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Transitivity of the Norm on Banach Spaces

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Contents

1. Introduction

In this paper we deal with those concepts and results generated around the Banach-Mazur "rotation" problem if every transitive separable Banach space is a Hilbert space. Basic references in this field are the book of S. Rolewicz [65] and the survey paper of F. Cabello [23]. We recall that a Banach space is called transitive whenever, given two points in its unit sphere, there exists a surjective linear isometry on the space mapping one of such points into the other. As far as we know, the Banach-Mazur rotation problem remains open to date. However, the literature dealing with transitivity of (possibly non separable) Banach spaces seemed to us very attractive and interesting. Therefore, in recent years we have revisited such a literature with the aim of refining some previously known results, as well as of developing some previously unexplored aspects. Most results got in this purpose apear in the papers [8], [9], [10], [11], [12], and [13]. In the present paper we review the main results obtained in the papers just quoted, and prove some new facts. In reviewing known facts, we even provide the reader with new proofs whenever such proofs are available.

In Section 2 we formally introduce the notion of transitivity, as well as the successive weak forms of such a notion appearing in the literature, namely almost-transitivity, convex transitivity, and maximality of the norm. Most classical results involving these concepts are also included, and some facts easily deducible from such results are proved. For instance, we show that the dual of a transitive separable Banach space is almost transitive (Corollary 2.9), and that a Banach space is a 1-complemented subspace of some transitive Banach space whenever it is a 1-complemented subspace of its bidual (Proposition 2.22). This last result depends on a folklore non-separable version of a theorem, due to W. Lusky [55] in the separable case, asserting that every Banach space is a 1-complemented subspace of some almost transitive Banach space (see Theorem 2.14). We provide the reader with a proof of Theorem 2.14 (most part courtesy of F. Cabello) because we have not found such a proof in the literature. By the sake of pleasantness, we introduce a new concept between convex transitivity and maximality of the norm, namely strong maximality of the norm (Definition 2.37). Indeed, l_1 has maximal but not strongly maximal norm (Corollary 2.40), and c_0 has strongly maximal norm but is not convex-transitive (Proposition 2.41).

Section 3 is devoted to review the results in [9] and [10] concerning characterizations of Hilbert spaces in terms of the "abundance" of isometric one-dimensional perturbations of the identity (see Theorems 3.1 and 3.10). Such characterizations refine previous ones of N. J. Kalton and G. V. Wood [48], A. Skorik and M. Zaidenberg [70], and F. Cabello [21], which are collected in Theorems 2.33, 2.16, and 2.17, respectively. As a newness, we prove a characterization of real Hilbert spaces involving strong maximality of the norm and and some kind of abundance of isometric one-dimensional perturbations of the identity (Theorem 3.7). For a characterization of Hilbert spaces in terms of the abundance of isometric finite-dimensional perturbations of the identity the reader is referred to [25].

In Section 4 we review the result in [9] characterizing real Hilbert spaces (among real Banach spaces) by the existence of a non rare subset in the unit sphere consisting of points which "act as units" (Theorem 4.1). A point of a Banach space acts as a unit whenever the geometry of the space at such a point is quite similar to that of a norm-unital Banach algebra at is unit. For a complex Banach space, the existence of a non rare subset in the unit sphere consisting of points which act as units is a too strong condition. Indeed, such a condition is characteristic of the complex field. Then we introduce the notion of point "which acts weakly as a unit", and prove that a real or complex Banach space is a Hilbert space if and only if there is a dense subset in its unit sphere consisting of points which act weakly as units (Theorem 4.4). This leads to the fact that a Banach space X is a Hilbert space if and only if there exists a Banach space Y and a symmetric bilinear mapping $f : X \times X \to Y$ satisfying $\|f(x, z)\| = \|x\|\|z\|$ for all x, z in X.

Section 5 deals with natural generalizations of Wood's conjecture [75] that almost transitive complex $C_0(L)$-spaces (for some locally compact Hausdorff topological space L) are one-dimensional, and of the Grein-Rajalopagan theorem [41] that almost transitive real $C_0(L)$-spaces actually are one-dimensional. In our opinion, the strongest form of Wood's conjecture is the one that almost transitive JB^*-triples coincide with complex Hilbert spaces (Problem 5.1). Concerning the Grein-Rajalopagan theorem, we proved in [11] a wide generalization by showing that almost transitive JB-algebras are one-dimensional (Corollary 5.20). In fact Corollary 5.20 follows straightforwardly from the Grein-Rajalopagan theorem and the fact shown in [11] that convex-transitive JB-algebras are $C_0(L)$-spaces (Theorem 5.19). In Theorem 5.25 we refine Theorem 5.19, providing as well a completely new proof of this last theorem. Concerning Problem 5.1 and the closely related question if transitive preduals of JBW^*-triples are Hilbert spaces (which in general answers negatively), we gave in [11] several partial affirmative answers, collected here in Propositions 5.3 and 5.6. As a newness, we prove that, if X is the predual of a purely atomic JBW^*-triple, and if the norm of X is strongly maximal, then X is a Hilbert space (Theorem 5.4). This generalizes the result previously obtained in [11] that convex-transitive preduals of atomic JBW^*-triples are Hilbert spaces (Corollary 5.5).

The last section of the paper, Section 6, is devoted to the study of the geometry of convex-transitive Banach spaces. We begin by proving a minor new result, namely that proper faces of the closed unit ball of a convex-transitive Banach space have empty interior relative to the unit sphere (Corollary 6.3). In the remaining part of the section we collect the results obtained in [12] and [13] under the philosophy that a convex-transitive Banach space fulfilling some "minor" isometric or isomorphic condition actually is almost transitive and superreflexive (see Theorem 6.8 and Corollary 6.9). Among the "minor" conditions suggested above, we emphasize the Radon-Nikodym property, to be an Asplund space, or merely the Fréchet differentiability of the norm at some point of the unit sphere. Keeping in mind these results, the difference between convex transitivity and maximality of the norm becomes very big. Indeed, c_0 has strongly maximal norm but has no convex-transitive equivalent renorming. When it has been possible, we have revisited the main tools in the proof of Theorem 6.8 quoted above. For instance, this has been the case of [12, Lemma 2.7], which in the present formulation appear greatly refined (see Proposition 6.7). Finally, let us note that the class of almost transitive superreflexive Banach spaces has been fully discussed by C. Finet [36] and F. Cabello [24].

2. Basic notions and results on transitivity of the norm

In this section we introduce the basic concepts and results related to the transitivity (of the norm) of a Banach space. We also present some new facts which are easily derived from previously known results.

Transitivity

Throughout this paper $\mathbb{K}$ will mean the field of real or complex numbers. Given a normed space X over $\mathbb{K}$, S_X, B_X, and X^* will denote the unit sphere, the closed unit ball, and the (topological) dual, respectively, of X, and $\mathcal{G} := \mathcal{G}(X)$ will stand for the group of all surjective linear isometries from X to X. The fundamental concept we are dealing with is the following.

Definition 2.1. A normed space X is said to be *transitive* if for every x, y in S_X there exists T in $\mathcal{G}$ such that $T(x) = y$.

It is well-known and easy to see that pre-Hilbert spaces are transitive. On the other hand, it is apparent that S. Banach knew about the existence of transitive, non separable, and non Hilbert Banach spaces. Therefore, in his book [4] he raises the following question, called the *Banach-Mazur rotation problem*.

Problem 2.2. Is every transitive separable Banach space a Hilbert space?

As well as we know, the above problem remains open to date. As we have just commented, the answer is negative if the assumption of separability is removed. We collect here the first published counter-example, which is due to A. Pelczynski and S. Rolewicz [60] (see also [65, Propositions 9.6.7 and 9.6.8]).

Example 2.3. Let Γ be the disjoint union of an uncountable family of copies of the closed real interval $[0,1]$, and μ the measure on Γ whose measurable sets are those subsets A of Γ whose intersection with each of such copies is measurable relative to the Lebesgue measure, with $\mu(A)$ equal to the sum of the measures of those intersections. Then, for $1 \leq p < \infty$, the Banach space $L_p(\Gamma, \mu)$ is transitive.

It is worth mentioning that the Banach-Mazur rotation problem has an affirmative answer whenever the assumption of separability is strengthened to that of finite dimensionality. In such a case, as we will see later, even the assumption of transitivity can be drastically relaxed.

It is clear that a transitive Banach space is smooth whenever it is smooth at some point of its unit sphere. In this way, thanks to a well-known theorem of S. Mazur, the transitive separable Banach space in Problem 2.2 must be smooth. Therefore Problem 2.2 has an affirmative answer into every class of Banach spaces whose smooth members are Hilbert spaces. For instance, this is the case for the class of spaces $C_0^{\mathbb{K}}(L)$, where L is a locally compact Hausdorff topological space, and $C_0^{\mathbb{K}}(L)$ stands for the Banach space of all $\mathbb{K}$-valued continuous functions on L vanishing at infinity. Actually it is easily seen that the unique smooth $C_0^{\mathbb{K}}(L)$-space is $\mathbb{K}$ ($= C_0^{\mathbb{K}}(L)$, with L reduced to a single point). Now that we know that the locally compact Hausdorff topological space L is reduced to a point whenever $C_0^{\mathbb{K}}(L)$ is separable and transitive, we can wonder if the result remains true when the assumption of separability is removed. The answer is affirmative if $\mathbb{K} = \mathbb{R}$, even with a slight relaxing of the assumption of transitivity [41] (to be more precise, see Theorem 2.11 below). It is a conjecture that the answer is also affirmative if $\mathbb{K} = \mathbb{C}$. In any case, the next characterization of the transitivity of $C_0^{\mathbb{C}}(L)$ seems to be interesting.

Theorem 2.4. ([41]) *Let $X := C_0^{\mathbb{C}}(L)$. Then X is transitive if and only if for x, y in S_X with $x, y \geq 0$ there exists a homeomorphism $\sigma : L \to L$ such that $x(l) = y(\sigma(l))$ for every l in L, and every element z in X has a "polar decomposition" $z = ut$ with $t \geq 0$ in X and u a continuous function from L into the unit sphere of $\mathbb{C}$.*

In relation to Problem 2.2, the class of $C_0^{\mathbb{C}}(L)$-spaces is not too instructive because of the enormous scarcity of Hilbert spaces in that class. In this direction, the largest class of the so-called JB^*-triples becomes more suggestive. In Section 5 we will provide the reader with the definition of JB^*-triples, and will comment about the importance of these Banach spaces in connection with complex Analysis. For the moment, let us limit ourselves to say that complex Hilbert spaces are JB^*-triples, and that, according to the main result in [71], every smooth JB^*-triple is a Hilbert space. Therefore we have the following partial answer to Problem 2.2.

