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Sequentially m-Barrelled Algebras

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1. Introduction

The concept of m-barrelled algebra was introduced in [5]. Using sequential convergence, we introduce, in this paper, sequentially m-barrelled algebras in the same fashion as s-barrelled spaces were introduced in [8].

An analogue of the Banach-Steinhaus theorem is proved. As an application, we obtain an interesting result in orthogonal bases, which is the analogue of the isomorphism theorem.

An algebra which is also a locally convex space is called a locally convex algebra if the multiplication in it is jointly continuous. A subset S of an algebra is called m-convex if it is convex and idempotent (i.e. $SS \subseteq S$). A locally convex algebra E is called a locally m-convex algebra if it has a neighbourhood basis of 0 consisting of closed, circled and m-convex sets [7]. A locally convex algebra E is called an m-barrelled algebra if every m-barrel (closed, circled, m-convex and absorbing set) is a neighbourhood of 0 in E [5]. A locally convex space is called a barrelled space (sequentially barrelled space) if every barrel, i.e. closed, circled, convex, absorbing set, is a neighbourhood of 0 (an S-barrel, i.e. sequentially closed, circled, convex, absorbing set, is a sequential neighbourhood of 0 [8]). A mapping $T : E \to F$ (E and F are algebras) is called multiplicative if $T(xy) = T(x)T(y)$. A set V in a topological vector space X is called a sequential neighbourhood of 0 if every sequence in X converging to 0 belongs to V eventually.

A sequence $\{x_i\}$ in a locally convex space E is called a topological basis (or, basis) for E if for each x in E, there is a unique sequence $\{\alpha_i\}$ in $\mathbb{K}$ such that

$$x = \lim_n \sum_{i=1}^{n} \alpha_i x_i$$

in the topology of E [6]. Each α_i, called expansion coefficient, defined by $\lambda_i(x) = \alpha_i$, defines a linear functional λ_i on E. If each λ_i is continuous (sequentially continuous) then $\{x_i\}$ is called a Schauder basis (S-Schauder basis [4])

Let E and F be locally convex spaces. A sequence $\{x_i\}$ in E is similar to a sequence $\{y_i\}$ in F if for all sequences $\{a_i\} \subset \mathbb{K}$, $\sum_{i=1}^{\infty} a_i x_i$ converges (in E) iff $\sum_{i=1}^{\infty} a_i y_i$ converges (in F) [2].

A mapping $T : E \to F$ is called sequential topological isomorphism if it is linear, one-one, onto, sequentially continuous and T^{-1} is sequentially continuous.

A basis $\{x_i\}$ in a locally convex algebra E is called orthogonal if $x_i x_j = 0$ for $i \neq j$ and $x_i^2 = x_i$ [1]. In a Hausdorff locally convex algebra (or even in Hausdorff Topological algebra) an orthogonal basis is a Schauder basis [1]. We always consider vector spaces over the field of complex numbers.

2. Sequentially m-barrelled algebras

In this section we introduce the concept of sequentially m-barrelled algebra with two examples and obtain some results.

Definitions 2.1. (a) Let E be a locally convex algebra. If a subset A is an S-barrel and idempotent, then it is called a sequentially m-barrel.

(b) If every sequentially m-barrel in E is a sequential neighbourhood of 0, then E is called a sequentially m-barrelled algebra.

Remarks 2.2. (a) Every m-barrel is a sequentially m-barrel.

(b) In a metrizable locally convex algebra, the concepts of m-barrelled algebra and sequentially m-barrelled algebra coincide.

Example 2.3. Let $C(I)$ be the Banach algebra of all continuous functions on $I = [0, 1]$ with the norm

$$\|f\| = \sup_{t \in I}\{|f(t)|\}, \qquad f \in C(I).$$

Let E be the vector subspace of $C(I)$, consisting of all elements $f \in C(I)$ which vanish in a neighbourhood (depending on f) of $t = 0$. Let

$$B = \left\{ f \in E \,:\, |f(1/n)| \leq 1/n \text{ for all } n \in \mathbb{N} \right\}.$$

Then B is a sequentially m-barrel in E. But B is not a sequential neighbourhood of 0 in E [3]. Hence E is not a sequentially m-barrelled algebra. However $C(I)$, being a Banach algebra, is sequentially m-barrelled algebra. Since E is an ideal in $C(I)$, it follows that an ideal of a sequentially m-barrelled algebra need not be of the same sort.

EXAMPLE 2.4. If E is an algebra, the family of all circled, convex, absorbing and idempotent sets is a basis of neighbourhoods of 0 for a locally m-convex topology on E which is the strongest locally m-convex topology on E. Now let E be the subalgebra of $\mathbb{K}[x]$ of all polynomials without constant term. If α is a positive real number, let $V(\alpha)$ be the circled convex envelope of $\{\alpha^m x^m \,:\, m \in \mathbb{N}\}$. The family $\{V(\alpha)\}$, with α rational and less than one, is a basis of neighbourhoods of 0 for the strongest locally m-convex topology on E. This topology is metrizable. Now, E, with this topology, is a sequentially m-barrelled algebra which is not S-barrelled, since it is metrizable but not barrelled [9].

OPEN PROBLEM 2.5. Is there a sequentially m-barrelled algebra which is not m-barrelled?

