinations of the elements of the original family $\{e_s : s \in S\}$ and so by (1) we have that $x = 0$. So we may and shall suppose that $e_s \notin P$ for $s \in S$.

Put $E_Y = \frac{1}{\|p_Y\|} p_Y \circ E$. We shall show that E_Y fulfils the corresponding analogue of the property (1) by a contradiction. Let

$$0 \neq e \in \bigcap\{\overline{\text{conv}}\ \big(E_Y(U \cap T)\big) : t \in U,\ U \text{ open}\}$$

for a $t \in T \setminus S$. Notice that the mapping p_Y restricted to the preimage $p^{-1}(\|p\| B_P)$ of the compact ball $\|p\| B_P$ in P is closed being a projection along a compact space. Hence $\overline{\text{conv}}\ E_Y(U \cap T) = \frac{1}{\|p_Y\|} p_Y\big(\overline{\text{conv}}\ E(U \cap T)\big)$ and the sets $K_U = p_Y^{-1}(\|p_Y\| e) \cap \overline{\text{conv}}\ E(U \cap T)$, $t \in U$, U open, form a centered family of nonempty compact sets in the two-dimensional space $p_Y^{-1}(\|p_Y\| e)$. Thus there is an f in $\bigcap\{K_U : t \in U,\ U \text{ open}\}$ with $p_Y(f) = \|p_Y\| e \neq 0$ and this contradicts (1). So the mapping $E_Y = \frac{1}{\|p_Y\|} p_Y \circ E$ fulfils the corresponding analogue of (1) and $E_Y(t) \subset Y \cap B_X$ for $t \in T$.

Hence we may and shall consider such an $E = E_Y$ in the following constructions of K and B.

We define $K = \overline{\text{conv}}\ \{t + y : t \in T,\ y \in E(t)\}$. The set K is obviously closed convex and bounded. We shall show that

$$P \cap \exp K = P \cap \operatorname{ext} K = T \setminus S.$$

Notice that $K \subset p^{-1}(D)$ and $P \cap K = D$, where $D = \{x \in P : \langle x, x\rangle \leq 1\}$, and thus $P \cap \exp K \subset P \cap \operatorname{ext} K \subset T$. Therefore it is sufficient to consider all $s \in T$ and show that $s \in S$ implies $s \notin \operatorname{ext} K$ and $s \in T \setminus S$ implies $s \in \exp K$.

If $s \in S$, then $s = \frac{1}{2}(s + e_s) + \frac{1}{2}(s - e_s)$ and therefore s is not an extreme point of K.

Let $s \in T \setminus S$. We are going to show that s is the only element of K for which $f_s(s) = \max_K f_s = 1$, where $f_s(x) = \langle s, p(x)\rangle$. Thence we shall see that s is an exposed point of K.

Notice that the set K is a subset of the closed convex set $\{x \in X : f_s(x) \leq 1\}$ and thus $f_s(s) = \max_K f_s = 1$. Let $x \in K$ be such that $f_s(x) = \max_K f_s = 1$. Thus $p(x) = s$. Then there is a sequence (μ_n) of probabilities with finite support (representing finite convex combinations) contained in $\{t + y : t \in T,\ y \in E(t)\}$ such that their barycenters $b\mu_n$ converge to $x = s + p_Y(x)$. Let μ_n^* be their images under p and let $U_k^* = P \cap B(s, \frac{1}{k})$. Due to Lemma 3, we have $\lim_{n\to\infty} \mu_n^*(P \setminus U_k^*) = 0$ for every $k \in \mathbb{N}$. Let us denote $U_k = p^{-1}(U_k^*)$.

It follows easily that we may choose an increasing sequence (n_k) of natural numbers such that for $\nu_k = \frac{1}{\mu_{n_k}(U_k)}\, \mu_{n_k}|_{U_k}$ we have that $b\mu_{n_k} - b\nu_k \to 0$ as $n \to \infty$, and thus $(b\nu_k)$ tends to x. (We may choose n_k such that $\mu^*_{n_k}(P \setminus U^*_k) \leq \frac{1}{k}$.) Since the support of ν_k is contained in $\{t + y \colon t \in T \cap B(s, \frac{1}{k}), y \in E(t)\}$, the point $p_Y(b\nu_k)$ is in $\overline{\text{conv}}\ E(B(s, \frac{1}{k}) \cap T)$ and so $p_Y(x) = \lim_{n\to\infty} p_Y(b\mu_n) = \lim_{k\to\infty} p_Y(b\nu_k) = 0$, using the property (1). Hence $x = s$ and therefore s is an exposed point of K.