Corollary 2.5. ([71]) *Every transitive separable JB^*-triple is a Hilbert space.*

We do not know if the above corollary remains true without the assumption of separability. If this were the case, then we would be provided with a nice characterization of complex Hilbert spaces involving transitivity. Before to pass to study some weak forms of transitivity, let us give a useful characterization of this property.

Proposition 2.6. *A normed space X is transitive if and only if there exists x in S_X such that $\mathcal{G}(x) := \{T(x) : T \in \mathcal{G}\}$ has nonempty interior relative to S_X.*

Proof. We may assume that the dimension of X over $\mathbb{R}$ is ≥ 2. Let x be in S_X such that there exists $\varepsilon > 0$ satisfying $\{y \in S_X : \|y - x\| < \varepsilon\} \subseteq \mathcal{G}(x)$. Given z in S_X, there are $y_0, \ldots, y_n$ in S_X with $y_0 = x$, $y_n = z$, and $\|y_i - y_{i-1}\| < \varepsilon$ for every $i = 1, \ldots, n$. Clearly $y_1 \in \mathcal{G}(x)$. Let $0 \leq k < n$ with $y_k \in \mathcal{G}(x)$. Choosing T in $\mathcal{G}$ such that $T(y_k) = x$, we have

$$\|T(y_{k+1}) - x\| = \|T(y_{k+1} - y_k)\| < \varepsilon,$$

and hence y_{k+1} belongs to $\mathcal{G}(x)$. Therefore $z = y_n$ belongs to $\mathcal{G}(x)$. ∎

Almost-transitivity

Now let us introduce the first weakening of the transitivity which appears in the literature.

Definition 2.7. A normed space X is said to be *almost transitive* if there exists a dense subset D of S_X such that the equality $\mathcal{G}(u) = D$ holds for every u in D.

Obviously, every transitive normed space is almost transitive. Almost-transitivity has several useful re-formulations which are collected in the next proposition. We refer the reader to either [10, Proposition 2.1] or [12, Proposition 3.1] for a proof. We recall that a subset R of a topological space E is said to be *rare* in E if the interior of the closure of R in E is empty.

PROPOSITION 2.8. *For a normed space X, the following assertions are equivalent:*

1. *X is almost transitive.*
2. *There exists x in S_X such that $\mathcal{G}(x)$ is dense in S_X.*
3. *For every x in S_X, $\mathcal{G}(x)$ is dense in S_X.*
4. *For every x in S_X, $\mathcal{G}(x)$ is non rare in S_X.*
5. *There exists x in S_X such that $\mathcal{G}(x)$ is non rare in S_X.*

The notion of almost-transitivity just introduced allows us to obtain some nontrivial information about the transitive separable Banach space in the Banach-Mazur rotation problem.

COROLLARY 2.9. *If X is a transitive separable Banach space, then X^* is almost transitive.*

Actually we have the following more general result.

PROPOSITION 2.10. *Let X be a transitive smooth Banach space. Then X^* is almost transitive.*

Proof. Let f be a norm-one linear functional on X attaining its norm. Since X is smooth and transitive, the set $\{T^*(f) : T \in \mathcal{G}\}$ contains the set of all norm-one linear functionals on X which attain their norms. But, by the Bishop-Phelps theorem, this last set is dense in S_{X^*}. It follows that $\mathcal{G}(X^*)(f)$ is dense in S_{X^*}. ∎

The question of almost-transitivity of $C_0^{\mathbb{R}}(L)$-spaces is definitively settled by the theorem of P. Greim and M. Rajalopagan which follows (see also [21]).

THEOREM 2.11. ([41]) *Let L be a locally compact Hausdorff topological space. If $C_0^{\mathbb{R}}(L)$ is almost transitive, then L reduces to a singleton.*

Transitivity and almost-transitivity of the norm on $L_p(\mu)$-spaces, for $1 \leq p < \infty$, has been fully studied in [40]. Among the results in that paper, we cite for convenience the following.

PROPOSITION 2.12. ([40, Proposition 1.1]) *If $1 \leq p < \infty$, if $p \neq 2$, if $L_p^{\mathbb{K}}(\mu)$ is almost transitive, and if μ has some atom, then $L_p^{\mathbb{K}}(\mu) = \mathbb{K}$.*

Now let us deal with nontrivial examples of almost transitive Banach spaces.

EXAMPLE 2.13. For $1 \leq p < \infty$ with $p \neq 2$, $L_p[0,1]$ is almost transitive [65, Theorems 9.6.3 and 9.6.4] but not transitive [40, Theorem 1.3]. V.I. Gurarij [42] builds a separable Banach space G with a "nice extension property" which we do not specify here. Later W. Lusky [54] shows that all separable Banach spaces enjoying such an extension property coincide up to isometric isomorphisms, and proves that Gurarij's space is almost transitive. We can realize that G is not transitive by arguing as follows. We know that $G^* = L_1(\mu)$ [74] for some measure μ which, obviously, has some atom. By Proposition 2.12, G^* is not almost transitive, and therefore, by Corollary 2.9, G is not transitive.

Banach spaces in the above example show that, if we relax, in the Banach-Mazur problem, the assumption of transitivity to that of almost-transitivity, then the answer is negative. Although examples of almost transitive classical Banach spaces are scarce, the abundance of almost transitive (non classical) Banach spaces is guaranteed by the result of W. Lusky which follows.

THEOREM 2.14. ([55]) *Every Banach space X can be isometrically regarded as a 1-complemented subspace of an almost transitive Banach space having the same density character as X.*

In Lusky's paper the reader can find only a particular version of the above theorem, namely *every separable Banach space can be regarded as a 1-complemented subspace of some almost transitive separable Banach space.* However, as remarked by several authors, minor changes in Lusky's proof allow to arrive in the most general result given in Theorem 2.14. Since no author specifies the changes needed to achieve this goal, we include here a complete proof of Theorem 2.14, most part of which is courtesy of F. Cabello.

LEMMA 2.15. *Let X be a Banach space with density character $\aleph$, $\{E_\alpha\}_{\alpha\in\Gamma}$ a family of subspaces of X, and for α in Γ let $T_\alpha : E_\alpha \to X$ be an isometry, and $P_\alpha : X \to E_\alpha$ and $Q_\alpha : X \to T_\alpha(E_\alpha)$ contractive projections. Then there exists a Banach space $\widetilde{X}$ containing X isometrically and whose density character is $\aleph$, together with a contractive projection $P : \widetilde{X} \to X$, and for*

α in Γ there is an isometric extension of T_α, $\widetilde{T}_\alpha : X \to \widetilde{X}$, together with a contractive projection $\widetilde{Q}_\alpha : \widetilde{X} \to \widetilde{T}_\alpha(X)$.

Proof. Let us consider the Banach space $Y := X \oplus^{\ell_1} (\oplus^{\ell_1}_{\alpha\in\Gamma} X)$, whose elements will be written in the form $(x; (x_\alpha)_{\alpha\in\Gamma})$, and the closed subspace V of Y generated by the set

$$\big\{\big(-T_\gamma(e); (\delta_{\gamma\alpha} e)_{\alpha\in\Gamma}\big) : \gamma \in \Gamma,\, e \in E_\gamma\big\},$$

where $\delta_{\gamma\alpha}$ means Dirac's function. Put $\widetilde{X} := Y/V$. Then the mapping $x \to (x; (0)_{\alpha\in\Gamma}) + V$ from X to $\widetilde{X}$ becomes a linear isometry, so that, up to such an isometry, we will see X as a subspace of $\widetilde{X}$. We note that, for γ in Γ and e in E_γ, we have

$$\big(T_\gamma(e); (0)_{\alpha\in\Gamma}\big) + V = \big(0; (\delta_{\gamma\alpha} e)_{\alpha\in\Gamma}\big) + V.$$

Now, for γ in Γ, we consider the contractive linear extension $\widetilde{T}_\gamma : X \to \widetilde{X}$ of $T_\gamma : E_\gamma \to X$ given by $\widetilde{T}_\gamma(x) := \big\{(0; (\delta_{\gamma\alpha} x)_{\alpha\in\Gamma}) + V\big\}$. Actually, for γ in Γ, $\widetilde{T}_\gamma$ is an isometry. Indeed, for every x in X and every quasi-null family $(e_\alpha) \in \prod_{\alpha\in\Gamma} E_\alpha$ we have

$$\begin{aligned} &\Big\|\sum_{\alpha\in\Gamma} T_\alpha(e_\alpha)\Big\| + \|x - e_\gamma\| + \sum_{\alpha\in\Gamma\setminus\{\gamma\}} \|e_\alpha\| \\ &\geq \|T_\gamma(e_\gamma)\| + \|x - e_\gamma\| + \sum_{\alpha\in\Gamma\setminus\{\gamma\}} \big(\|e_\alpha\| - \|T_\alpha(e_\alpha)\|\big) \\ &= \|e_\gamma\| + \|x - e_\gamma\| \geq \|x\|, \end{aligned}$$

and hence

$$\begin{aligned} \|\widetilde{T}_\gamma(x)\| &= \|(0; (\delta_{\gamma\alpha} x)_{\alpha\in\Gamma}) + V\| \\ = \inf\Big\{\Big\|\sum_{\alpha\in\Gamma} T_\alpha(e_\alpha)\Big\| &+ \|x - e_\gamma\| + \sum_{\alpha\in\Gamma\setminus\{\gamma\}} \|e_\alpha\| : \ (e_\alpha) \text{ as above}\Big\} \geq \|x\|. \end{aligned}$$

Finally, the proof is concluded by considering the contractive projection $P : \widetilde{X} \to X$ given by

$$P\big\{\big(x; (x_\alpha)_{\alpha\in\Gamma}\big) + V\big\} = \big\{\big(x; (P_\alpha(x_\alpha))_{\alpha\in\Gamma}\big) + V\big\},$$

and for γ in Γ the contractive projection $\widetilde{Q}_\gamma : \widetilde{X} \to \widetilde{T}_\gamma(X)$ defined by

$$\widetilde{Q}_\gamma\big\{\big(x; (x_\alpha)_{\alpha\in\Gamma}\big) + V\big\} = \big\{\big(Q_\gamma(x + \sum_{\alpha\in\Gamma\setminus\{\gamma\}} T_\alpha P_\alpha x_\alpha); (\delta_{\gamma\alpha} x_\gamma)_{\alpha\in\Gamma}\big) + V\big\}.$$

∎

Proof of Theorem 2.14. Let X be a Banach space over $\mathbb{K}$ with density character $\aleph$. For $X_0 := X$, choose a dense subset $D_0 := \{x_\lambda^0\}_{\lambda \in \Lambda_0}$ of S_{X_0} with cardinal equal to $\aleph$. Put $\Gamma_0 := \Lambda_0 \times \Lambda_0$ and, for $\alpha := (\lambda, \mu)$ in Γ_0, consider the subspace $E_\alpha^0 := \mathbb{K}\, x_\lambda^0$ of X_0, and the unique (automatically isometric) linear mapping $T_\alpha^0 : E_\alpha^0 \to X_0$ satisfying $T_\alpha^0(x_\lambda^0) = x_\mu^0$. By the Hahn-Banach theorem, for α in Γ_0 we can choose contractive projections P_α^0 and Q_α^0 from X_0 onto E_α^0 and $T_\alpha^0(E_\alpha^0)$, respectively. Let R^0 denote the identity mapping on X, which, by methodological reasons, should be seen as a contractive projection from X_0 onto X. Applying the above lemma, we obtain:

1. A Banach space X_1 containing X_0 and having $\aleph$ as density character.
2. An isometric linear extension $\widetilde{T}_\alpha^0 : X_0 \to X_1$ of T_α^0 for each $\alpha \in \Gamma_0$.
3. A contractive projection $\widetilde{Q}_\alpha^0 : X_1 \to \widetilde{T}_\alpha^0(X_0)$ for each $\alpha \in \Gamma_0$.
4. A contractive projection $P^0 : X_1 \to X_0$.

