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Generalized Homotopy in C-Categories

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0. Introduction

The main aim of the *Generalized Homotopy Theory* is to embody the different pointed homotopy theories associated to a homotopy structure. A notion similar to the one of point in the ordinary homotopy theory of topological spaces has been used in other homotopy theories. So in the proper homotopy theory of topological spaces, different homotopy groups occur depending on the object used as point: a sequence of points in Brown homotopy groups [3], a base ray in Steenrod homotopy groups [4]. Moreover Baues and Quintero [2] use, in proper homotopy theory, spaces under a tree as pointed spaces in ordinary topology.

In this sense the Generalized Homotopy Theory can be developed without using cofibrant objects and zero morphisms. Generalized homotopy groups are defined as homotopy groups relative to cofibrations based on arbitrary morphisms. In this way the different pointed homotopy groups are particular cases of them.

The authors of this paper Díaz and Rodríguez-Machín introduce in "Homotopy theory induced by cones" [6] a homotopy structure denominated *Category with a Natural Cone* or *C-category*, based on a cone functor, obtaining pointed homotopy groups and exact homotopy sequences of them. Now generalized homotopy theory is developed in these categories. Generalized homotopy groups are obtained and exact sequences of them are built.

Moreover, properties of a C-category let one to obtain Kan complexes associated to a category with a natural cone, in a way similar to the one used

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by Rodríguez-Machín [10] and Huber [8] to associate Δ-groups and simplicial sets to dual standard constructions in additive categories and arbitrary categories, respectively. So pointed [6] and generalized homotopy groups in the C-category are isomorphic to homotopy groups of certain Kan complexes. In this way, higher pointed and generalized homotopy groups of a category with a natural cone are abelian. Also, isomorphic exact homotopy sequences are obtained in the C-category and the category of Kan complexes.

The mentioned relation between both categories allows the translation of homotopy developments from a category into the other.

A study of the different pointed categories associated to a C-category ratifies the fact described above that pointed homotopy groups are generalized homotopy groups. In this way one has a wider view of the homotopy theory associated to a category with a natural cone.

1. Generalized homotopy groups

In this section, generalized homotopy groups are defined as homotopy groups relative to a cofibration $i : B \rightarrowtail A$ based on a morphism $h : CA \to X$. The usual properties are given.

First, a brief reminder of the structure of a C-category [6] is made:

A *C-category*, or *a category with a natural cone*, is a category $\mathbf{C}$ together with a class "cof" of morphisms in $\mathbf{C}$, called *cofibrations*, a *cone functor* $C : \mathbf{C} \to \mathbf{C}$ and natural transformations $k : 1 \to C$ and $p : CC \to C$ satisfying the following axioms:

C1. (Cone axiom) $p(kC) = p(Ck) = 1C$ and $p(pC) = p(Cp)$.

C2. (Push out axiom) For any pair of morphisms $X \overset{f}{\leftarrow} B \overset{i}{\rightarrowtail} A$, where i is a cofibration, there exists the push out square

$$\begin{array}{ccc} B & \overset{i}{\rightarrowtail} & A \\ \downarrow f & & \downarrow \bar{f} \\ X & \overset{\bar{i}}{\rightarrowtail} & X \cup_B A \end{array}$$

and $\bar{i}$ is also a cofibration. The cone functor carries this push out diagram into a push out diagram, that is $C(X \cup_B A) = CX \cup_{CB} CA$.

C3. (Cofibration axiom) For each object X, the morphisms 1_X and k_X are cofibrations. The composition of cofibrations is a cofibration. Moreover, there is a retraction r for the cone of each cofibration i ($r(Ci) = 1$). This latter property is called the *nullhomotopy extension property* (NEP).

C4. (Relative cone axiom) For a cofibration $i : B \rightarrowtail A$, the morphism $i_1 = \{Ci, k\} : \Sigma^i = CB \cup_B A \rightarrowtail CA$ is a cofibration.

Given morphisms $i : B \to A$ and $u : B \to X$, the set of *extensions of the morphism* u *relative to* i is defined by $Hom(A,X)^{u(i)} = \{f : A \to X \ / \ fi = u\}$.