If $E(T)$ is relatively weakly compact, then $\{t + y \colon t \in T, y \in E(t)\}$ is contained in the relatively weakly compact set $T + E(T)$ and thus K, being the convex closure of $\{t + y \colon t \in T,\ y \in E(t)\}$, is weakly compact.

If $S = S^+ \cup S^-$, with $S^+ \cap S^- = \emptyset$ and $S^+ = -S^-$, we fix a closed ball B_0 centered at zero in X such that its projection $p(B_0)$ to P is contained in the disc $\{x \in P \colon \langle x, x\rangle \leq \frac{1}{2}\}$. We put $E^*(t) = E(t)$ for $t \in S^+$, and $E^*(t) = E(-t)$ for $t \in S^-$, and finally we put $B = \overline{\text{conv}}\ (K \cup B_0)$, where $K = \{t + y \colon t \in T, y \in E^*(t)\}$ as above. Now E^* fulfils the analogue of (1) and $E^*(t) = E^*(-t)$.

It is almost obvious that $B \cap P = K \cap P = D$. So we have that

$$P \cap \exp B \subset P \cap \operatorname{ext} B \subset P \cap \operatorname{ext} K = P \cap \exp K = T \setminus S.$$

It remains to check that $T \setminus S \subset \exp B$. Let $t \in T \setminus S$. We know that $f_t(t) = 1 > f_t(k)$ for every $k \in K \setminus \{t\}$.

If $x \in B \setminus \{t\}$, then there are $\alpha_n \in [0, 1]$, $b_n \in B_0$, and $k_n \in K$ such that $x_n = \alpha_n b_n + (1 - \alpha_n)k_n \to x$. We may suppose without loss of generality that $\alpha_n \to \alpha \in [0, 1]$.

If $\alpha = 0$, then $\alpha_n b_n \to 0$ as the sequence (b_n) is bounded. So $k_n \to x$ and $x \in K \setminus \{t\}$. Thus $f_t(x) < 1$.

Otherwise, $\alpha_n \to \alpha > 0$. Then

$$f_t(x) = \lim_{n\to\infty} f_t(x_n) \leq \lim_{n\to\infty} \alpha_n \cdot \frac{1}{2} + (1 - \alpha_n) = 1 - \frac{\alpha}{2} < 1.$$

In any case $f_t(x) < 1$, and so t is an exposed point of B as well.

Since B is convex closed bounded and symmetric, and since it contains the ball B_0, it is the closed unit ball of the norm $|\cdot|$, defined as the Minkowski functional of B, that is equivalent to the norm $\|\cdot\|$ of X. ∎

As corollaries of Main Lemma 4 we obtain examples in concrete spaces. We need to investigate the possibility to find E fulfilling (1) from Main Lemma 4 in particular nonseparable Banach spaces.

Remark 3. Let us point out that having E with property (1) with respect to some closed linear subspace X of Z, we have E with the same property for Z.

First we discuss the property (1) in the following two propositions.

The first one gives a characterization that is almost obvious and therefore we omit its proof.

PROPOSITION 1. *Let X be a Banach space and S_0, T_0 be as in Main Lemma 4. The existence of E with the property (1) from Main Lemma 4 is equivalent to the following condition:*

There is a family $e_s \in B_X \setminus \{0\}$, $s \in S_0$, such that if (F_n) is any sequence of pairwise disjoint finite subsets of S_0 tending to $\{t\}$, $t \in T_0 \setminus S_0$, in the Hausdorff metric, $x_n \in \operatorname{conv} \bigcup \{[-e_t, e_t] \colon t \in F_n\}$, and $x_n \to x$, then $x = 0$.

The following observation gives a sufficient condition for the possibility to apply our Main Lemma 4 in Theorems 2 and 3 below.