Now we write $R^1 := R^0 \circ P^0$, which is a contractive projection from X_1 onto X, and for α in Γ_0 we define:

1. $E_\alpha^1 := \widetilde{T}_\alpha^0(X_0)$
2. $T_\alpha^1 := j_0 \circ (\widetilde{T}_\alpha^0)^{-1} : E_\alpha^1 \to X_1$, where j_0 denotes the inclusion of X_0 into X_1.
3. $Q_\alpha^1 := P^0$.
4. $P_\alpha^1 := \widetilde{Q}_\alpha^0$.

A new application of Lemma 2.15 gives the existence of:

1. A Banach space X_2 containing X_1 and having density character equal to $\aleph$.
2. An isometric linear extension $\widetilde{T}_\alpha^1 : X_1 \to X_2$ of T_α^1 for each $\alpha \in \Gamma_0$.
3. A contractive projection $\widetilde{Q}_\alpha^1 : X_2 \to \widetilde{T}_\alpha^1(X_1)$ for each $\alpha \in \Gamma_0$.
4. A contractive projection $P^1 : X_2 \to X_1$.

Let $D_2 := \{x_\lambda^2\}_{\lambda \in \Lambda_2}$ be a dense subset of S_{X_2} of cardinality $\aleph$, and let Γ_2 denote the disjoint union of Γ_0 and $\Lambda_2 \times \Lambda_2$. For α in Γ_2 we consider:

1. The subspace E_α^2 of X_2 given by $E_\alpha^2 := \mathbb{K}\, x_\lambda^2$ whenever $\alpha = (\lambda, \mu)$ belongs to $\Lambda_2 \times \Lambda_2$, and $E_\alpha^2 := X_0$ otherwise.
2. The linear isometry $T_\alpha^2 : E_\alpha^2 \to X_2$ determined by $T_\alpha^2(x_\lambda^2) = x_\mu^2$ whenever $\alpha = (\lambda, \mu)$ belongs to $\Lambda_2 \times \Lambda_2$, and given by $T_\alpha^2 := \widetilde{T}_\alpha^0$ otherwise.
3. The contractive projection P_α^2 from X_2 onto E_α^2 given by $P_\alpha^2 := P^0 \circ P^1$ if α belongs to Γ_0, and arbitrarily chosen otherwise.

4. The contractive projection Q_α^2 from X_2 onto $T_\alpha^2(E_\alpha^2)$ given by $Q_\alpha^2 := \widetilde{Q}_\alpha^0 \circ P^1$ if α belongs to Γ_0, and arbitrarily chosen otherwise.

Moreover, we write $R^2 := R^0 \circ P^1$, which is a contractive projection from X_2 onto X. Now the sextuple

$$(X_2, \{E_\alpha^2\}_{\alpha\in\Gamma_2}, \{T_\alpha^2\}_{\alpha\in\Gamma_2}, \{P_\alpha^2\}_{\alpha\in\Gamma_2}, \{Q_\alpha^2\}_{\alpha\in\Gamma_2}, R^2)$$

is in the same situation as the one

$$(X_0, \{E_\alpha^0\}_{\alpha\in\Gamma_0}, \{T_\alpha^0\}_{\alpha\in\Gamma_0}, \{P_\alpha^0\}_{\alpha\in\Gamma_0}, \{Q_\alpha^0\}_{\alpha\in\Gamma_0}, R^0),$$

and therefore we can continue the process, obtaining in this way a sequence

$$(X_n, \{E_\alpha^n\}_{\alpha\in\Gamma_n}, \{T_\alpha^n\}_{\alpha\in\Gamma_n}, \{P_\alpha^n\}_{\alpha\in\Gamma_n}, \{Q_\alpha^n\}_{\alpha\in\Gamma_n}, R^n)_{n\geq 0}$$

whose properties can be easily guessed by the reader. We remark that, in the inductive definition of the above sequence, the passing from the n-th term to the subsequent one must be made in a way similar to that followed in passing from the 0-th term to the 1-th whenever n is even, whereas otherwise we must follow a process similar to that applied in obtaining the 2-th term from the 1-th. Finally, the almost transitive Banach space we are searching is nothing but $Z := \overline{\cup_{n\in\mathbb{N}} X_n}$, and the contractive projection from Z onto X is the one R determined by $R(z) = R^n(z)$, whenever z is in X_n for some n in $\mathbb{N}$. ∎

One of the classical topics in the matter we are developing is that of characterizing Hilbert spaces by means of some type of transitivity together with a suitable added natural property. In what follows we review some results in this direction. A linear operator F on a Banach space X is said to be a *reflection* if there is a maximal subspace M of X, together with a nonzero element e in X such that F fixes the elements of M and satisfies $F(e) = -e$. We note that every reflection is a (linear) one-dimensional perturbation of the identity. If X is a Hilbert space, then isometric reflections on X are abundant. Indeed, for each norm-one element e in X consider the mapping $x \mapsto x - 2(x|e)e$ from X into S_X.

Theorem 2.16. ([70]) *A real Banach space is a Hilbert space if (and only if) it is almost transitive and has an isometric reflection*

In his Thesis, F. Cabello observes that, given a real Banach space X, there are no one-dimensional isometric perturbations of the identity on X other than the isometric reflections on X. Moreover, he proves the following result, which becomes a complex version of Theorem 2.16.

THEOREM 2.17. ([21]) *A complex Banach space is a Hilbert space if (and only if) it is almost transitive and has an isometric one-dimensional perturbation of the identity*

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Theorem 2.14, together with ultraproduct techniques [44], will allow us to realize that transitive Banach spaces are quite more abundant than what one could suspect. To this end, we recall some simple ideas from the theory of ultraproducts. Let $\mathcal{U}$ be an ultrafilter on a nonempty set I, and $\{X_i\}_{i\in I}$ a family of Banach spaces. We can consider the Banach space $\oplus_{i\in I}^{\ell_\infty} X_i$, together with its closed subspace

$$N_{\mathcal{U}} := \big\{\{x_i\}_{i\in I} \in \oplus_{i\in I}^{\ell_\infty} X_i : \lim_{\mathcal{U}} \|x_i\| = 0\big\}.$$

The *ultraproduct* $(X_i)_{\mathcal{U}}$ of the family $\{X_i\}_{i\in I}$ relative to the ultrafilter $\mathcal{U}$ is defined as the quotient space $\oplus_{i\in I}^{\ell_\infty} X_i/N_{\mathcal{U}}$. Denoting by (x_i) the element of $(X_i)_{\mathcal{U}}$ containing $\{x_i\}$, it is easily seen that the equality $\|(x_i)\| = \lim_{\mathcal{U}} \|x_i\|$ holds. If, for each i in I, Y_i is a closed subspace of X_i, then we can apply the above formula to naturally identify $(Y_i)_{\mathcal{U}}$ with a closed subspace of $(X_i)_{\mathcal{U}}$. In the particular case that $X_i = X$ for every i in I, where X is a prefixed Banach space, the ultraproduct $(X_i)_{\mathcal{U}}$ will be called the *ultrapower* of X relative to the ultrafilter $\mathcal{U}$, and will be denoted by $X_{\mathcal{U}}$. In such a case, the mapping $x \mapsto \widehat{x}$ from X to $X_{\mathcal{U}}$, where $\widehat{x} = (x_i)$ with $x_i = x$ for every i in I, is a linear isometry. An ultrafilter $\mathcal{U}$ on a set I is called *trivial* if there exists i_0 in I such that $\{i_0\} \in \mathcal{U}$. If this is the case, then a subset U of I belongs to $\mathcal{U}$ if and only if i_0 belongs to U. Clearly, nontrivial ultrafilters on a set I contain the filter of all co-finite subsets of I. The ultrafilter $\mathcal{U}$ is called *countably incomplete* if there is a sequence $\{U_n\}_{n\geq 1}$ in $\mathcal{U}$ such that $J := \cap_{n=1}^{\infty} U_n$ does not belong to $\mathcal{U}$. If this is the case, then it is enough to inductively define $I_1 = I, I_{n+1} = I_n \cap U_n \cap (I \setminus J)$, to have in fact a sequence $\{I_n\}$ in $\mathcal{U}$ such that $I = I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots\ldots \supseteq \bigcap_{n\in\mathbb{N}} I_n = \emptyset$. It is clear that countably incomplete ultrafilters are non trivial. Is is also clear that, on countable sets, nontrivial ultrafilters and countably incomplete ultrafilters coincide. The following result is folklore in the theory (see [22, Lemma 1.4] or [40, Remark p. 479]).

