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Generalized Homotopy Theory

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1. Introduction

Usually, in homotopy theory with cofibrations, cofibrant objects and a suspension functor are necessary to obtain homotopy groups and exact sequences of them. In this sense H.J. Baues defines in his book "Algebraic Homotopy" [1] the concept of "Category with a Natural Cylinder". His definition requires the existence of initial object, conservation of it by cylinder functor and that objects be cofibrant. The latter is a necessary condition for the "Relative Cylinder Axiom".

The homotopy theory obtained here is called "generalized" since initial and cofibrant objects are not necessary axioms to build the homotopy groups and exact sequences of them. Also, the term "generalized" can be understood in the sense of the Generalized Homology theories since the homotopy groups are based on any morphism not necessary the zero morphism, obtaining different groups according to the base morphism.

The concept of Generalized Homotopy in categories with a natural cylinder is obtained by suppressing the above mentioned conditions, as the relative cylinder axiom can be described in another way without using cofibrant objects.

The interchange axiom originates transformations as described by Kamps and Porter [4], similar to the product of the double into the simple cylinder of a topological space. This homotopy theory replaces the interchange axiom by products of this type.

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Baues in [2] uses several base points to obtain proper homotopy theories: "Spaces under a tree correspond to pointed spaces in ordinary topology". This idea of point can be generalized for any object. Pointed objects are necessary to obtain homotopy groups. Thus, this theory is more general.

Exact sequences of generalized homotopy groups are built. Moreover, classic homotopy groups based on the zero morphism and exact sequences of them are a particular case of the generalized homotopy groups and their sequences.

2. Category with a generalized natural cylinder

In a category with a natural cylinder as described by Baues, is possible to obtain transformations χ_0 and χ_1 verifying the same properties on the faces of the cylinder that products in the topological cylinder: $\chi_0(x,t,s) = (x, 1-(1-t)(1-s))$ and $\chi_1(x,t,s) = (x, ts)$. On the other hand the push out diagram given by Baues in the relative cylinder axiom (I4) can be also defined without using the initial object. So, the following definition generalizes the concept of natural cylinder for a category:

Definition 1. A generalized I-Category is a category $\mathbf{C}$ with a class of morphisms called cofibrations, a functor $I : \mathbf{C} \to \mathbf{C}$, which will be called the cylinder functor, together with natural transformations $i_0, i_1 : 1_{\mathbf{C}} \to I$ and $p : I \to 1_{\mathbf{C}}$ verifying the axioms GI1, GI2, GI3, GI4 and GI5:

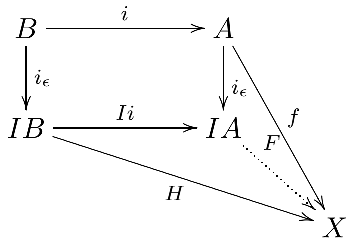
(GI1) Cylinder Axiom. $pi_\epsilon = 1_{\mathbf{C}}$, $\epsilon \in \{0,1\}$.

(GI2) Push out Axiom. For any pair of morphisms $X \overset{f}{\leftarrow} B \overset{i}{\rightarrowtail} A$, where i is a cofibration, there exists the push out square

$$\begin{array}{ccc} B & \overset{i}{\rightarrowtail} & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \bar{f}} \\ X & \overset{\bar{i}}{\rightarrowtail} & X \cup_B A \end{array} \qquad (1)$$

and $\bar{i}$ is also a cofibration. Any push out square (1) is carried by the cylinder functor I into a push out square.

(GI3) Cofibration Axiom. For every object X, $1_X, i_{0_X}$ and i_{1_X} are cofibrations, the composition of cofibrations is a cofibration and every cofibration $i : B \rightarrowtail A$ has the Homotopy Extension Property (HEP), namely: if $(B \overset{i_\epsilon}{\to} IB \overset{H}{\to} X) = (B \overset{i}{\rightarrowtail} A \overset{f}{\to} X)$, with $\epsilon \in \{0,1\}$, is a commutative square, then there exists a morphism $F : IA \to X$ making commutative the following diagram



that is, with $FIi = H$ and $Fi_\epsilon = f$.