PROPOSITION 2. *Let X be a Banach space. Let F be a subset of $B_X \setminus \{0\}$ such that every injective sequence of elements of F converges to zero in a Hausdorff locally convex topology τ which is comparable with (weaker or stronger than) the norm topology. If S_0 and T_0 are as in Main Lemma 4 and S_0 has the cardinality less or equal to the cardinality of F, then there exists E with the property (1).*

Proof. We use the equivalent formulation from Proposition 1. Let $e_t, t \in S_0$, be any injective "ordering" of elements of F. Let $x_n \in \operatorname{conv} \left(\bigcup \{\{e_s\} \cup \{-e_s\} \colon s \in F_n\}\right)$, where F_n is a sequence of pairwise disjoint finite subsets of S_0 tending to $\{t\}$, $t \in T_0 \setminus S_0$. The elements of $\bigcup_{n \in \mathbb{N}} F_n$ can be arranged into an injective sequence $(f_m), m \in \mathbb{N}$. If U is an absolutely convex neighbourhood of zero in τ, then $f_m \in U$ for sufficiently large m, thus $x_n \in U$ for n sufficiently large. Hence x_n converge to zero in τ. If x_n's converged also in the norm topology, the limits necessarily coincide. ∎

Now we formulate statements showing in particular that in many "nice" nonseparable Banach spaces the convex subsets with "bad" sets of exposed and extreme points exist. Notice that the class of weakly compactly generated spaces includes all reflexive spaces etc. We include also some separable cases although the claims do not say much for them.

THEOREM 2. *Let X be a weakly compactly generated (WCG) subspace of a Banach space Z with $\dim X \geq 3$. Let P be a two-dimensional subspace of X, and $T \subset P$ be the unit sphere with respect to some scalar product on P. Then, for every, respectively every symmetric, subset S of T that has the cardinality less or equal to the density of X, there are a weakly compact convex subset K of Z, and an equivalent norm with the closed unit ball B on Z, respectively, such that*

$$P \cap \operatorname{ext} K = P \cap \exp K = P \cap \operatorname{ext} B = P \cap \exp B = T \setminus S.$$

In particular, if X is nonseparable, there is a weakly compact convex set K, and an equivalent norm on Z with the closed unit ball B, such that the sets $\operatorname{ext} K$, $\exp K$, $\operatorname{ext} B$, and $\exp B$ are not Borel.

If the density of X is greater or equal to the cardinality of the continuum, then $K \subset Z$ and $B \subset Z$ can be chosen such that $P \cap \operatorname{ext} K = P \cap \exp K = P \cap \operatorname{ext} B = P \cap \exp B$ and $P \cap \operatorname{ext} K$ does not have the Baire property in T.

Proof. Every WCG space X contains a weakly compact subset A such that it has the only accumulation point zero and the linear span of A is dense in X [1, Theorem 1.2.5.]. It follows that the cardinality of A is equal to the density of X and that every injective sequence in A converges weakly to zero.

Let S_0 and T_0 be the images of S and T under some linear isometry of P (with the considered scalar product) onto $\mathbb{R}^2$. So the cardinality of S_0 is smaller or equal to the cardinality of A. By Proposition 2, there is a set-valued mapping E from T_0 to X with the property (1) of Main Lemma 4. We mentioned already in Remark 3 that E has the property (1) also as a mapping to subsets of Z. Hence it follows from Main Lemma 4 that both the weakly compact set $K \subset Z$ and the ball $B \subset Z$ constructed there fulfil the first statement of the theorem.

If X is nonseparable, then the set S may be chosen symmetric non-Borel and thus the second claim follows.

If the density of X is greater or equal to the cardinality of the continuum, then moreover S may be chosen so that it does not have the Baire property in T. ∎

COROLLARY 1. *The following statements on a Banach space Z are equivalent.*

(a) *Z contains a nonseparable weakly compact subset (i.e. Z contains a nonseparable weakly compactly generated subspace).*

(b) *Z contains a weakly compact convex subset K with* $\operatorname{ext} K$ *non-Borel.*

(c) *Z contains a weakly compact convex subset K with* $\exp K$ *non-Borel.*

Proof. The implications (a) implies (b) and (a) implies (c) follow from Theorem 2.

If the statement (a) does not hold, then by Theorem 1 also the statements (b) and (c) do not hold. ∎

The next theorem points out several other examples of Banach spaces to which Main Lemma 4 can be applied to get closed unit balls B of some equivalent norms with strange sets of exposed and extreme points.

The first example (a) of spaces that admit a Markushevich basis includes many standard spaces. E.g. the weakly Lindelöf determined spaces, or even all "Plichko spaces", have a Markushevich basis. This is a result of M.Valdivia [7, Corollary 2.2.]. One may find this result using the above terminology in the survey paper [5, Theorems 4.2.5 and 4.2.6]. We may notice that the space $\mathcal{M}(H)$ of all Radon signed measures with the finite variation endowed with the variational norm on a Hausdorff space H contains $\ell_1(H)$ as a closed subspace and since there is a Markushevich basis on $\ell_1(H)$ we may apply the following Theorem 3(a) to the space $\mathcal{M}(H)$ and obtain in particular an analogical example to the mentioned one from [4, Example 2]. In fact, it is easy to apply Proposition 1 directly in this case. Also all spaces $L^1(\mu)$ with μ a (not necessarily finite) nonnegative measure have a Markushevich basis. This can be obtained using [6, Corollary on page 136] and [5, Theorem 4.4.1.].