A morphism $f : X \to Y$ is said to be *nullhomotopic* if there exists an extension F of f relative to k. An object X is said to be *contractible* when 1_X is nullhomotopic.

The equivalence relation called "*relative homotopy*" is defined as follows:

Given a cofibration $i : B \rightarrowtail CA$, a morphism $f_0 : CA \to X$ is said to be *homotopic to* $f_1 : CA \to X$ *relative to* i if there exists an extension F of the morphism $\{f_0 p(Ci), f_1\}$ relative to i_1.

Hence, $[CA, X]^{u(i)} = Hom(CA, X)^{u(i)}/\simeq$.

Next, generalized homotopy groups are defined.

Definition 1. Given a cofibration $i : B \rightarrowtail A$ and a morphism $h : CA \to X$, the $n+1^{st}$ *homotopy group relative to the cofibration* i *based on the morphism* h is defined by

$$\pi^i_{n+1}(X,h) = [C^{n+1}A, X]^{hp^n i_{n+1}(i_{n+1})},$$

where $n \in \mathbb{N}$ and $i_{n+1} = (i_n)_1$ is a cofibration defined by induction using the relative cone axiom, with $i_0 = i$.

If $A = CA'$ and $h' : A \to X$ then $\pi^i_{n+1}(X, h')$ exists even for $n = 0$.

Observe that $\pi^i_{n+1}(X,h) = \pi^{i_r}_{n+1-r}(X, hp^{r-1})$, for $0 \le r \le n$. Hence, $\pi^{i_r}_{n+1}(X,h)$ can be defined by extension for $n \ge 1-r$. In particular, $\pi^i_{n+1}(X,h)$ $= \pi^{i_n}_1(X, hp^{n-1})$. Therefore, it is enough to define the multiplication in the first group.

Given a cofibration $i : B \rightarrowtail CA$ and a morphism $h : CA \to X$, by NEP there exists an extension $\mu : C^2A \to C(CA \cup_B CA)$ of the morphism $k(pCi \cup_1 1)$ relative to the cofibration i_1. The multiplication of $\pi^i_1(X,h)$ is defined using this extension, and does not depend on the choice of μ:

$[F]^{-1} = [\overline{F}]$, where $\overline{F} = \{F, hp\}\mu$.

$[F].[G] = [F * G]$, where $F * G = \{\overline{F}, G\}\mu = \{\{F, hp\}\mu, G\}\mu$.

$[hp]$ is the unit element.

Proposition 1. *For each* $n \in \mathbb{N}$,

a) *every morphism* $f : X \to Y$ *induces a homomorphism of groups* $f_* : \pi^i_{n+1}(X,h) \to \pi^i_{n+1}(Y, fh)$.

b) *every commutative square* $fi = gj$ *relating cofibrations* i *and* j *induces a homomorphism of groups* $(C^{n+1}f)^* : \pi^j_{n+1}(X, h) \to \pi^i_{n+1}(X, hCf)$*. If the square is a push out then* $(C^{n+1}f)^*$ *is an isomorphism of groups.*

When the C-category is pointed [6], the pointed homotopy group $\pi^i_{n+1}(X)$ agrees with the generalized homotopy group $\pi^i_{n+1}(X, 0)$, where $0 : CA \to X$ is the zero morphism.

Next the relation between categories with a natural cone and the category of Kan complexes is established.

Definition 2. Given two objects A and X of a C-category, the *simplicial set associated to* X *referred to* A is defined by:

$X^A_{n-1} = Hom(C^n A, X)$, for $n \in \mathbb{N}$.

$d_i = (C^{n-i}k)^* : X^A_n = Hom(C^{n+1}A, X) \to X^A_{n-1} = Hom(C^n A, X)$, for $0 \leq i \leq n$.

$s_i = (C^{n-i-1}p)^* : X^A_{n-1} = Hom(C^n A, X) \to X^A_n = Hom(C^{n+1}A, X)$, for $0 \leq i \leq n-1$.