(GI4) RELATIVE CYLINDER AXIOM. For any cofibration $i : B \rightarrowtail A$, the induced morphism

$$i^1 = \{Ii, i_0, i_1\} : IB \cup_B A \cup_B A \to IA$$

is also a cofibration.

(GI5) PRODUCTS AXIOM. There are natural transformations $\chi_0, \chi_1 : II \to I$ verifying

$$\chi_\epsilon(Ii_\nu) = \chi_\epsilon i_\nu = \begin{cases} 1\,, & \nu = \epsilon \\ i_\nu p\,, & \nu \neq \epsilon \end{cases} \quad \text{and } p\chi_\epsilon = p^2;$$

with $\nu, \epsilon \in \{0, 1\}$.

$IB \cup_B A \cup_B A$ is also obtained interchanging in the definition i_ϵ by i_ν, hence $\{Ii, i_1\}$ is also cofibration.

THEOREM 1. *Given a cofibration $i : B \rightarrowtail A$, the following statements are equivalent:*

a) *i verifies HEP.*

b) *$\{Ii, i_\epsilon\} : IB \cup_B A \to IA$ are sections for $\epsilon \in \{0, 1\}$.*

The HEP will be the main tool used along this paper to obtain homotopies. The relative cylinder for a cofibration is used to define homotopy:

DEFINITION 2. Let $i : B \rightarrowtail A$ a cofibration, and let $f_0, f_1 : A \to X$ be morphisms. We say that f_0 is homotopic to f_1 relative to the cofibration i, if there exists a morphism $F : IA \to X$ making commutative the following triangle

$$\begin{array}{ccc} IB \cup_B A \cup_B A & \xrightarrow{\;i^1\;} & IA \\ {\scriptstyle \{f_0 ip, f_0, f_1\}}\downarrow & \swarrow{\scriptstyle F} & \\ X & & \end{array}$$

in such a case we write $F : f_0 \simeq f_1 \;\; rel \;\; i$.

By the relative cylinder axiom, it is defined $i^n = (i^{n-1})^1$ with $i^0 = i$. If $fi = jg$ then $I^n f i^n = j^n(I^n g \cup I^{n-1} f \cup I^{n-1} f \cup \overset{(2n}{\cdots\cdots} \cup I^{n-1} f)$. Moreover, if $fi = jg$ is a push out then so is the second commutative square.

For any cofibration $i : B \rightarrowtail A$ and morphism $u : B \to X$, the homotopy relation relative to i is an equivalence relation on the set $Hom(A, X)^{u(i)} = \{f \in Hom(A, X)/fi = u\}$ compatible with the composition of morphisms:

$[A, X]^{u(i)} = Hom(A, X)^{u(i)}/ \simeq$.

If $F : f_0 \simeq f_1 \;\; rel \;\; i$ then $hF : hf_0 \simeq hf_1 \;\; rel \;\; i$.

If $fi = jg$ and $H : h_0 \simeq h_1 \;\; rel \;\; j$ then $HIf : h_0 f \simeq h_1 f \;\; rel \;\; i$.

If $fi = jg$ is a push out diagram: $h_0 \simeq h_1$ *rel* j if and only if $h_0 f \simeq h_1 f$ *rel* i.

PROPOSITION 1. *Given a cofibration $i : B \rightarrowtail A$ every morphism $H : IB \to X$ induces a bijection $H^{\#} : [A, X]^{Hi_\epsilon(i)} \to [A, X]^{Hi_\nu(i)}$ defined by $H^{\#}([f]) = [Fi_\nu]$, where $F : IA \to X$ is any morphism such that $FIi = H$ and $Fi_\epsilon = f$, which exists by (HEP) ; $\epsilon, \nu \in \{0, 1\}$, $\epsilon \neq \nu$.*

3. GENERALIZED HOMOTOPY GROUPS

First one defines the homotopy groupoids of an object X relative to a cofibration $i : B \rightarrowtail A$ and further one obtains the homotopy groups based on a morphism $h : A \to X$ relative to the cofibration i.