PROPOSITION 2.18. *If I is a countable set, if $\mathcal{U}$ is a nontrivial ultrafilter on I, and if $\{X_i\}_{i\in I}$ is a family of almost transitive Banach spaces, then the Banach space $(X_i)_{\mathcal{U}}$ is transitive.*

A reasonable extension of the above result is the following

PROPOSITION 2.19. *Let $\mathcal{U}$ be a countably incomplete ultrafilter on a set I, and $\{X_i\}_{i\in I}$ a family of almost transitive Banach spaces. Then $(X_i)_{\mathcal{U}}$ is transitive.*

Proof. Take a decreasing sequence $\{I_n\}_{n\in\mathbb{N}}$ in $\mathcal{U}$ such that $I_1 = I$ and $\bigcap_{n\in\mathbb{N}} I_n = \emptyset$. Consider the mapping $\sigma : I \to \mathbb{N}$ given by

$$\sigma(i) := \max\{n \in \mathbb{N} : i \in I_n\}.$$

We are showing that, for (x_i) and (y_i) in $S_{(X_i)_{\mathcal{U}}}$, there exists T in $\mathcal{G}((X_i)_{\mathcal{U}})$ satisfying $T((x_i)) = (y_i)$. Without loss of generality we may assume that x_i and y_i belong to S_{X_i} for every i in I. Given $i \in I$, there is T_i in $\mathcal{G}(X_i)$ with $\|y_i - T_i(x_i)\| \leq \frac{1}{\sigma(i)}$. Now, the mapping T from $(X_i)_{\mathcal{U}}$ into $(X_i)_{\mathcal{U}}$ defined by $T((z_i)) := (T_i(z_i))$ is a surjective linear isometry satisfying $T((x_i)) = (y_i)$. To verify this last equality, note that it is equivalent to the fact that, for every $\varepsilon > 0$, the set $\{i \in I : \|y_i - T_i(x_i)\| \leq \varepsilon\}$ belongs to $\mathcal{U}$. But this is true because, given $\varepsilon > 0$, the set $\{i \in I : \|y_i - T_i(x_i)\| \leq \varepsilon\}$ contains $\{i \in I : \sigma(i) \geq p\}$ for p in $\mathbb{N}$ with $\frac{1}{p} < \varepsilon$, and this last set contains I_p. ∎

A first consequence of Proposition 2.18 is that, if a class of Banach spaces is closed under ultrapowers, and if some of its members is almost transitive but non Hilbert, then there exists a member in the class which is transitive and non Hilbert. As a bi-product of the commutative Gelfand-Naimark theorem [66, Corollary 1.2.2], the class of $C_0^{\mathbb{C}}(L)$-spaces is closed under ultraproducts. Therefore the conjecture early commented, that an almost transitive $C_0^{\mathbb{C}}(L)$-space must be equal to $\mathbb{C}$, is equivalent to the apparently stronger which follows. Such a conjecture is called *Wood's conjecture*.

CONJECTURE 2.20. ([75]) If L is a locally compact Hausdorff topological space, and if the Banach space $C_0^{\mathbb{C}}(L)$ is almost transitive, then L reduces to a singleton.

It is well-known (see [75, Section 3]) that the answer to the above conjecture is affirmative whenever L is in fact compact. A straightforward consequence of Theorem 2.14 and Proposition 2.18 is the following.

COROLLARY 2.21. *Every Banach space can be isometrically regarded as a subspace of a suitable transitive Banach space.*

The next result becomes a useful variant of Corollary 2.21.

PROPOSITION 2.22. *If the Banach space X is 1-complemented in its bidual, then also X is 1-complemented in some transitive Banach space.*

Proof. Since the relation "to be 1-complemented in" is transitive, it is enough to show that every dual Banach space is 1-complemented in some transitive Banach space. (By the way, this is also necessary because every dual Banach space is 1-complemented in its bidual.) Let X be a dual Banach space. Take a nontrivial ultrafilter $\mathcal{U}$ on $\mathbb{N}$. By Theorem 2.14, there exist an almost transitive Banach space Z and a contractive projection $Q_1 : Z \to X$. Then the mapping $\widehat{Q_1} : (x_i) \mapsto (Q_1(x_i))$ is a contractive projection from $Z_{\mathcal{U}}$ onto $X_{\mathcal{U}}$. On the other hand, regarding naturally X as a subspace of $X_{\mathcal{U}}$, the mapping $Q_2 : (x_n) \mapsto w^* - \lim_{\mathcal{U}}\{x_n\}$ becomes a contractive projection from $X_{\mathcal{U}}$ onto X. Now $Z_{\mathcal{U}}$ is transitive (by Proposition 2.18) and $Q_2 \circ \widehat{Q_1}$ is a contractive projection from $Z_{\mathcal{U}}$ onto X. ∎

We recall that a Banach space X has the *approximation property* if for every compact subset K of X and every $\varepsilon > 0$ there exista a finite rank bounded linear operator T on X such that $\|T(x) - x\| \leq \varepsilon$, for all x in K. In [55] W. Lusky observes how Theorem 2.14 can be applied to obtain the existence of almost transitive separable Banach spaces failing to the approximation property. Since today we know about the existence of reflexive Banach spaces without the approximation property (see for instance [53, Theorem 2.d.6]), the next result follows from Proposition 2.22.

COROLLARY 2.23. *There exist transitive Banach spaces failing to the approximation property.*

The assertion in the above corollary can be found in [21, p. 57], where a proof is missing. Now that we know how almost transitive Banach spaces give rise to transitive Banach spaces, let us consider the converse question. Of course, such a question becomes interesting only whenever the almost transitive Banach space built from a transitive one is drastically "smaller". A relevant result in this line is the one of F. Cabello [22] that, *if X is a transitive Banach space, and if M is a closed separable subspace of X, then there exists an almost transitive closed subspace of X containing M.* Actually the result just quoted is a particular case of Theorem 2.24 below, which has shown very useful in the applications. Let $\mathcal{J}$ be a subcategory of the category of Banach spaces (see [68, p. 161, Definition 9.13]). By a *$\mathcal{J}$-space* we mean an object of

$\mathcal{J}$, and by a *$\mathcal{J}$-subspace* of a given $\mathcal{J}$-space X we mean a closed subspace Y of X such that Y is a $\mathcal{J}$-space and the inclusion $Y \hookrightarrow X$ is a $\mathcal{J}$-morphism. The subcategory $\mathcal{J}$ is said to be *admissible* whenever the following conditions are fulfilled:

1. For every $\mathcal{J}$-space X and every separable subspace Z of X, there exists a $\mathcal{J}$-subspace Y of X which is separable and contains Z.
2. For every $\mathcal{J}$-space X and every increasing sequence $\{Y_n\}$ of $\mathcal{J}$-subspaces of X, the space $\overline{\bigcup_{n\in\mathbb{N}} Y_n}$ is a $\mathcal{J}$-space.

THEOREM 2.24. ([22]) *Let $\mathcal{J}$ be an admissible subcategory of Banach spaces, X a transitive $\mathcal{J}$-space, and M a separable subspace of X. Then there exists an almost transitive separable $\mathcal{J}$-subspace of X containing M.*

Actually, replacing the density character $\aleph_0$ of separable Banach spaces with an arbitrary cardinal number $\aleph$, a reasonable concept of $\aleph$-admissible subcategory of Banach spaces can be given, and the corresponding generalization of Theorem 2.24 can be proved (see [7, pp. 19-22]). As a relevant application of Theorem 2.24, F. Cabello observes that the subcategory of $C_0^{\mathbb{C}}(L)$-spaces, with morphisms equal to algebra homomorphisms, is admissible, and consequently reduces Wood's conjecture (Conjecture 2.20) to the bleeding one which follows.

CONJECTURE 2.25. ([22]) If L is a locally compact Hausdorff topological space whose one-point compactification is metrizable, and if $C_0^{\mathbb{C}}(L)$ is almost transitive, then L reduces to a point.

CONVEX TRANSITIVITY

After the almost-transitivity, the subsequent weakening of the transitivity arising in the literature is the so-called convex transitivity, given in the following definition.

DEFINITION 2.26. A normed space X is said to be *convex-transitive* if for every x in S_X we have $\overline{co}(\mathcal{G}(x)) = B_X$, where $\overline{co}$ means closed convex hull.

By Proposition 2.8, almost transitivity implies convex-transitivity. In the next Theorem we collect a very useful characterization of convex transitivity, due to E. R. Cowie.

THEOREM 2.27. ([28]) *For a normed space X, the following assertions are equivalent:*

1. *X is convex-transitive.*
2. *Every continuous norm on X enlarging the group of surjective linear isometries of X is a positive multiple of the norm of X.*
3. *Every equivalent norm on X enlarging the group of surjective linear isometries of X is a positive multiple of the norm of X.*

In Cowie's original formulation of Theorem 2.27, only the equivalence of Conditions 1 and 3 above is stated. On the other hand, the implication $2 \Rightarrow 3$ is clear. Let us realize that 3 implies 2. Let $|||\cdot|||$ be a continuous norm on X enlarging $\mathcal{G}$. Then the norm $|\cdot|$ on X given by $|x| := \|x\| + |||x|||$ is equivalent to the natural norm of X and enlarges $\mathcal{G}$. Now, if we assume Condition 3, then $|\cdot|$ is a positive multiple of $\|\cdot\|$, and hence $|||\cdot|||$ is also a positive multiple of $\|\cdot\|$. A standard application of the Hahn-Banach theorem provides us with the following folklore but useful characterization of convex transitivity.

PROPOSITION 2.28. *A Banach space X is convex-transitive if and only if, for every f in S_{X^*}, the convex hull of $\{T^*(f) : T \in \mathcal{G}\}$ is w^*-dense in B_{X^*}.*

Convex transitivity in $C_0^{\mathbb{K}}(L)$-spaces has been studied in some detail by G. V. Wood [75]. The main results in this line are collected in the two theorem which follow.

THEOREM 2.29. ([75]) *Let L be a locally compact Hausdorff topological space. Then $C_0^{\mathbb{C}}(L)$ is convex-transitive If and only if, for every probability measure μ on L and every t in L, there exists a net $\{\gamma_\alpha\}$ of homeomorphisms of L such that the net $\{\mu \circ \gamma_\alpha\}$ is w^*-convergent to the Dirac mesure δ_t.*

In the above theorem (as well as in what follows), for a measure μ on L and a homeomorphism γ of L, $\mu \circ \gamma$ denotes the measure on L defined by $\int x d(\mu \circ \gamma) := \int (x \circ \gamma) d(\mu)$, for every x in $C_0^{\mathbb{K}}(L)$.

THEOREM 2.30. ([75]) *Let L be a locally compact Hausdorff topological space. Then $C_0^{\mathbb{R}}(L)$ is convex-transitive if and only if L is totally disconnected and, for every probability measure μ on L and every t in L, there exists a net $\{\gamma_\alpha\}$ of homeomorphisms of L such that the net $\{\mu \circ \gamma_\alpha\}$ is w^*-convergent to δ_t.*

The next result follows straightforwardly from the two above theorems.

Corollary 2.31. *$C_0^{\mathbb{C}}(L)$ is convex-transitive whenever $C_0^{\mathbb{R}}(L)$ is.*

In our opinion, Theorems 2.29 and 2.30 are theoretically nice, but not too useful to provide us with examples of convex-transitive Banach spaces. The interest of those theorems actually relies in Corollary 2.31 above, and the fact noticed in [21] that Theorem 2.11 can be easily derived from Theorem 2.30. As far as we know, before our work (see Corollary 5.15 below), all known examples of convex-transitive non almost transitive Banach spaces are $C_0^{\mathbb{K}}(L)$-spaces. All these examples were discovered by A. Pelczynski and S. Rolewicz [60] (see also [65] and [75]), and are collected in what follows.

Example 2.32. The following Banach spaces are convex-transitive:

1. $C_0^{\mathbb{C}}(L)$, with $L = (0, 1)$.
2. $C^{\mathbb{K}}(K)$, where K denotes Cantor's set.
3. $C^{\mathbb{C}}(K)$, with $K = \{z \in \mathbb{C} : |z| = 1\}$.
4. $L_\infty^{\mathbb{K}}([0, 1], \gamma)$, where γ stands for Lebesgue's measure.