Proposition 2. *The simplicial set* X^A *is a Kan complex.*

Theorem 1. *For any* 0*-simplex* $h : CA \to X$*, there is an isomorphism between* $\pi^A_{n+1}(X, h)$ *and* $\pi_n(X^A, h)$*.*

Corollary 1. *The homotopy groups* $\pi^A_{n+1}(X, h)$ *are abelian, for every pair of objects* A*,* X *of a* C*-category and* $n \in \mathbb{N} - \{1\}$*.*

Every cofibration $i : B \rightarrowtail A$ induces a simplicial morphism $i_X : X^A \to X^B$ defined by $(i_X)_{n-1} = (C^n i)^*$.

Proposition 3. i_X *is a Kan fibration.*

Theorem 2. $\pi^i_{n+1}(X, h)$ *is isomorphic to* $\pi_n(F, h)$*, where* F *is the fibre of the Kan fibration* i_X*,* $h : CA \to X$ *is any* 0*-simplex and* $n \in \mathbb{N}$*.*

Corollary 2. *The homotopy groups* $\pi^i_{n+1}(X, h)$ *are abelian, for every cofibration* $i : B \rightarrowtail A$*, morphism* $h : CA \to X$ *and* $n \in \mathbb{N} - \{1\}$*.*

2. Exact homotopy sequences

The method used in [6] to obtain the exact sequence of pointed homotopy groups associated to a pair will now be extended to build the respective exact sequence of generalized homotopy groups.

Given a C-category $\mathbf{C}$, the full subcategory of **pair C** whose objects are cofibrations is also a C-category, denoted by **cof C**. The cofibrations of pairs are morphisms $(u,v):(X,Y)\rightarrowtail(X',Y')$ verifying that v and $\{f',u\}:Y'\cup_Y X\rightarrowtail X'$ are cofibrations in the original category $\mathbf{C}$, where (X,Y), (X',Y'), ... symbolize objects in **cof C** with associated cofibrations $f:Y\rightarrowtail X$, $f':Y'\rightarrowtail X'$, ... respectively.

Homotopy groupoids of **cof C** are related with respective ones of $\mathbf{C}$ in the following sense:

If $(f_0,g_0)\simeq(f_1,g_1)$ rel. (u,v), then $f_0\simeq f_1$ rel. u and $g_0\simeq g_1$ rel. v.
$[(F_0,G_0)].[(F_1,G_1)]=[(F_0,G_0)*(F_1,G_1)]=[(F_0*F_1,G_0*G_1)]$.

Definition 3. *The $n+2^{nd}$ homotopy group relative to the cofibration $i:B\rightarrowtail A$ of the pair (X,Y) based on the morphism $h:CA\to Y$* is defined by

$$\pi^i_{n+2}((X,Y),h)=\pi^{(Ci,i)}_{n+1}((X,Y),(fhp,h)),\quad(n\in\mathbb{N}).$$

Lemma 1. *There is an isomorphism of groups*

$$\theta_n:\pi^i_{n+2}(X,fh)\to\pi^{(i_1,1)}_{n+1}((X,Y),(fhp,h))$$

for every pair (X,Y) and $n\in\mathbb{N}$.

Theorem 3. *The following sequence of groups is exact:*

$$....\to\pi^i_3(Y,h)\xrightarrow{f_*}\pi^i_3(X,fh)\xrightarrow{j_1}\pi^i_3((X,Y),h)\xrightarrow{\delta_1}\pi^i_2(Y,h)\xrightarrow{f_*}\pi^i_2(X,fh),$$

where f_* is the homomorphism given in Proposition 1; $j_n=(1,1)^*\theta_n$, with θ_n the isomorphism given in Lemma 1, and $(1,1)^*$ is defined using Proposition 1 and the commutative square $(1,1)(Ci,i)=(i_1,1)(\bar{i},i)$, where $\bar{i}$ is the induced cofibration in the relative cone Σ^i; δ_n is defined by $\delta_n([(F,G)])=[G]$.

If $f:Y\to X$ is not a cofibration, the group $\pi^i_{n+2}(f,h)$ is defined in a similar way to $\pi^i_{n+2}((X,Y),h)$, and the above sequence is also exact replacing (X,Y) by f.

