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Homotopy Theory Induced by Cones

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0. Introduction

Generally, in homotopy theory a cylinder object is used to define homotopy between morphisms. A cone object is necessary to build exact sequences of homotopy groups. So, the factorization through cone objects characterizes the nullhomotopic morphisms.

Here an abstract homotopy theory based on a cone functor that allows extend nullhomotopy to homotopy is given. Fundamental properties of the topological cone to develop homotopy theory are generalized in dual standard constructions by Huber [4]. Some axioms about cofibrations given by Baues [1] in categories with a natural cylinder are adapted in the case of a cylinder whose base is collapsed to a single point. The axiomatic theory obtained in this way allows one to obtain homotopy groups of based objects and cofibrations. Also, exact homotopy sequences of these groups can be created.

Principal axiomatic theories based on a cone, as homotopy theories given by Huber [4], Kleisli [6], Seebach [10] and Rodríguez-Machín [8], are particular cases of this theory. Moreover, known homotopy theories are examples of this axiomatic theory or its dual: the classical homotopy of topological spaces, pointed topological spaces and chain complexes [5]; projective and injective homotopy theories of R-modules [3]. Others less known are examples too: some tensorial homotopy theories and proper homotopy theory of exterior spaces [2].

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1. Category with a natural cone

In this section, one gives the axioms that a category must have to obtain an homotopy theory based on a cone functor, creating the necessary tools to develop the homotopy theory and obtaining some basic results.

Definition 1. A C-category, or category with a natural cone, is a category $\mathbf{C}$ together with a class "cof" of morphisms in $\mathbf{C}$, called cofibrations, a functor $C : \mathbf{C} \to \mathbf{C}$, which will be called the cone functor, and natural transformations $k : 1 \to C$ and $p : CC \to C$ satisfying the following axioms:

C1. (Cone axiom) $p(kC) = p(Ck) = 1C$ and $p(pC) = p(Cp)$.

C2. (Push out axiom) For any pair of morphisms $X \xleftarrow{f} B \overset{i}{\rightarrowtail} A$, where i is a cofibration, there exists the push out square

$$\begin{tikzcd} B \arrow[r, tail, "i"] \arrow[d, "f"] & A \arrow[d, "\overline{f}"] \\ X \arrow[r, tail, "\overline{i}"] & X \cup_B A \end{tikzcd}$$

and $\overline{i}$ is also a cofibration. The cone functor carries this push out diagram (called cofibration push out) into a push out diagram, that is $C(X \cup_B A) = CX \cup_{CB} CA$.

C3. (Cofibration axiom) For each object X the morphisms 1_X and k_X are cofibrations. The composition of cofibrations is a cofibration. Moreover, there is a retraction r for the cone of each cofibration i ($r(Ci) = 1$). This latter property is called *nullhomotopy extension property* (NEP).

C4. (Relative cone axiom) For a cofibration $i : B \rightarrowtail A$ the morphism $i_1 = \{Ci, k\} : \Sigma^i = CB \cup_B A \rightarrowtail CA$ is a cofibration.

By this definition, isomorphisms and cone of cofibrations are cofibrations. Moreover the cone functor carries cofibration push outs into cofibration push outs.

Theorem 1. *Given the commutative diagram*

$$\begin{tikzcd} Y \arrow[d, "\alpha"] & X \arrow[l, "f"'] \arrow[r, "g"] \arrow[d, "\gamma"] & Z \arrow[d, "\beta"] \\ Y' & X' \arrow[l, "f'"] \arrow[r, "g'"'] & Z' \end{tikzcd}$$

if α, β, γ and one of the morphisms $\{g', \beta\} : X' \cup_X Z \to Z'$ or $\{f', \alpha\} : X' \cup_X Y \to Y'$ are cofibrations, then $\alpha \cup \beta : Y \cup_X Z \to Y' \cup_{X'} Z'$ so is.

DEFINITION 2. A morphism $f : X \to Y$ is said to be nullhomotopic if there exists an extension F of f relative to k, that is, $F : CX \to Y$ such that $Fk = f$. An object X is said to be contractible when 1_X is nullhomotopic.