The next example (b) concerns dual spaces $X = Y^*$ of many (nonseparable) Banach spaces. It uses the existence of bounded and sufficiently large sets $F \subset X$ having the property (Z_0). Let us recall that $F \subset Y^*$ has the property (Z_0) if each injective sequence of elements of F converges to zero in the w^* topology. This property (Z_0) was introduced and discussed in detail in [3]. Let us mention that we cover by this theorem e.g. the dual spaces $\ell_\infty(\Gamma)$ with uncountable Γ, although it is easy to notice that the characteristic functions of singletons give the asked set F of Proposition 2 with respect to the w^* topology immediately in this case.

We do not see if it is possible to apply Proposition 2 to the case of the Banach space $\ell_\infty(\mathbb{N})$. Therefore we apply Main Lemma 4, or rather Proposition 1, directly in this case (c) below.

THEOREM 3. *Let Z be a Banach space that contains a closed linear subspace X with* $\dim X \geq 3$ *that fulfils one of the following assumptions* (a) - (c).

Let P be a two-dimensional subspace of Z, and $T \subset P$ be the unit sphere with respect to some scalar product on P. Let S be any symmetric subset of T that has the cardinality κ.

(a) *There is a Markushevich basis $\{(x_a, f_a) \in X \times X^* : a \in A\}$ in X such that* $\operatorname{card} A \geq \kappa$.

(b) *Let $X = Y^*$ be a dual Banach space, $F \subset X$ have the property (Z_0) mentioned above, and* $\operatorname{card} F \geq \kappa$.

(c) *Let $X = \ell_\infty(\mathbb{N})$.*

Then there is an equivalent norm $|\cdot|$ on Z such that

$$P \cap \operatorname{ext} B = P \cap \exp B = T \setminus S,$$

where B is the closed unit ball with respect to the norm $|\cdot|$.

In the particular cases, when S can be taken as an arbitrary subset of T with the cardinality $\aleph_1$ or even an arbitrary subset of T, we may obtain B with the sets $\operatorname{ext} B \cap P$ *and* $\exp B \cap P$ *non-Borel or even without the Baire property in T.*

Proof. In all what follows we suppose that $S_0 \subset T_0$ be the image of S under some linear isometry of P (with the mentioned scalar product) onto $\mathbb{R}^2$ and we identify S with S_0 and T with T_0. According to the mentioned fact that the property (1) of E from Main Lemma 4 being fulfilled with respect of X is automatically fulfilled for Z, it remains to verify the existence of such an E for X.

(a) Let $F^* = \operatorname{span}\{f_a : a \in A\}$ and $\tau = \sigma(X, F^*)$ be the topology on X generated by F^*. Then τ and $F = \{x_a : a \in A\}$ fulfil the assumptions of Proposition 2.

(b) There is an E into X with the property (1) due to Proposition 2.

(c) By Main Lemma 4, it is sufficient to find the set-valued mapping E with the property (1). Let $\mathbb{Q} = \{q_n : n \in \mathbb{N}\}$ be a countable dense subset of T. For simplicity we construct our example in $\ell_\infty(\mathbb{Q})$. We consider the inner metric ρ on T and we use $B_\rho(t, r)$ to denote the set $\{s \in T : \rho(s, t) < r\}$ for $r > 0$. For $t \in T$, let $S_t = \mathbb{Q} \cap B_\rho(t, 1)$. We denote by $e_t : \mathbb{Q} \to \{0, 1\}$ the characteristic function of S_t.

We shall show that the family $e_t, t \in S$, fulfils the conditions of Proposition 1 by contradiction. Let $t_0 \in T \setminus S$, let F_n be a sequence of pairwise disjoint finite subsets of S tending to $\{t_0\}$ in the Hausdorff metric, let x_n be of the form $\sum_{t \in F_n} \lambda(t) e_t$, where $\sum_{t \in F_n} |\lambda(t)| \leq 1$, and let x_n converge to a non-zero $x \in \ell_\infty(\mathbb{Q})$.