PROPOSITION 2.6. *Let E be a sequentially m-barrelled algebra and F a locally m-convex algebra. If f is a multiplicative linear mapping of E into F, then f is almost sequentially continuous.*

Proof. Let V be a circled m-convex neighbourhood of 0 in F. Then $\overline{f^{-1}(V)}^{\mathrm{S}}$, the smallest sequentially closed set containing $f^{-1}(V)$, is a sequential m-barrel in E and hence a sequential neighbourhood of 0 in E. This proves that f is almost sequentially continous. ∎

PROPOSITION 2.7. *Let E be a sequentially m-barrelled algebra and F a locally convex algebra. If f is a sequentially continous and almost sequentially open, multiplicative, linear mapping of E into F, then F is sequentially m-barrelled.*

Proof. Let B be sequential m-barrel in F. Then $f^{-1}(B)$ is a sequential m-barrel in E and hence a sequential neighbourhood of 0 in E. Since f is almost sequentially open, it follows that $\overline{f\{f^{-1}(B)\}}^{\mathrm{S}}$ is a sequential neighbourhood of 0 in F. But

$$\overline{f\{f^{-1}(B)\}}^{\mathrm{S}} \subseteq \overline{B}^{\mathrm{S}} = B$$

so that B is a sequential neighbourhood of 0 in F. Hence F is a sequentially m-barrelled algebra. ∎

3. Main results

In this section, we obtain an analogue of Banach-Steinhaus theorem for sets of multiplicative linear mappings on sequentially m-barrelled algebras and we use it to prove an analogue of the isomorphism theorem by using the orthogonal basis.

Let E and F be locally convex spaces. Then a set H of linear mappings from E to F is called equi-sequentially continuous if for each neighbourhood V of 0 in F, $\cap_{f\in H} f^{-1}(V)$ is a sequential neighbourhood of 0 in E.

THEOREM 3.1. *Let E be a sequentially m-barrelled algebra and F any locally m-convex algebra. If H is a simply bounded set of sequentially continuous multiplicative linear mappings, then H is equi-sequentially continuous.*

Proof. Let V be a closed, circled and m-convex neighbourhood of 0 in F. Then $\cap_{f\in H} f^{-1}(V)$ is a sequentially m-barrel in E and hence a sequential neighbourhood of 0 in E. Thus H is equi-sequentially continuous. ∎

COROLLARY 3.2. *Let E and F be as in* 3.1. *Suppose $\{f_n\}$ is a pointwise bounded sequence of sequentially continuous multiplicative linear mappings from E to F. Then $\{f_n\}$ is equi-sequentially continuous.*

COROLLARY 3.3. *Let E and F be as in* 3.1. *If $\{f_n\}$ is a sequence of sequentially continuous multiplicative linear mappings from E to F such that it converges pointwise to a mapping $f : E \to F$, then f is linear, multiplicative and sequentially continuous.*

As an application of 3.3, we have the following analogue of the isomorphism theorem.

Theorem 3.4. *Let E and F be sequentially m-barrelled algebras. Suppose $\{x_i, \lambda_i\}$ and $\{y_i, \mu_i\}$ be orthogonal S-Schauder bases in E and F respectively. Then $\{x_i, \lambda_i\}$ is similar to $\{y_i, \mu_i\}$ if and only if there exists a multiplicative sequentially topological isomorphism $T : E \to F$ such that $T(x_i) = y_i$ for all $i \in \mathbb{N}$.*

Proof. If such a T exists, then for all sequences $\{a_i\} \subset C$, $\sum_{i=1}^{\infty} a_i x_i$ converges (in E) iff

$$T\left(\sum_{i=1}^{\infty} a_i x_i\right) = \sum_{i=1}^{\infty} a_i T(x_i) = \sum_{i=1}^{\infty} a_i y_i$$

converges (in F). Hence we get similarity. Conversely, we assume that the bases are similar. For each $x \in E$, $x = \sum_{i=1}^{\infty} \lambda_i(x) x_i$.

We define T_n by

$$T_n(x) = \sum_{i=1}^{n} \lambda_i(x) x_i\,, \qquad n \in \mathbb{N}\,,$$

and T by

$$T(x) = \sum_{i=1}^{\infty} \lambda_i(x) x_i\,;$$

T is well-defined, one-one, onto, each T_n is sequentially continuous, linear, multiplicative, and $\{T_n\}$ converges pointwise to T. Hence, by 3.3, T is sequentially continuous, linear and multiplicative. Similarly T^{-1} is sequentially continuous. Hence T is multiplicative sequentially topological isomorphism. ∎

Corollary 3.5. *Suppose E and F in 3.4 are Hausdorff, and $\{x_i, \lambda_i\}$ and $\{y_i, \mu_i\}$ are orthogonal bases in E and F respectively. Then the result of 3.4 follows.*

Proof. Since E and F are Hausdorff, $\{x_i, \lambda_i\}$ and $\{y_i, \mu_i\}$ are Schauder bases [1] and hence S-Schauder bases. ∎

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References

[1] Husain, T.,"Orthogonal Schauder Bases", Marcel Dekker, New York, 1991.

[2] Jones, O.T., Retherford, J.R., On similar bases in barrelled spaces, *Proc. Amer. Math. Soc.* **22** (1969), 330 – 334.

[3] Khaleelulia, S.M., Countably m-barrelled algebras, *Tamkang J. Math.* **6** (2) (1975), 185 – 190.

[4] Khaleelulia, S.M., Some results in S-barrelled spaces, *Tamkang J. Math.* **18** (2) (1987), 43 – 48.

[5] Mallios, A., On the spectra of topological algebras, *J. Functional Analysis* **3** (1969), 301 – 309.

[6] Mart, J., "Introduction to the Theory of Bases", Springer Tracts in Natural Philo. Vol. 18, New York, 1969.

[7] Michael, E.A., "Locally Multiplicatively-Convex Topological Algebras", Memoirs of the A.M.S. Vol. 11, Providence, R.I., 1952.

[8] Snipes, R.F., S-barrelled topological vector spaces, *Canada Math. Bull.* **21** (2) (1978), 221 – 227.

[9] Warner, S., Inductive limits of normed algebras, *Trans. Amer. Math. Soc.* **82** (1956), 190 – 216.