If the category $\mathbf{C}$ is pointed, the pointed exact homotopy sequence associated to the pair (X,Y) [6] is obtained taking $h=0$.

Isomorphisms of groups obtained in the preceding section let one to translate the exact homotopy sequence of the Kan fibration induced by a cofibration into an *exact homotopy sequence associated to the cofibration and based on a morphism* in the category with a natural cone.

THEOREM 4. *Given a cofibration $i : B \rightarrowtail A$ and a morphism $h : CA \to X$ in a category with a natural cone, the following sequence of groups is exact:*

$$... \to \pi_3^B(X, hCi) \overset{\partial_1}{\to} \pi_2^i(X, h) \overset{(j_*)_1}{\to} \pi_2^A(X, h) \overset{(i_{X*})_1}{\to} \pi_2^B(X, hCi)$$

where:

a) $(i_{X*})_n = (C^{n+1}i)^* : \pi_{n+1}^A(X, h) \to \pi_{n+1}^B(X, hCi)$ *are the homomorphisms induced by the commutative square $ki = Cik$.*

b) $(j_*)_n = (C^{n+1}1)^* : \pi_{n+1}^i(X, h) \to \pi_{n+1}^A(X, h)$ *are the homomorphisms induced by the commutative square $i_1\overline{k} = 1k$, where $\overline{k}$ is the morphism induced by k in the push out of Σ^i.*

c) $\partial_n : \pi_{n+2}^B(X, hCi) \to \pi_{n+1}^i(X, h)$ *is defined by $\partial_n([F]) = [GC^{n+1}k]$, where G is an extension of the morphism $\{F, hp^n, ..., hp^n\}$ relative to the cofibration $(Ci)_{n+1}$.*

The process used in categories with a natural cone to obtain the exact homotopy sequence relative to a cofibration of a morphism and based on other morphism can be developed similarly in the category of Kan complexes, and an *exact homotopy sequence associated to a simplicial morphism and based on a 0-simplex* is obtained.

PROPOSITION 4. *Given a simplicial morphism $f : K \to L$ between Kan complexes, the simplicial set (K, L) defined by $(K, L)_{n-1} = \{(x, y) \ / \ x \in K_{n-1},\ y \in L_n$ and $d_0(y) = f_{n-1}(x)\}$, with $d_i = (d_i, d_{i+1})$ and $s_i = (s_i, s_{i+1})$, is a Kan complex.*

DEFINITION 4. Given a simplicial morphism $f : K \to L$ between Kan complexes, for $n \in \mathbb{N}$ the $n+1^{st}$ homotopy group of f based on the 0-simplex x_0 of K is defined by

$$\pi_{n+1}(f, x_0) = \pi_n((K, L), (x_0, s_0 f_0(x_0))).$$

THEOREM 5. *Given a simplicial morphism $f : K \to L$ between Kan complexes, for every 0-simplex x_0 of K the following sequence is exact:*

$$... \to \pi_2(L, f_0(x_0)) \xrightarrow{j_1} \pi_2(f, x_0) \xrightarrow{\partial_1} \pi_1(K, x_0) \xrightarrow{f_1} \pi_1(L, f_0(x_0))$$

where $f_n([x]) = [f_n(x)]$, $\partial_n([(x, y)]) = [x]$ *and* $j_n([y]) = [(s_0^n(x_0), y)]$.

Observe that every commutative square $fp = qf'$, with p and q Kan fibrations, induces a simplicial morphism $f'' : F_p \to F_q$ between the fibres associated to p and q, respectively, such that $f'i = jf''$, where i and j are the natural inclusions. On the other hand every morphism $g : X \to Y$ in a category with a natural cone originates in a natural way a simplicial morphism $g^A : X^A \to Y^A$, for any object A. Therefore given a cofibration $i : B \rightarrowtail A$, there exists a simplicial morphism $g^i : F_{i_X} \to F_{i_Y}$.