We note that, since $L_\infty^{\mathbb{C}}([0, 1], \gamma)$ is a unital commutative C^*-álgebra, we have $L_\infty^{\mathbb{K}}([0, 1], \gamma) = C^{\mathbb{K}}(K)$, for some compact Hausdorff topological space K. As shown in [75, pag. 180], $C_0^{\mathbb{K}}(L)$ cannot be almost transitive if L is not reduced to a point and either L is compact or L has a compact connected subset with nonempty interior. It follows that all Banach spaces enumerated above are not almost transitive.

Let us now review a nice characterization of complex Hilbert spaces in terms of convex transitivity, due to N. J. Kalton y G. V. Wood. We recall that a bounded linear operator F on a complex Banach space X is said to be *hermitian* if, for every λ in $\mathbb{R}$, the operator $\exp(i\lambda F)$ is an isometry. If X is a Hilbert space, then orthogonal projections onto the closed subspaces of X become examples of hermitian projections.

Theorem 2.33. ([48, Theorem 6.4]) *Let X be a convex-transitive complex Banach space having a one-dimensional hermitian projection. Then X is a Hilbert space*

To close our review on convex transitivity, let us notice some mistaken results in [65], which can produce confusion to an inexpert reader. Indeed, in [65, Theorems 9.7.3 and 9.7.7] (referring to a forerunner of [75]), Assertions 1 and 2 which follow are formulated.

1. $C_0^{\mathbb{C}}(L)$ is convex-transitive if and only if, for every s in L, the set $\{\gamma(s) : \gamma \text{ homeomorphism of } L\}$ is dense in L.
2. $C_0^{\mathbb{R}}(L)$ is convex-transitive if and only if L is totally disconnected and, for every s in L, the set $\{\gamma(s) : \gamma \text{ homeomorphism of } L\}$ is dense in L.

Clearly, both assertions are not true. Indeed, if those assertions were correct, then the real or complex Banach spaces ℓ_∞^n $(n \in \mathbb{N} \setminus \{1\})$ and c_0 would be convex-transitive (note that, by Theorem 2.33, this would imply in the complex case that those spaces are Hilbert spaces). The mistake in the proof of Assertions 1 and 2 above is very elemental: a simple erroneous change of direction in an inequality. It seems to us that Assertions 1 and 2 above are nothing but non corrected preliminary versions of Theorems 2.29 and 2.30, respectively. In any case, as we show in the next proposition, a little true can be found in such assertions.

Proposition 2.34. *Let L be a locally compact Hausdorff topological space. Put $X := C_0^{\mathbb{C}}(L)$, and denote by Γ the group of all homeomorphisms of L. Then the following assertions are equivalent:*

1. *For every x in S_X, the linear hull of $\mathcal{G}(x)$ is dense in X.*
2. *For every s in L, the set $\Gamma(s) := \{\gamma(s) : \gamma \in \Gamma\}$ is dense in L.*

Proof. 1 $\Rightarrow$ 2.- Assume that for some s in L we have $\overline{\Gamma(s)} \neq L$. Then $I := \{x \in X : x(\Gamma(s)) = 0\}$ is a nonzero proper closed subspace of X. Moreover it follows from the well-known theorem of M. H. Stone, describing the surjective linear isometries of $C_0^{\mathbb{C}}(L)$, that I is $\mathcal{G}$-invariant. Now, for x in $I \cap S_X$, the linear hull of $\mathcal{G}(x)$ is not dense in X.

2 $\Rightarrow$ 1.- Let x be in S_X. Let I denote the closed linear hull of $\mathcal{G}(x)$. We claim that I is an ideal of X. Indeed, for α in $\mathbb{R} \setminus \{0\}$ and z in X with $x(L) \subseteq \mathbb{R}$, the operator of multiplication by $\exp(i\alpha z)$ on X belongs to $\mathcal{G}$. Therefore, since I is $\mathcal{G}$-invariant, for y in I we have

$$\frac{\exp(i\alpha z)y - y}{\alpha} \in I,$$

and hence, letting $\alpha \to 0$, we obtain $zy \in I$. Now that we know that I is an ideal of X, we can find a closed subset E of L such that

$$I = \{y \in X : y(E) = 0\}.$$

Then a new application of Stone's theorem, together with the $\mathcal{G}$-invariance of I, leads that E is Γ-invariant. Now, if Assertion 2 holds, then E is empty (since $I \neq 0$), and hence $I = X$. ∎

For future discussions, se recall that Stone's theorem just applied asserts that, given a surjective linear isometry T on $C_0^{\mathbb{K}}(L)$, there exist a continuous unimodular $\mathbb{K}$-valued function θ on L and a homeomorphism γ of L satisfying $T(x)(s) = \theta(s)x(\gamma(s))$ for all x in X and s in L.

Maximality of norm

Now, let us introduce the weakest form of transitivity arising in the literature.

DEFINITION 2.35. Given a normed space X, we say that *the norm of X is maximal* whenever there is no equivalent norm on X whose group of surjective linear isometries strictly enlarges $\mathcal{G}$.

In relation to the philosophy of transitivity, the notion of maximality of norm is somewhat less intuitive than the ones previously given. In any case, maximality of the norm is implied by convex transitivity (in view of the implication 1 $\Rightarrow$ 3 in Theorem 2.27). Summarizing, we are provided with the following chain of implications between transitivity conditions on a Banach space:

$$\boxed{\text{Hilbert}} \Rightarrow \boxed{\text{Transitive}} \Rightarrow \boxed{\text{Almost transitive}}$$
$$\Rightarrow \boxed{\text{Convex-transitive}} \Rightarrow \boxed{\text{Maximal norm}}$$

We already know that none of the first three implications in the above chain is reversible. As we see immediately below, also the last implication cannot be reversed.

EXAMPLE 2.36. We recall that a Schauder basis $\{e_n\}_{n\in\mathbb{N}}$ of a Banach space X is said to be (1-)*simmetric* whenever, for every bijection $\sigma : \mathbb{N} \to \mathbb{N}$ and every couple of finite sequences $\alpha_1, \ldots, \alpha_n$, $\beta_1, \ldots, \beta_n$ with $|\beta_1| = \cdots = |\beta_n| = 1$, we have

$$\Big\| \sum_{i=1}^{n} \beta_i \alpha_i e_{\sigma(i)} \Big\| = \Big\| \sum_{i=1}^{n} \alpha_i e_i \Big\|.$$

Let X be a Banach space with a symmetric basis $\{e_n\}_{n\in\mathbb{N}}$. Given a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ and a sequence $\{\beta_n\}_{n\in\mathbb{N}}$ of unimodular numbers, for each $x = \sum_{n=1}^{\infty} \alpha_n e_n$ in X the series $\sum_{n=1}^{\infty} \beta_n \alpha_n e_{\sigma(n)}$ is convergent, and the mapping $x \mapsto \sum_{n=1}^{\infty} \beta_n \alpha_n e_{\sigma(n)}$ is a surjective linear isometry on X. A. Pelczinski and S. Rolewicz show that, on a Banach space $X \neq \ell_2$ with a symmetric basis, there are no surjective linear isometries other than the ones described above (see [65, Chapter 9, Section 8]). As a consequence, the norm of every Banach space non isomorphic to ℓ_2 and having a symmetric basis is maximal. For instance, this is the case of the spaces c_0 y ℓ_p $(1 \leq p < \infty, p \neq 2)$, which of course are not convex-transitive.

J.R. Partington [57, Theorem 1] shows that the natural norm of $C^{\mathbb{R}}([0,1])$ is not maximal. On the other hand, N. J. Kalton and G. V. Wood [48, Theorem 8.2] prove that, if L locally compact Hausdorff topological space, then the norm of $C_0^{\mathbb{C}}(L)$ is maximal whenever one of the following conditions is fulfilled:

1. L is infinite and contains a dense subset consisting of isolated points.
2. There exists a dense subset of L consisting of points admitting a neighbourhood homeomorphic to an open subset of an Euclidean space.

Consequently, although $C^{\mathbb{C}}([0,1])$ is not convex-transitive [28, Example 3], the norm of $C^{\mathbb{C}}([0,1])$ is maximal. For recent advances about maximality of the norm in $C_0(L)$-spaces, the reader is referred to [52].

Now, we are introducing an intermediate notion between convex transitivity and maximality of norm, which, as far as we know, has been not previously considered in the literature.

Definition 2.37. Given a normed space X, we say that *the norm of X is strongly maximal* whenever there is no continuous norm on X whose group of surjective linear isometries strictly enlarges $\mathcal{G}$.

By the implication $1 \Rightarrow 2$ in Theorem 2.27, convex transitivity implies strong maximality of norm, and, clearly, strong maximality of norm implies maximality of norm. The notion just introduced will become specially useful when it is put together with the following one.

Definition 2.38. By an *invariant inner product* on a normed space X we mean a continuous inner product $(\cdot|\cdot)$ on X satisfying $(F(x)|F(x)) = (x|x)$ for every x in X and every F in $\mathcal{G}$.

Now, keeping in mind that pre-Hilbert spaces are transitive, the proof of the following result becomes an exercise, which is left to the reader.

Proposition 2.39. *Let X be a normed space. Assume that the norm of X is strongly maximal and that there exists an invariant inner product on X. Then X is a pre-Hilbert space.*

As a first application of the above proposition, we find examples of Banach spaces whose norm is maximal but not strongly maximal (compare Example 2.36).

Corollary 2.40. *Let $1 \leq p < 2$. Then the norm of ℓ_p is not strongly maximal.*

Proof. For $x = \{\alpha_n\}$, $y = \{\beta_n\}$ in ℓ_p, put $(x|y) := \sum_{n=1}^{\infty} \alpha_n \overline{\beta_n}$. Then $(\cdot|\cdot)$ is an invariant inner product on ℓ_p. If the norm of ℓ_p were strongly maximal, then, by Proposition 2.39, ℓ_p would be a Hilbert space. ∎

Now, it seems reasonable to exhibit a Banach space failing to convex transitivity but having strongly maximal norms. The example cannot be easier.