Given a cofibration $i : B \rightarrowtail A$, the sets $H_i(f_0, f_1) = [IA, X]^{\{f_0 ip, f_0, f_1\}(i^1)}$, where $f_0, f_1 : A \to X$, and the push out diagram

$$\begin{array}{ccc} IB \cup_B A \cup_B A & \xrightarrow{\;i^1\;} & IA \\ {\scriptstyle \{ip,1\}\cup 1}\downarrow & & \downarrow{\scriptstyle \overline{\{ip,1\}\cup 1} = \omega} \\ A \cup_B A & \xrightarrow{\;\overline{i^1} = \{j_0, j_1\}\;} & I^i \end{array}$$

are used to obtain the mentioned groupoid. By HEP there is $\nu' : II^i \to I^i \cup_A I^i$ such that $\nu' i_0 = \widetilde{j_0} : I^i \to I^i \cup_A I^i$ (inclusion in the second component of the union) and $\nu' I\{j_0, j_1\} = \omega \cup j_1 p$. Thus

$$\nu^* = (\nu' i_1)^* : [I^i \cup_A I^i, X]^{\{f_0,f_1\}(j_1 \cup j_1)} \to [I^i, X]^{\{f_0,f_1\}(\{j_0,j_1\})}$$

is a bijection for all object X and morphisms $f_0, f_1 : A \to X$. Using ν^*, the following theorem can be proved.

THEOREM 2. *$H_i(X)$ is a groupoid whose objects are the elements of $Hom(A, X)$; morphisms from f_0 to f_1, the elements of $H_i(f_0, f_1)$; identity for f, $[fp]$; inverse of $[F] \in H_i(f_0, f_1)$, $[\{\{f_0, f_1, F\}, \{f_0, f_0, f_0 p\}\}\nu\omega]$ and if $[F] \in H_i(f_0, f_1)$, $[G] \in H_i(f_1, f_2)$, the composed*

$$[F] * [G] = [\{\{\{f_0, f_1, F\}, \{f_0, f_0, f_0 p\}\}\nu, \{f_1, f_2, G\}\}\nu\omega].$$

Let us remark that, by Proposition 1, multiplication on the groupoid does not depend on the choice of the morphism ν'.

DEFINITION 3. Given a cofibration $i : B \rightarrowtail A$ and a morphism $h : A \to X$, the nth-homotopy group, $n \geq 1$, of X based on h relative to i, is defined by $\pi_n^i(X, h) = H_{i^{n-1}}(hp^{n-1}, hp^{n-1})$.

By the above definition $\pi_n^i(X, h) = \pi_{n-s}^{i^s}(X, hp^s)$, and by the properties given to the end of the section 2:

> Every morphism $f : X \to Y$ induces a homomorphism of groups $f_* : \pi_n^i(X, h) \to \pi_n^i(Y, fh)$.
>
> Every commutative square $fi = jg$ induces a homomorphism of groups $(I^n f)^* : \pi_n^j(X, h) \to \pi_n^i(X, hf)$. If the commutative square is a push out then $(I^n f)^*$ is an isomorphism.

4. EXACT SEQUENCES OF GENERALIZED HOMOTOPY GROUPS

Given a category $\mathbf{C}$, the full subcategory Cof $\mathbf{C}$ of Pair $\mathbf{C}$ whose objects are cofibrations, is used to define homotopy groups in the category of pairs. $f : Y \rightarrowtail X$ will be the associated cofibration of the pair (X, Y) along this section.

THEOREM 3. *If $\mathbf{C}$ is a generalized I-Category then so is Cof $\mathbf{C}$, with $(u, v) : (X, Y) \to (X', Y')$ cofibration if and only if v and $\{f', u\} : Y' \cup_Y X \to X'$ are cofibrations in $\mathbf{C}$.*

Given a cofibration $(u,v):(X,Y)\to(X',Y')$, for any pair (X'',Y''), the homotopy relation relative to (u,v) and the multiplication in the groupoid $H_{(u,v)}(X'',Y'')$ are related with the respective ones in $\mathbf{C}$:

If $(F,G):(f_0,g_0)\simeq(f_1,g_1)\ \ rel\ (u,v)$ then $F:f_0\simeq f_1\ \ rel\ u$ and $G:g_0\simeq g_1\ \ rel\ v$.