COROLLARY 3. *Given a morphism* $g : X \to Y$ *in a category with a natural cone, then:*

a) *For every object* A *of the* C*-category, the exact homotopy sequence referred to the object* A *associated to the morphism* g *based on the morphism* $h : CA \to X$ *is isomorphic to the exact homotopy sequence associated to the simplicial morphism* g^A *based on the 0-simplex* h.

$$\begin{array}{ccccccc} ... \to \pi_3^A(Y, gh) & \to & \pi_3^A(g, h) & \to & \pi_2^A(X, h) & \to & \pi_2^A(Y, gh) \\ \updownarrow\cong & & \updownarrow\cong & & \updownarrow\cong & & \updownarrow\cong \\ ... \to \pi_2(Y^A, gh) & \to & \pi_2(g^A, h) & \to & \pi_1(X^A, h) & \to & \pi_1(Y^A, gh) \end{array}$$

b) *For every cofibration* $i : B \rightarrowtail A$, *the exact homotopy sequence relative to the cofibration* i *associated to the morphism* g *based on the morphism* $h : CA \to X$ *is isomorphic to the exact homotopy sequence associated to the simplicial morphism* g^i *based on the 0-simplex* h.

$$\begin{array}{ccccccc} ... \to \pi_3^i(Y, gh) & \to & \pi_3^i(g, h) & \to & \pi_2^i(X, h) & \to & \pi_2^i(Y, gh) \\ \updownarrow\cong & & \updownarrow\cong & & \updownarrow\cong & & \updownarrow\cong \\ ... \to \pi_2(F_{i_Y}, gh) & \to & \pi_2(g^i, h) & \to & \pi_1(F_{i_X}, h) & \to & \pi_1(F_{i_Y}, gh). \end{array}$$

3. Pointed categories

In this section one can see that the generalized homotopy theory embodies all the homotopy theories of the different pointed categories generated by a C-category.

For every object ∇ of a C-category $\mathbf{C}$, the object $C\nabla$ behaves like a point. In this sense, $\mathbf{C}^*$ denotes the full subcategory of $\mathbf{C}^{C\nabla}$ whose objects, (X,x), are cofibrations $x : C\nabla \rightarrowtail X$.

Definition 5. A push out diagram

$$\begin{array}{ccc} S & \xrightarrow{\quad s \quad} & C\nabla \\ \big\downarrow{\scriptstyle j} & & \big\downarrow{\scriptstyle \bar{j} = x} \\ T & \xrightarrow{\quad \bar{s} \quad} & T \cup_S C\nabla = X \end{array}$$

will be termed a *push out associated to the pointed object* (X,x).

Observe that every pointed object (X,x) has associated a canonical push out $X \cup_{C\nabla} C\nabla$.

The use of push out associated to pointed objects is fundamental in the development of this section. The definitions of pointed cones, natural transformations and relative cones do not depend on the choice of the push out associated to the pointed objects.

A cone functor in $\mathbf{C}^*$ is defined for an object (X,x) with associated push out $T \cup_S C\nabla$ by the following push out

$$\begin{array}{ccc} CS & \xrightarrow{\quad pCs \quad} & C\nabla \\ \big\downarrow{\scriptstyle Cj} & & \big\downarrow{\scriptstyle \overline{Cj} = \overset{1)}{\bar{x}}} \\ CT & \xrightarrow{\quad \bar{p}C\bar{s} \quad} & CT \cup_{CS} C\nabla = C_*X \end{array}$$

and $C_*(f \cup_g 1) = Cf \cup_{Cg} 1 : CT \cup_{CS} C\nabla \to CT' \cup_{CS'} C\nabla$.

Note that, if $f : (X,x) \to (X',x')$ is a pointed morphism, and $T \cup_S C\nabla$ is a push out associated to (X,x), then $f = f\bar{s} \cup_s 1 : T \cup_S C\nabla \to X' \cup_{C\nabla} C\nabla$.

Proposition 5. *Given a push out $T \cup_S C\nabla$ associated to the pointed object (X,x), the following square is a push out:*

$$\begin{array}{ccc} C^nS & \xrightarrow{\qquad p^nC^ns \qquad} & C\nabla \\ \big\downarrow{\scriptstyle C^nj} & & \big\downarrow{\scriptstyle \overset{n)}{\bar{x}}} \\ C^nT & \xrightarrow{\bar{p}C\bar{p}...C^{n-1}\bar{p}C^n\bar{s}} & C^n_*X = C^nT \cup_{C^nS} C\nabla \end{array}$$

where $\overset{(n}{\bar{x}} = \overline{C \overset{(n-1}{\bar{x}}}$ *is obtained by induction, using the definition of the pointed cone* C_*, *with* $\overset{(0}{\bar{x}} = x$.