Proposition 2.41. *The norm of c_0 is strongly maximal.*

Proof. Since the norm of c_0 is maximal, it is enough to show that every continuous norm on c_0 whose group of linear surjective isometries enlarges $\mathcal{G}(c_0)$ is in fact equivalent to the natural norm. Actually we will prove something better, namely that every norm on c_0 enlarging the group of surjective linear isometries generates a topology stronger than that of the natural norm. Let $|||\cdot|||$ be a norm on c_0 such that $\mathcal{G}(c_0) \subseteq \mathcal{G}(c_0, |||\cdot|||)$. Let us denote by $U := \{u_n : n \in \mathbb{N}\}$ the canonical basis of c_0, and by $V := \{v_n : n \in \mathbb{N}\}$ the canonical basis of ℓ_1 regarded as a subset of c_0^*. Since $U \subseteq \mathcal{G}(c_0)(u_1)$, we have $U \subseteq \mathcal{G}(c_0, |||\cdot|||)(u_1)$, and therefore there is no loss of generality in assuming that $|||u_n||| = 1$ for every n in $\mathbb{N}$. On the other hand, for n in $\mathbb{N}$, the mapping $x \mapsto x - 2v_n(x)u_n$ from c_0 to c_0 belongs to $\mathcal{G}(c_0)$, and hence it also belongs to $\mathcal{G}(c_0, |||\cdot|||)$. It follows that, for every n in $\mathbb{N}$, v_n belongs to $(c_0, |||\cdot|||)^*$ and the equality $|||v_n||| = 1$ holds. Let $f := \{\lambda_n\}$ be in ℓ_1. Then we have

$$\sum_{n=1}^{\infty} |||\lambda_n v_n||| \leq \sum_{n=1}^{\infty} |\lambda_n|,$$

and therefore the series $\sum \lambda_n v_n$ converges in $(c_0, |||\cdot|||)^*$. But, regarded as an element of the algebraic dual of c_0, the sum of that series both in $(c_0, |||\cdot|||)^*$ and c_0^* must coincide because both $|||\cdot|||$-convergence in $(c_0, |||\cdot|||)^*$ and $\|\cdot\|$-convergence in c_0^* imply point-wise convergence. Now we have shown that $c_0^* \subseteq (c_0, |||\cdot|||)^*$ and that for f in c_0^* we have

$$|||f||| \leq \sum_{n=1}^{\infty} |||\lambda_n v_n||| \leq \|f\|.$$

Finally, for x in c_0, we can choose f in $S_{c_0^*}$ with $f(x) = \|x\|$ to obtain

$$\|x\| = f(x) \leq |||f|||\, |||x||| \leq \|f\|\, |||x||| = |||x|||.$$

∎

To conclude our review on maximality of the norm, let us realize that, as we commented early, the Banach-Mazur theorem has an affirmative answer in a finite-dimensional setting. Actually the better result which follows holds.

Corollary 2.42. ([65, Proposition 9.6.1]) *Every finite–dimensional space over $\mathbb{K}$ with a maximal norm is a Hilbert space.*

The above corollary follows straightforwardly from Proposition 2.39 and the celebrated theorem of H. Auerbach ([2], [3]) which follows.

Theorem 2.43. *On every finite-dimensional Banach space there exists an invariant inner product.*

For a collection of elegant proofs of Auerbach's theorem the reader is referred to [23].

Transitivity and isomorphic conditions

We devote the last part of this section to consider one of the most open aspects of the matter we are dealing with, namely that of the relationship between transitivity conditions previously introduced and the most familiar isomorphic conditions on Banach spaces. In Diagrams I and II which follow we recall the first and second ones, respectively. By the sake of shortness, we write "hilbertizable" instead of "isomorphic to a Hilbert space".

Diagram I

$$\begin{array}{c} \boxed{\text{Hilbert}} \\ \Downarrow \\ \boxed{\text{Transitive}} \\ \Downarrow \\ \boxed{\text{Almost transitive}} \\ \Downarrow \\ \boxed{\text{Convex-transitive}} \\ \Downarrow \\ \boxed{\text{Maximal norm}} \end{array}$$

Diagram II

$$\begin{array}{ccc} & \boxed{\text{Finite-dimensional}} & \\ & \Downarrow & \\ & \boxed{\text{Hilbertizable}} & \\ & \Downarrow & \\ & \boxed{\text{Superreflexive}} & \\ & \Downarrow & \\ & \boxed{\text{Reflexive}} & \\ \Downarrow & & \Downarrow \\ \boxed{\text{Asplund}} & & \boxed{\text{Radon-Nikodym}} \end{array}$$

It is quite natural to raise the following two questions:

1. If a Banach space satisfies some of the properties in Diagram I, does it satisfy someone of those in Diagram II?
2. If the Banach space X satisfies some of the properties in Diagram II, must some of the conditions in Diagram I be fulfilled by a suitable equivalent renorming of X?

With the unique exception of the obvious fact that Hilbert spaces are hilbertizable, the answer to Question 1 above is a total disaster because of Corollary 2.21 and the fact that all conditions in Diagram II are hereditary. Such a disaster led some authors (see [75], [57], and [24]) to conjecture that the weakest condition in Diagram I could be isomorphically innocuous. In other words, the following problem remains open.

PROBLEM 2.44. Can every Banach space be equivalently renormed in such a way that the new norm is maximal?

The above problem can be re-formulated as follows. *Has every Banach space X a maximal bounded subgroup of the group of all automorphisms of X?* The equivalence of this new question with Problem 2.44 follows from the fact that, if G is a bounded group of automorphisms of a Banach space X, then, for the equivalent new norm $\|\cdot\|_G$ on X given by $\|x\|_G := \sup\{\|F(x)\| : F \in G\}$, we have $G \subseteq \mathcal{G}(X, \|\cdot\|_G)$. Concerning Question 2, things behave no much better. For instance, it seems to be unknown if every superreflexive Banach space has an equivalent maximal renorming. However, in the opinion of some authors (see [36] and [30]), the possibility that superreflexive Banach spaces could be almost transitively renormed must not be discarded. Now that we know that the answers to Question 1 and 2 are negative and essentially unknown, respectively, a reasonable way to leave the "impase" consists in putting together some of the properties in Diagram I and someone of those in Diagram II, hoping to obtain some nontrivial additional information. With some chance, a successful work in the direction just mentioned could at least show the isomorphic non-innocuousness of transitivity, almost-transitivity, and convex-transitivity. One of the pioneering paper in this line is that of V. P. Odinec [56], where it is proven that the dual of a transitive reflexive Banach space is transitive. Other interesting contribution is that of C. Finet [36] (see also [30]), who shows that superreflexive almost transitive Banach spaces are uniformly convex and uniformly smooth. Later, F. Cabello [24] proves that almost transitive Banach spaces which either are Asplund or have the Radon-Nikodym property actually are superreflexive (graphically, they ascend two steps in Diagram II). As a consequence, almost-transitivity is not an isomorphically innocuous property. It is shown also in [24] that superreflexive convex-transitive Banach spaces are in fact almost transitive (graphically, they ascend one step in Diagram I). As a consequence of our results (see Corollary 6.9 below), convex-transitive Banach spaces which either are Asplund or have the Radon-Nikodym property are in fact superreflexive (they ascend two steps in Diagram II) and almost transitive (they ascend one step in Diagram I). Consequently, convex-transitivity is not an isomorphically innocuous property.

3. Isometric one-dimensional perturbations of the identity

In this section we review some results in [9] and [10] characterizing Hilbert spaces among Banach spaces in terms of the abundance of isometric one-dimensional perturbations of the identity. Let us say that an element e in a Banach space X is an *isometric reflection vector* (respectively, a *vector of isometric one-dimensional perturbation of the identity*) if $\|e\| = 1$ and there exists an isometric reflection (respectively, an isometric one-dimensional perturbation of the identity) F on X such that e belongs to the range of $\mathbf{1}-F$.

We recall that, for real Banach spaces, isometric one-dimensional perturbations of the identity are nothing but isometric reflections. Therefore Theorems 2.16 and 2.17 can be unified by saying that *a Banach space over $\mathbb{K}$ is a Hilbert space if (and only if) it is almost transitive and has an isometric one-dimensional perturbation of the identity.* Now, putting together [9, Theorem 2.2] and [10, Theorem 2.1], we are provided with the following reasonable generalization of the result just quoted.

Theorem 3.1. *A Banach space X over $\mathbb{K}$ is a Hilbert space if (and only if) there exists a non rare set in S_X consisting of vectors of isometric one-dimensional perturbation of the identity.*

In [9] we proved the real version of the above theorem by applying some results on isometric reflections taken from [70]. Now we are giving an alternative proof of such a real version, which is nothing but a simplification of the argument used in [10] to cover the complex case. For later discussion, we put some emphasis in the following lemma.

Lemma 3.2. *Let H be a real Hilbert space, and e be in S_H. Let Ω denote the open subset of S_H given by $\Omega := \{y \in S_H : (e|y) > 0\}$. Then there exists* *a continuous function x from Ω to S_H satisfying $x(e) = e$ and*

$$e - 2(e|x(y))x(y) = -y$$

for all y in Ω.

Proof. Define $x(y) := \frac{e+y}{\|e+y\|}$ for y in Ω. ∎

Proof of Theorem 3.1 in the case $\mathbb{K} = \mathbb{R}$. Assume that, for the real Banach space X, there exists a non rare set in S_X consisting of isometric reflection vectors. Since the set of all isometric reflection vectors of X is closed in X (see [9, Lemma 2.1] for details), it follows that actually there is a non-empty open subset ω of S_X consisting of isometric reflection vectors. Now, fixing e in ω, it is enough to show that every finite-dimensional subspace of X containing e is a Hilbert space. Let Y be such a subspace of X, take an invariant inner product $(\cdot|\cdot)$ on Y (Theorem 2.42) satisfying $(e|e) = 1$, denote by $|\!|\!| \cdot |\!|\!|$ the Hilbertian norm on Y deriving from $(\cdot|\cdot)$, and consider the Hilbert space $H := (Y, |\!|\!| \cdot |\!|\!|)$ together with the homeomorphism $h : y \to |\!|\!| y |\!|\!|^{-1} y$ from S_Y onto S_H. For the Hilbert space H and the element e in S_H arising above, let Ω be the non empty open subset of S_H, and x be the continuous function from Ω to S_H, given by Lemma 3.2. Then the set

$$L := \{y \in S_Y : \, h(y) \in \Omega \text{ and } h^{-1}\big(x(h(y))\big) \in \omega\}$$

is open in S_Y and non empty (since $e \in L$). Let y be in L. Then there exists an isometric reflection on X having $x(h(y))$ as an eigenvector corresponding to the eigenvalue -1, and such an isometric reflection can be seen as an isometric reflection (say F_y) on Y. Since F_y is also an isometric reflection on H and $x(h(y))$ is in S_H, it follows that F_y is nothing but the mapping

$$z \to z - 2\big(z|x(h(y))\big)x(h(y))$$

from Y to Y. By the properties of the function x in Lemma 3.2, we have $F_y(e) = -h(y)$, so $h(y) = y$ (since F_y is an isometry on Y), and so $-F_y(e) = y$. Since y is arbitrary in L, we have $L \subseteq \mathcal{G}(Y)(e)$, so $\mathcal{G}(Y)(e)$ has non-empty interior in S_Y, and so the norm of Y is transitive (by Proposition 2.6). Therefore, the norm of H coincides with that of Y on $S(Y)$, hence both norms are equal, and Y is a Hilbert space, as required. ∎

The proof of the real case of Theorem 3.1 just given can be useful to illustrate that of the complex case. One of the key tools in the proof of such a complex case is the following (more complicated) variant of Lemma 3.2 above.