Observe that all morphism factored through a contractible object is nullhomotopic.

THEOREM 2. *Given a morphism $i : B \to A$, the following sentences are equivalent:*

a) *i verifies NEP.*
b) *Every nullhomotopic morphism $f : B \to X$ has a nullhomotopic extension $\widetilde{f} : A \to X$ relative to i.*
c) *Every nullhomotopic morphism $f : B \to X$ has an extension $\widetilde{f} : A \to X$ relative to i.*
d) *$k : B \to CB$ has an extension $\widetilde{k} : A \to CB$ relative to i.*

By NEP all morphism with contractible domain or codomain can be extended relative to any cofibration with the same domain. This will be the main tool used along this paper to obtain homotopies.

DEFINITION 3. Given a cofibration $i\colon B \rightarrowtail CA$, a morphism $f_0 : CA \to X$ is said to be homotopic to $f_1 : CA \to X$ relative to i if there exists an extension F of the morphism $\{f_0 pCi, f_1\}$ relative to i_1, that is, a morphism $F : C^2A \to X$ such that $Fi_1 = \{f_0 pCi, f_1\} : CB \cup_B CA \to X$.

The relative homotopy relation is an equivalence relation compatible with the composition of morphisms. Given morphisms $i : B \to A$ and $u : B \to X$, $Hom(A, X)^{u(i)} = \{f\colon A \to X \ / \ fi = u\}$ denotes the set of extensions of the morphism u relative to i.

$[CA, X]^{u(i)} = Hom(CA, X)^{u(i)}/ \simeq$.

If $F : f_0 \simeq f_1$ rel. i then $hF : hf_0 \simeq hf_1$ rel. i.

If $Cfi = jg$ and $H : h_0 \simeq h_1$ rel. j then $HC^2f : h_0Cf \simeq h_1Cf$ rel. i.

If $Cfi = jg$ is a push out diagram: $h_0 \simeq h_1$ rel. j if and only if $h_0Cf \simeq h_1Cf$ rel. i.

2. Homotopy groups

Here, homotopy groups relative to a cofibration or referred to an object are built, and usual properties of them are given.

Along this section, the C-category will be pointed, that is, $*_X$ is a cofibration for all object X and $C* = *$, where $*$ is the initial object.

By NEP, given a based cofibration $i : B \rightarrowtail CA$, there is $\mu : C^2\mathrm{A} \rightarrow C^2A \cup_{CB} C^2A = C(CA \cup_B CA)$ such that $\mu i_1 = k(pCi \cup 1)$. If $F : 0 \simeq 0$ rel. i then $\overline{F} = \{F, 0\}\mu : 0 \simeq 0$ rel. i. Moreover, if $G : 0 \simeq 0$ rel. i then $F * G = \{\overline{F}, G\}\mu : 0 \simeq 0$ rel. i.

Using that $\mu^* : [C^2A \cup_{CB} C^2A, X]^{0(k)} \rightarrow [C^2A, X]^{0(i_1)}$ is a bijection one can prove the following

Theorem 3. $\pi_1^i(X) = [C^2A, X]^{0(i_1)}$ *is a group with multiplication and inverse induced by "$*$" and "$-$", respectively.*

Let us remark that the multiplication on $\pi_1^i(X)$ does not depend on the choice of the extension μ.

Definition 4. The n-th homotopy group of an object X relative to a cofibration $i : B \rightarrowtail CA$ is defined by $\pi_n^i(X) = \pi_1^{i_{n-1}}(X)$.

Given a cofibration $i : B \rightarrowtail A$, the codomain of the cofibration i_n is C^nA. In this way, homotopy groups relative to any cofibration $i : B \rightarrowtail A$ can be defined for $n \geq 2$.

The homotopy group $\pi_n^{*_A}(X)$ is also denoted by $\pi_n^A(X)$ and called the *n-th homotopy group of the object X referred to the object A.*

By Definition 4 $\pi_n^i(X) = \pi_{n-s}^{i_s}(X)$. Moreover, all morphism $f : X \rightarrow Y$ induces $f_* : \pi_n^i(X) \rightarrow \pi_n^i(Y)$, and all commutative square $fi = jg$ induces an homomorphism $(C^nf)^* : \pi_n^j(X) \rightarrow \pi_n^i(X)$. If the square is a push out diagram then $(C^nf)^*$ is an isomorphism.