The natural transformations are defined for a pointed object (X,x) with associated push out $T \cup_S C\nabla$ by $k_{*(X,x)} = k_T \cup_{k_S} 1 : T \cup_S C\nabla \to CT \cup_{CS} C\nabla$ and $p_{*(X,x)} = p_T \cup_{p_S} 1 : C^2T \cup_{C^2S} C\nabla \to CT \cup_{CS} C\nabla$.

DEFINITION 6. A *pointed cofibration* $i : (B, b) \rightarrowtail (A, a)$ is a pointed morphism such that $i : B \rightarrowtail A$ is a cofibration in $\mathbf{C}$.

THEOREM 6. *The category* $\mathbf{C}^*$, *with the structure described above, is a pointed category with a natural cone.*

Observe that $k_{*C_*^n(X,x)} = k_{C^nT} \cup_{k_{C^nS}} 1 : C^nT \cup_{C^nS} C\nabla \to C^{n+1}T \cup_{C^{n+1}S} C\nabla$ and $p_{*C_*^n(X,x)} = p_{C^nT} \cup_{p_{C^nS}} 1 : C^{n+2}T \cup_{C^{n+2}S} C\nabla \to C^{n+1}T \cup_{C^{n+1}S} C\nabla$. Moreover, given $f \cup_g 1 : T \cup_S C\nabla \to T' \cup_{S'} C\nabla$, one has $C_*^n(f \cup_g 1) = C^nf \cup_{C^ng} 1 : C^nT \cup_{C^nS} C\nabla \to C^nT' \cup_{C^nS'} C\nabla$.

A pointed cofibration $i : (B, b) \rightarrowtail (A, a)$ relates canonical push outs as follows: $i \cup_1 1 : B \cup_{C\nabla} C\nabla \rightarrowtail A \cup_{C\nabla} C\nabla$. In this sense one can extend this idea:

DEFINITION 7. A pointed cofibration $u : (B, b) \rightarrowtail (A, a)$ is said to be associated with a *push out cofibration* when $u = i \cup_1 1 : T \cup_S C\nabla \rightarrowtail T' \cup_S C\nabla$, where i is a cofibration in $\mathbf{C}$.

These push out cofibrations are fundamental to relate pointed and generalized homotopy.

THEOREM 7. *Given a push out cofibration* $i \cup_1 1 : T \cup_S C\nabla \to T' \cup_S C\nabla$, *the following square is a push out:*

$$\begin{array}{ccc}
C^{n+1}S & \xrightarrow{\quad p^{n+1}C^{n+1}s \quad} & C\nabla \\
\big\downarrow{\scriptstyle \overline{i_n}C\overline{i_{n-1}}...C^n\bar{i}C^{n+1}j} & & \big\downarrow \\
\Sigma^{i_n} & \xrightarrow{\overline{p}C\overline{p}...C^n\overline{p}C^{n+1}\overline{s} \,\cup\, \overline{p}C\overline{p}...C^{n-1}\overline{p}C^n\overline{s} \,\cup\, \overset{(n+1}{\cdots\cdots} \,\cup\, \overline{p}C\overline{p}...C^{n-1}\overline{p}C^n\overline{s}} & \Sigma_*^{(i\cup_1 1)_{n*}}
\end{array}$$

COROLLARY 4. $(i \cup_1 1)_{n+1*} = i_{n+1} \cup_1 1 : \Sigma^{i_n} \cup_{C^{n+1}S} C\nabla \rightarrowtail C^{n+1}T' \cup_{C^{n+1}S} C\nabla$.

Based objects are necessary to define pointed homotopy groups.

DEFINITION 8. A pointed object (X, x) is said to be *based* when there is a morphism $\alpha : X \to C\nabla$ such that $\alpha x = 1$.