LEMMA 3.3. ([10, Lemma 2.4]) *Let H be a complex Hilbert space, α in $S_{\mathbb{C}} \setminus \{1\}$, and e in S_H. Let Ω denote the non-empty open subset of S_H given by*

$$\Omega := \left\{ y \in S_H : |(e|y)| > \left(\frac{\mathrm{Re}(\alpha)+1}{2} \right)^{1/2} \right\}.$$

Then there exist continuous functions γ and x from Ω to $S_{\mathbb{C}}$ and S_H, respectively, satisfying $x(e) = e$ and

$$e + (\alpha - 1)(e|x(y))x(y) = \gamma(y)y$$

for all y in Ω.

Sketch of proof of Theorem 3.1 *in the case* $\mathbb{K} = \mathbb{C}$. For each point e of isometric one-dimensional perturbation of the identity in the complex Banach space X, denote by V_e the set of those (uni-modular) complex numbers α such that there exists an isometric one-dimensional perturbation of the identity F having α as an eigen-value and satisfying $e \in (\mathbf{1} - F)(X)$. As a consequence of [10, Lemma 2.2], V_e is a closed subgroup of $S_{\mathbb{C}}$ different from $\{1\}$. This remark is useful to show that the set I of all points of isometric one-dimensional perturbation of the identity of X is closed in X [10, Lemma 2.3], and implies that $V_e = S_{\mathbb{C}}$ whenever there is some α in V_e whose argument is irrational modulo π. Put

$$Q := \{\beta \in S(\mathbb{C})\backslash\{1\} : \text{ the argument of } \beta \text{ is rational modulo } \pi\},$$

and for β in $S_{\mathbb{C}} \setminus \{1\}$ consider the closed subset of X given by

$$I_\beta := \{e \in I : \beta \in V_e\}.$$

It follows that $I = \cup_{\beta \in Q} I_\beta$. Now assume that there exists a non rare set in S_X contained in I. Then actually I has non-empty interior in S_X, so that Bair's theorem provides us with some α in Q such that I_α contains a non-empty open subset ω of S_X. Finally, applying Lemma 3.3 instead of Lemma 3.2, we can repeat with minor changes the proof of Theorem 3.1 in the case $\mathbb{K} = \mathbb{R}$ to obtain that X is a Hilbert space. ∎

Let X be a complex Banach space. It is easy to see that, with the notation in the above proof, the elements e in I satisfying $V_e = S_{\mathbb{C}}$ are nothing but those elements e in S_X such that $\mathbb{C}e$ is the range of a hermitian projection on X (compare Theorem 2.33). Now, for e in S_X, consider the following three conditions:

(a) $\mathbb{C}e$ is the range of a hermitian projection on X.

(b) e is an isometric reflection vector in X.

(c) e is a vector of isometric one-dimensional perturbation of the identity in X.

Then we have the chain of implications $(a) \Rightarrow (b) \Rightarrow (c)$, but no implication in such a chain is reversible (see for instance [10, p. 150]). We note that Theorem 3.1 contains the result of E. Berkson [15] that *a complex Banach space is a Hilbert space if (and only if), for every element e in S_X, $\mathbb{C}e$ is the range of a hermitian projection on X.*

Now let X be a Banach space over $\mathbb{K}$. A subspace M of X is said to be an *L^2-summand* of X if M is the range of a linear projection π satisfying

$$\|x\|^2 = \|\pi(x)\|^2 + \|x - \pi(x)\|^2$$

for every x in X. If e is in S_X and if $\mathbb{K}e$ is an L^2-summand of X, then e is an isometric reflection vector. Therefore the next corollary follows directly from Theorem 3.1.

Corollary 3.4. *A Banach space over $\mathbb{K}$ is a Hilbert space if (and only if) there exists a non rare subset of S_X consisting of elements e such that $\mathbb{K}e$ is an L^2-summand of X.*

The above corollary contains the actually elemental result in [27] that *a real Banach space is a Hilbert space if (and only if), for every element e in S_X, $\mathbb{R}e$ is an L^2-summand of X.*

Now we deal with the question of obtaining the appropriate version for real spaces of Kalton-Wood's Theorem 2.33 (see Corollary 3.8 below). Such a version will become an "almost" affirmative answer to the natural question if almost-transitivity can be relaxed to convex transitivity in Theorem 2.16. Given a nonzero Banach space X over $\mathbb{K}$, we denote by $\mathcal{G}_0 = \mathcal{G}_0(X)$ the connected component of the identity in $\mathcal{G}$ relative to the strong operator topology. We say that $\mathcal{G}_0$ is trivial if $\mathcal{G}_0 = \{\mathbf{1}\}$ (a situation that only can happen if $\mathbb{K} = \mathbb{R}$). Our argument relies on some of the main results of the Skorik-Zaidenberg paper [70], which are collected in the theorem and lemma which follow.

Theorem 3.5. ([70, Theorem 1]) *If e is an isometric reflection vector in a real Banach space X and if $\mathcal{G}_0(e) \neq \{e\}$, then the closed linear hull of $\mathcal{G}_0(e)$ (say H) is a Hilbert space, S_H coincides with $\mathcal{G}_0(e)$, and there exists a* *unique projection p from X onto H such that $\mathbf{1}-2p$ is an isometry. Moreover, putting $N := Ker(p)$, every element of $\mathcal{G}(H)$ can be extended to an element of $\mathcal{G}$ whose restriction to N is the identity, and every element of $\mathcal{G}$ leaving H invariant also leaves N invariant.*

The uniqueness of the projection $p : X \to H$ under the condition that $\mathbf{1} - 2p$ is an isometry is not explicitly stated in [70], but can be proved as follows. Let p, q be linear projections on a Banach space X with the same range and such that $\mathbf{1} - 2p$ and $\mathbf{1} - 2q$ are isometries. Then $T := 2(q - p)$ satisfies $T^2 = 0$ and $\mathbf{1} + T = (\mathbf{1} - 2p)(\mathbf{1} - 2q)$. It follows that, for every natural number n, $\mathbf{1} + nT = (\mathbf{1} + T)^n$ is an isometry, and therefore we have $T = 0$. As a consequence of the above fact, for an isometric reflection vector e in a Banach space X, there exists a unique element f in S_{X^*} such that the mapping $x \to x - 2f(x)e$ is an isometric reflection on X. Elements f as above will be called *isometric reflection functionals* on X.

We recall that a family $\mathcal{F}$ of linear functionals on a vector space X is said to be *total* if for every x in $X \setminus \{0\}$ there exists f in $\mathcal{F}$ satisfying $f(x) \neq 0$.

LEMMA 3.6. ([70, Lemma 5.4]) *Let X be a real Banach space having a total family of isometric reflection functionals. Let e_1, e_2 be vectors of isometric reflection vectors in X satisfying $\mathcal{G}_0(e_i) \neq \{e_i\}$ for $i = 1, 2$ and*

$$\mathcal{G}_0(e_1) \cap \mathcal{G}_0(e_2) = \emptyset.$$

Then the linear projections p_i from X onto $H_i := \overline{Lin}\mathcal{G}_0(e_i)$ such that $\mathbf{1} - 2p_i \in \mathcal{G}$ $(i = 1, 2)$ are orthogonal.

Now we can prove the following theorem.

THEOREM 3.7. *A real Banach space X of dimension ≥ 2 is a Hilbert space if (and only if) the following conditions are fulfilled:*

1. *$\mathcal{G}_0$ is non trivial.*
2. *The norm of X is strongly maximal.*
3. *X has no non trivial $\mathcal{G}$-invariant closed subspace.*
4. *There exists an isometric reflection vector e in X such that $co(\mathcal{G}(e))$ is non rare in X.*

Proof. Let X be a real Banach space satisfying Conditions 1 to 4 above. Conditions 1 and 3 imply that $\mathcal{G}_0(x) \neq \{x\}$ for every x in S_X [9, Lemma 3.1], whereas Condition 3 and the existence of isometric reflections on X assured by Condition 4 imply the existence of a total family of isometric reflection functionals on X [9, Lemma 3.3]. Put $\mathcal{B} := \{\mathcal{G}_0(x) : x \in \mathcal{G}(e)\}$. Since $\mathcal{G}_0$ is a group acting on $\mathcal{G}(e)$, $\mathcal{B}$ is a partition of $\mathcal{G}(e)$. By Theorem 3.5, for β in $\mathcal{B}$, $H_\beta := \overline{Lin}(\beta)$ is a Hilbert space, and there exists a unique linear projection p_β from X onto H_β such that $I - 2p_\beta$ is an isometry. Moreover, by Lemma 3.6, we have $p_\beta \circ p_\alpha = 0$ whenever α and β are in $\mathcal{B}$ with $\beta \neq \alpha$. On the other hand, the non rarity of $co(\mathcal{G}(e))$ in X assumed in Condition 4 provides us with a positive number δ satisfying $\overline{co}(\mathcal{G}(e)) \supseteq \delta B_X$. It follows that $X = \overline{\bigoplus_{\beta\in\mathcal{B}} H_\beta}$. If F is in $\mathcal{G}$, then $F(e)$ belongs to H_γ for some γ in $\mathcal{B}$ (take $\gamma := \mathcal{G}_0(F(e))$), and hence we have $\sum_{\beta\in\mathcal{B}}\|p_\beta(F(e))\| = \|F(e)\| = 1$. Now, for every finite subset Γ of $\mathcal{B}$, the set $\{x \in X : \sum_{\beta\in\Gamma}\|p_\beta(x)\| \leq 1\}$ is closed and convex, and contains $\mathcal{G}(e)$. Since $\overline{co}\{\mathcal{G}(e)\} \supseteq \delta B_X$, it follows that, for every x in X, the family $\{\|p_\beta(x)\|\}_{\beta\in\mathcal{B}}$ is sumable in $\mathbb{R}$ and the inequality $\delta\sum_{\beta\in\mathcal{B}}\|p_\beta(x)\| \leq \|x\|$ holds. Now, for x in X with $\|x\| \leq \delta$, the family $\{\|p_\beta(x)\|^2\}_{\beta\in\mathcal{B}}$ is sumable with $\sum_{\beta\in\mathcal{B}}\|p_\beta(x)\|^2 \leq 1$. Therefore, for x, y in X, the family $\{(p_\beta(x)|p_\beta(y))\}_{\beta\in\mathcal{B}}$ is sumable with $\delta^2|\sum_{\beta\in\mathcal{B}}(p_\beta(x)|p_\beta(y))| \leq \|x\|\|y\|$. In this way, the mapping $(x, y) \to \sum_{\beta\in\mathcal{B}}(p_\beta(x)|p_\beta(y))$ becomes a continuous inner product on X. But, actually, such an inner product is invariant because, for F in $\mathcal{G}$ and β in $\mathcal{B}$, we have $F(\beta) = \gamma$ for some γ in $\mathcal{B}$ (use that $\mathcal{G}_0$ is a normal subgroup of $\mathcal{G}$), and hence $F(H_\beta) = H_\gamma$. Finally, by Condition 2 and Proposition 2.39, X is a Hilbert space. ∎

Recalling the definition of convex transitivity, and the fact that convex transitivity implies strong maximality of norm, the next corollary follows straightforwardly from Theorem 3.7 above.