We find a sequence of real numbers $r(k) \in (0,1)$ and an increasing sequence n_k of positive integers such that $F_{n_k} \subset B_\rho(t_0,1) \setminus B_\rho(t_0,r(k))$ and such that $(\sum_{t\in F_{n_k}} \lambda(t))_{k\in\mathbb{N}}$ converges to some $c \in \mathbb{R}$.

Now we notice that $x_{n_k}(q) \to c$ for $q \in \mathbb{Q} \cap B_\rho(t_0,1)$ and $x_{n_k}(q) \to 0$ for $q \in \mathbb{Q} \setminus \overline{B_\rho(t_0,1)}$ since $F_{n_k} \to \{t_0\}$.

Suppose first that $c = 0$. Thus $x(q) = 0$ for $q \in (\mathbb{Q} \cap B_\rho(t_0,1)) \cup (\mathbb{Q} \setminus \overline{B_\rho(t_0,1)})$. Let $q_0 \in \mathbb{Q}$ be such that $|x(q_0)| > 0$ (necessarily $\rho(q_0,t) = 1$). Then obviously $\|x_{n_k} - x\| \geq \frac{|x(q_0)|}{2}$ as x_{n_k} attains each of its values on an infinite subset of $\mathbb{Q}$ whereas x attains a nonzero value in at most two points. This contradicts the convergence of x_{n_k} to x and therefore $c \neq 0$.

Let b_k^+, b_k^- be two distinct elements of $\mathbb{Q}$ with $\{\rho(b_k^+,t_0), \rho(b_k^-,t_0)\} \subset (1-r(k),1)$ and $\rho(b_k^+,b_k^-) > 1$ for every $k \in \mathbb{N}$. Thus $\{b_k^+,b_k^-\} \subset x^{-1}(c)$ and $\{b_k^+,b_k^-\} \cap \operatorname{supp} e_t$ is a singleton for $t \in F_{n_k}$. It follows that

$$x_{n_k}(b_k^+) + x_{n_k}(b_k^-) = \sum\{\lambda(t) \colon t \in F_{n_k}\} \quad \text{and} \quad x(b_k^+) + x(b_k^-) = 2c.$$

Hence

$$\lim_{n\to\infty}[(x(b_k^+) + x(b_k^-)) - (x_{n_k}(b_k^+) + x_{n_k}(b_k^-))] = c \neq 0$$

and this is a contradiction with the convergence $x_n \to x$ in $\ell_\infty(\mathbb{Q})$.

So we may use Main Lemma 4 by Proposition 1.

In the particular cases, when arbitrary $S \subset T$ with cardinality $\aleph_1$ or $\mathfrak{c}$ can be chosen, we may find a symmetric S that is non-Borel, or even without the Baire property in T in the other case, like in Main Lemma 4. ∎

Remark 4. We should point out an evident shortcoming of the above results. We did not show any example of a nonseparable Banach space X in which we cannot use Main Lemma 4 or in which even no bounded closed convex set K with non-Borel set of extreme or/and exposed points exists. This is of course because we do not know the answer to these questions.

Acknowledgements

We would like to express our gratitude to L. Zajíček whose observation attracted our attention to the constructions presented in the second section. His remarks and also discussions with O. Kalenda, J. Orihuela, and M. Zelený lead to improvements of the presentation of our results.

References

[1] Fabian, M., "Gâteaux Differentiability of Convex Functions and Topology, Weak Asplund Spaces", John Wiley & Sons, New York, 1997.

[2] Choquet, G., Corson, H., Klee, V., Exposed points of convex sets, *Pacific J. Math.,* **17** (1996), 33 – 43.

[3] Holický, P., Šmídek, M., Zajíček, L., Convex functions with non-Borel set of Gâteaux differentiability points, *Comment. Math. Univ. Carolinae,* **39** (1998), 469 – 482.

[4] Jayne, J.E., Rogers, C.A., The extremal structure of convex sets, *J. Funct. Anal.,* **26** (1977), 251 – 288.

[5] Kalenda, O., Valdivia compact spaces in topology and Banach space theory, *Extracta Math.,* **15(1)** (2000), 1 – 85.

[6] Lacey, H.E., "The Isometric Theory of Classical Banach Spaces", Springer Verlag, New York – Heidelberg, 1974.

[7] Valdivia, M., Simultaneous resolutions of the identity operator in normed spaces, *Collect. Math.,* **42(3)** (1991), 265 – 285.