If (X, x, α) is a based pointed object, then $\alpha : X \to C\nabla$ is said to be a *base morphism for the push out* $X \cup_{C\nabla} C\nabla$. In general, given a push out $T \cup_S C\nabla$ associated to a pointed object (X, x), a morphism $\alpha : T \to C\nabla$ such that $\alpha j = s$ is said to be a *base morphism* for $T \cup_S C\nabla$.

Observe that a push out cofibration $i \cup_1 1 : T \cup_S C\nabla \to T' \cup_S C\nabla$ is based when $\beta i = \alpha$, where α and β are base morphisms for $T \cup_S C\nabla$ and $T' \cup_S C\nabla$ respectively.

The zero will be suppressed in the notation of the homotopy brackets.

THEOREM 8. *Given a based push out cofibration* $i \cup_1 1 : T \cup_S C\nabla \to T' \cup_S C\nabla$, *with* $\beta i = \alpha$,

a) $[CT' \cup_{CS} C\nabla, (X,x)]^{(i\cup_1 1)_{1*}}$ *is bijective to* $[CT', X]^{xp(C\beta)(i_1)}$.

b) $\pi_{n*}^{i\cup_1 1}((X,x))$ *is a group isomorphic to* $\pi_n^i(X, xp(C\beta))$.

c) $\pi_{n*}^{T'\cup_S C\nabla}((X,x))$ *is a group isomorphic to* $\pi_n^{j'}(X, xp(C\beta))$, *where* $T' \cup_S C\nabla$ *is a push associated to the based pointed object* $(B, b, \{\beta, 1\})$.

COROLLARY 5. *Given a pointed morphism* $f : (X,x) \to (Y,y)$, *the following exact homotopy sequences are isomorphic:*

$$... \xrightarrow{f_*} \pi_3^{i\cup_1 1}((Y,y)) \xrightarrow{j} \pi_3^{i\cup_1 1}(f) \xrightarrow{\delta} \pi_2^{i\cup_1 1}((X,x)) \xrightarrow{f_*} \pi_2^{i\cup_1 1}((Y,y))$$

$$... \xrightarrow{f_*} \pi_3^{i}(Y, yp(C\beta)) \xrightarrow{j} \pi_3^{i}(f, xp(C\beta)) \xrightarrow{\delta} \pi_2^{i}(X, xp(C\beta)) \xrightarrow{f_*} \pi_2^{i}(Y, yp(C\beta)).$$

The above Theorem 8 allows the definition of pointed homotopy groups to be extended to objects such that $x : C\nabla \to X$ is not a cofibration.

Observe that in the category with a natural cone of topological spaces, the cone of the initial object is a point, and the process described here originates the homotopy theory of pointed topological spaces. In this sense, for the category $\mathbf{C}^{C\nabla}$ it is possible to define *spherical objects* by a similar method to the one used to obtain topological spheres.

DEFINITION 9. The $0-$ *sphere of* $\mathbf{C}^{C\nabla}$ is defined by $S^0 = C\nabla \cup_\nabla C\nabla$. Thus, the $n-$ *sphere* S^n is the n^{th} pointed suspension of S^0, that is, $S^n = S_*^n(S^0, \overline{k})$.

Observe that $(S^0, \overline{k}, \{1,1\})$ is a based pointed object.

Remark 1. Note that, for $n \geq 2$,

$$\pi_{n*}^{S^0}((X,x)) = [S^n, (X,x)] = \pi_n^{\overline{k}}(X, \{xp, xp\}) \cong \pi_n^{k_\nabla}(X,x).$$

In this way one can define, by extension, spherical homotopy groups even for n=1:

DEFINITION 10. The n^{th} spherical homotopy group of the pointed object (X,x) is defined by $\pi_{n*}^{S^0}((X,x)) = \pi_n^{k_\nabla}(X,x)$, for $n \in \mathbb{N}$.

In the category of topological spaces, the classic homotopy groups of a pointed topological space (X, x) can be obtained as spherical homotopy groups in $\mathbf{Top}^* = \mathbf{Top}^{C\phi}$.

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