3. Exact sequences of homotopy groups

Now, the exact homotopy sequence associated to a pair is built.

Let cof $\mathbf{C}$ be the full subcategory of Pair $\mathbf{C}$ whose objects are cofibrations. (X, Y) will denote an object of cof $\mathbf{C}$ with associated cofibration $f : Y \rightarrowtail X$.

Theorem 4. *cof* $\mathbf{C}$ *is a C-category, with $C :$ cof $\mathbf{C} \rightarrow$ cof $\mathbf{C}$ defined by $C(X, Y) = (CX, CY)$, where the associated cofibration is Cf, and* $C(g, h) =$

(Cg, Ch); $k = (k, k)$, $p = (p, p)$ and cofibrations defined as those morphisms $(u, v) : (X, Y) \rightarrowtail (X', Y')$ such that v and $\{f', u\} : Y' \cup_Y X \rightarrowtail X'$ are cofibrations in $\mathbf{C}$.

DEFINITION 5. The $(n+1)$-th homotopy group of the pair (X, Y) relative to a cofibration i is defined by $\pi^i_{n+1}((X, Y)) = \pi^{(Ci,i)}_n((X, Y))$, where $(Ci, i) : (CB, B) \rightarrowtail (CA, A)$ with associated cofibrations k_B and k_A, respectively.

THEOREM 5. *The following sequence of groups is exact:*

$$..... \to \pi^i_3(Y) \xrightarrow{f_*} \pi^i_3(X) \xrightarrow{j} \pi^i_3((X, Y)) \xrightarrow{\delta} \pi^i_2(Y) \xrightarrow{f_*} \pi^i_2(X)$$

where f_* is defined in the section 2, $j([F]) = [(F, 0)]$ and $\delta([(F, G)]) = [G]$.

The above sequence is denominated *the exact homotopy sequence of* (X, Y) *relative to i.* If $i = *_A$ one obtains the *exact homotopy sequence of* (X, Y) *referred to A.*

4. EXAMPLES

The developed theory has an obvious dual version with cocones and fibrations. Adjoint functors $(cone, cocone)$ give the same homotopy theory. The adjunction between the tensorial and Hom functors gives homotopy theories induced by cones and cocones on the categories of Abelian Groups, R-quasi-modules ($\equiv$ abelian groups verifying the properties of R-modules save the external associative property) and R-modules:

If X denotes an object in any of such categories, R is an unitary ring and $f : R \to S$ is an homomorphism of unitary rings, then $CX = X \otimes_{\mathbb{Z}} R$, $CX = X \otimes_R R$ and $CX = X \otimes_R S$, respectively; $C'X = Hom_{\mathbb{Z}}(R, X)$, $C'X = Hom_R(R, X)$ and $C'X = Hom_R(S, X)$, respectively; $k : X \to CX$ is defined by $k(x) = x \otimes 1$ and $k' : C'X \to X$ by $k'(\alpha) = \alpha(1)$; $p : C^2X \to CX$ is defined by $p(x \otimes r \otimes s) = x \otimes rs$ and $p' : C'X \to C'^2X$ by $p'(\alpha)(r \otimes s) = \alpha(rs)$.

Homotopy theories obtained by Baues [1] on categories with natural cylinders or natural path objects are also induced by cones or cocones, respectively. In this way, the homotopy theory defined by Kamps [5] on chain complexes of an abelian category can be induced by cones and cocones; the classical homotopy theory of topological spaces and the homotopy theory of exterior spaces [2] are induced by cones.

The classical homotopy theory on pointed topological spaces is also generated by cones and cocones, where cones are known and cocones are the spaces of the paths with initial point the base point of the topological space.

The free and $Hom\,(Hom\,(-, Q_1), Q_1)$ functors on the category of R-modules are cocone and cone functors that induces, respectively, the projective and injective homotopy theories defined by Hilton [3], where Q_1 is the additive group of the rational numbers modulo the integers.

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