$[F_0,G_0]*[F_1,G_1]=[F_0*F_1,G_0*G_1]$.

Given an object A in $\mathbf{C}$ the category of the objects pointed by A, $\mathbf{C}^{*A}$, has triples (j,X,q) as objects, where $j:A\to X$ is cofibration with $qj=1$; and $f:X\to X'$ verifying $fj=j'$, $q'f=q$ as morphisms from (j,X,q) to (j',X',q').

Definition 4. The cone of a pointed object (j,X,q), CX, is defined by the push out square

$$\begin{array}{ccc} IA\cup_A X\cup_A X & \xrightarrow{\ j^1\ } & IX \\ {\scriptstyle\{jp,jq,1\}}\big\downarrow & & \big\downarrow{\scriptstyle\overline{\{jp,jq,1\}}=\rho} \\ X & \xrightarrow{\ \overline{j^1}=k\ } & CX \end{array}$$

In this way the associated cofibration to the pair (CX,X) is k.

Given a morphism $f:(j,X,q)\to(j',X',q')$, $Cf=f\cup If:CX\to CX'$.

If f is cofibration then so is $\{Cf,k\}$ since $\rho f^1\overset{(1)}{=}\{Cf,k\}\{\overline{f}\rho,\overline{k}j'q',\overline{k}\}$ is a push out diagram. Hence $(Cf,f):(CX,X)\to(CX',X')$ is also cofibration.

Given a cofibration $i:B\rightarrowtail A$, then each i^n is actually a morphism in $\mathbf{C}^{*A}$, $i^n:(l_0i_0^{n-1},IJ\cup_J I^{n-1}A\cup_J I^{n-1}A,p^{n-1}r)\to(i_0^n,I^nA,p^n)$ with $J=$ *domain* i^{n-1}, $l_0:I^{n-1}A\to IJ\cup_J I^{n-1}A\cup_J I^{n-1}A$ is the induced cofibration in the first $I^{n-1}A$ and $r=\{i^{n-1}p,1,1\}:IJ\cup_J I^{n-1}A\cup_J I^{n-1}A\to I^{n-1}A$.

Definition 5. The $(n+2)th$-homotopy group relative to a cofibration $i:B\rightarrowtail A$ of the pair (X,Y) based on a morphism $h:A\to Y$ is defined by $\pi^i_{n+2}((X,Y),h)=\pi_1^{(Ci^n,i^n)}((X,Y),(\{fhp^n,fhp^{n+1}\},hp^n))$ for $n\geq 1$.

Theorem 4. *There is an isomorphism of groups*

$$\alpha:\pi^i_{n+2}(X,fh)\to\pi_1^{(\{Ci^n,k\},1)}((X,Y),(\{fhp^n,fhp^{n+1}\},hp^n)).$$

Theorem 5. *The sequence*

$$... \xrightarrow{j_2} \pi_4^i((X,Y),h) \xrightarrow{\delta} \pi_3^i(Y,h) \xrightarrow{f_*} \pi_3^i(X,fh) \xrightarrow{j_1}$$

$$\xrightarrow{j_1} \pi_3^i((X,Y),h) \xrightarrow{\delta} \pi_2^i(Y,h) \xrightarrow{f_*} \pi_2^i(X,fh).$$

is exact; where $f_*([F]) = [fF]$, $\delta([F,G]) = [G]$ *and* $j_n([G]) = [G', hp^{n+1}]$ *with* G' *the induced morphism by* G *in the push out generated by* (1) *for* $f = i^n$*:* $I\rho i^{n+2} = \{Ci^n, k\}^1(I\{\overline{i^{n+1}}\rho, \overline{k}i_0^n p^n, \overline{k}\} \cup \rho \cup \rho)$.

This sequence is denominated homotopy sequence associated to the pair (X,Y) based on the morphism h relative to the cofibration i. The exact sequence associated to a pair given by Baues [1] in a category with a natural cylinder is a particular case of this one, taking the initial cofibration $*_A : * \rightarrowtail A$.

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