COROLLARY 3.8. ([9, Corollary 3.7]) *Let X be a convex transitive real Banach space having an isometric reflection and such that $\mathcal{G}_0$ is non trivial. Then X is a Hilbert space.*

Let X be a nonzero complex Banach space, and let us denote by $X_\mathbb{R}$ the real Banach space underlying X. We remark that $\mathcal{G}_0(X_\mathbb{R}) \supseteq \mathcal{G}_0(X)$, and therefore $\mathcal{G}_0(X_\mathbb{R})$ is automatically non trivial. Moreover convex transitivity of X implies that of $X_\mathbb{R}$. Keeping in mind these facts, Kalton-Wood's Theorem 2.33 follows from Corollary 3.8 above and the following proposition.

PROPOSITION 3.9. ([9, Proposition 4.2]) *Let X be a complex Banach space. Then there is a hermitian projection on X with one-dimensional range if and only if there is an isometric reflection on $X_\mathbb{R}$. More precisely, for e*

in S_X, $\mathbb{C}e$ is the range of a hermitian projection on X if and only if e is an isometric reflection vector in $X_{\mathbb{R}}$.

In fact, partially, the proof of Theorem 3.7 is strongly inspired in that of Kalton-Wood's Theorem 2.33. With a similar philosophy, we proved in [9] the following variant of Theorem 3.7.

THEOREM 3.10. ([9, Theorem 3.4]) *Let X be a real Banach space. Assume that there is an isometric reflection on X, that $\mathcal{G}_0$ is non trivial, and that the exists $\delta > 0$ such that $\overline{co}\{\mathcal{G}(x)\} \supseteq \delta B_X$ for every x in S_X. Then X is isomorphic to a Hilbert space. More precisely, there exists a natural number $n \leq \delta^{-2}$, together with pair-wise isomorphic Hilbertian subspaces $H_1, H_2, \ldots, H_n$ of X, satisfying:*

1. *$X = \bigoplus_{i=1}^{n} H_i$.*
2. *For $i = 1, 2, \ldots, n$ and F in $\mathcal{G}$, there is $j = 1, 2, \ldots, n$ with $F(H_i) = H_j$.*
3. *For $i = 1, 2, \ldots, n$, H_i is $\mathcal{G}_0$-invariant.*

Now, *let X be a real Banach space having an isometric reflection and such that $\mathcal{G}_0$ is non trivial.* If follows from Theorem 3.10 that, *if there exists $\delta > 1/\sqrt{2}$ such that $\delta B_X \subseteq \overline{co}(\mathcal{G}(x))$ for every x in S_X, then X is a Hilbert space* [9, Corollary 3.5]. Moreover, the constant $1/\sqrt{2}$ above is sharp [9, Proposition 3.10], and the assumption that $\mathcal{G}_0$ is non trivial cannot be removed (see [9] for details). With some additional effort, it can be also derived from Theorem 3.10 that, *if the norm of X is maximal and there is $\delta > 0$ such that $\delta B_X \subseteq \overline{co}(\mathcal{G}(x))$ for every x in S_X, then X is a Hilbert space* [9, Corollary 3.6].

Either from [9, Corollary 3.5] or [9, Corollary 3.6], just reviewed, Corollary 3.8 follows again. We do not know if the assumption in [9, Corollary 3.6] and Corollary 3.8 that $\mathcal{G}_0$ is non trivial can be removed.

Thanks to Proposition 3.9 and the comments preceding it, the following result follows from Theorem 3.10.

COROLLARY 3.11. ([9, Corollary 4.4]) *Let X be a complex Banach space. Assume that there exists a hermitian projection on X with one-dimensional range, and that there is $\delta > 0$ such that $\overline{co}\{\mathcal{G}(x)\} \supseteq \delta B_X$ for every x in S_X. Then X is isomorphic to a Hilbert space. More precisely, there exists a natural number $n \leq \delta^{-2}$, together with pair-wise isomorphic Hilbertian subspaces $H_1, H_2, \ldots, H_n$ of X, satisfying:*

1. $X = \bigoplus_{i=1}^{n} H_i$.
2. *For* $i = 1, 2, \ldots, n$ *and* F *in* $\mathcal{G}$, *there is* $j = 1, 2, \ldots, n$ *with* $F(H_i) = H_j$.
3. *For* $i = 1, 2, \ldots, n$, H_i *is* $\mathcal{G}_0$*-invariant.*

Let X *be a complex Banach space having a hermitian projection with one-dimensional range.* It can be derived from the above corollary that, *if either there exists* $\delta > 1/\sqrt{2}$ *such that* $\delta B_X \subseteq \overline{co}(\mathcal{G}(x))$ *for every* x *in* S_X, *or there is* $\delta > 0$ *such that* $\delta B_X \subseteq \overline{co}(\mathcal{G}(x))$ *for every* x *in* S_X *and the norm of* X *is maximal, then* X *is a Hilbert space* [9, Corollary 4.5].

To conclude this section, let us mention that, from [9, Corollary 4.5] just reviewed, we re-encounter again Kalton-Wood's Theorem 2.33.

4. Multiplicative characterizations of Hilbert spaces

The geometry of norm-unital (possibly non associative) Banach algebras at their units is very peculiar (see for instance [17], [18], and [64]). Therefore it must be expected that Banach spaces possessing "many" points in their unit spheres enjoying such a peculiarity (for instance, almost transitive norm-unital Banach algebras) have to be very "special". We devote the present section to this topic.

Since most properties of the geometry of a norm-unital Banach algebra at its unit are inherited by subspaces containing the unit, we in fact consider the following chain of conditions on a norm-one element e of a Banach space X over $\mathbb{K}$:

(α) There exists a norm-one bounded bilinear mapping $f : X \times X \to X$ satisfying $f(e, x) = f(x, e) = x$ for every x in X.

(β) $m(X, e) = 1$, where $m(X, e)$ means the infimum of the set of numbers of the form $\|f\|$ when f runs over the set of all bounded bilinear mappings $f : X \times X \to X$ satisfying $f(e, x) = f(x, e) = x$ for every x in X.

(γ) $sm(X, e) = 1$, where $sm(X, e)$ stands for the infimum of the set of numbers of the form $\max\{\|f\|, 1 + \|L_e^f - \mathbf{1}\|, 1 + \|R_e^f - \mathbf{1}\|\}$ when f runs over the set of all bounded bilinear mappings from $X \times X$ to X. (Here, for u in X, L_u^f and R_u^f denote the operators on X given by $x \mapsto f(u, x)$ and $x \mapsto f(x, u)$, respectively.)

(δ) There exists a Banach space Y over $\mathbb{K}$ containing X isometrically, together with a norm-one bounded bilinear mapping $f : Y \times Y \to Y$, such that the equality $f(e, y) = f(y, e) = y$ holds for every y in Y.

We actually have $(\alpha) \Rightarrow (\beta) \Rightarrow (\gamma) \Rightarrow (\delta)$ (for the last implication see the proof of [64, Proposition 2.4]).

Every norm-one element e in a real Hilbert space X satisfies Condition α above (with $f(x,y) := (x|e)y + (y|e)x - (x|y)e$) [64, Observation 1.3]. Conversely, if a real Banach space X is smooth at some norm-one element e satisfying Condition γ, then X is a Hilbert space [64, Theorem 2.5]. However, the fact just reviewed does not remains true if Condition γ is relaxed to Condition δ (see [9]).

Let us say that an element e of a Banach space over $\mathbb{K}$ *acts as a unit* on X if e belongs to S_X and fulfills Condition δ. It follows from the above comments that a characterization of real Hilbert spaces in terms of elements which act as units seems to require some kind of "abundance" of such points. To achieve such a characterization, we argued in [9] as follows. Let X be a real Hilbert space having a non rare subset of S_X consisting of elements acting as units on X. Since the set of all elements of X which act as units is closed in X [9, Lemma 2.3], we are provided in fact with a nonempty open subset ω of S_X consisting of elements which act as units. On the other hand, the theory of the geometry of norm-unital Banach algebras at their units gives easily that $\mathbb{R}e$ is an L^2-summand of X whenever e is in ω and X is smooth at e [9, Lemma 2.4], and that the norm of X is uniformly strongly subdifferentiable on ω (in the sense of [37]). By means of an adaptation of the proof of [37, Proposition 4.1], this last fact shows that X is smooth at every point of ω, and hence $\mathbb{R}e$ is an L^2-summand of X whenever e is in ω. In this way, it is enough to apply Corollary 3.4 to obtain the following result.

Theorem 4.1. ([9, Theorem 2.5]) *A real Banach space X is a Hilbert space if and only if there is a non rare subset of S_X consisting of elements which act as units on X.*

It follows from the above theorem that *if X is an almost transitive closed subspace of a norm-unital real Banach algebra A containing the unit of A, then X is a Hilbert space* [9, Corollary 2.6]. We note that, in the result just reviewed, almost-transitivity cannot be relaxed to convex transitivity (compare Example 2.32). Now let A be an almost transitive norm-unital real Banach algebra. Then, since the norm of A derives from an inner product $(\cdot|\cdot)$, the structure of A is given by [61, Theorem 27] (see also [62, Section 2]). As a consequence, for x, y in A, we have

$$\frac{1}{2}(xy + yx) = (x|e)y + (y|e)x - (x|y)e,$$

where e denotes the unit of A. Therefore A is a quadratic algebra, and every element x of A has an inverse in the (associative) subalgebra of A generated by x. It follows from the Frobenius-Zorn theorem [34, p. 262] that, if A is alternative, then A is equal to either $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ (the absolute-valued algebra of Hamiltons's quaternions), or $\mathbb{O}$ (the absolute-valued algebra of Cayley's octonions). This extends results of L. Ingelstam in [45] and [46].

Now, let X be a complex Banach space. Because of the non-associative Bohnenblust-Karlin theorem ([16], [64, Theorem 1.5]), if X is smooth at some element e acting as a unit on X, then $X = \mathbb{C}e$. Moreover, if there is a non rare subset of X consisting of elements which act as units, then we arrive again in the equality $X = \mathbb{C}e$ (see [9] or [7, Theorem III.2.2] for independent proofs). Despite the above facts, as the following example shows, elements acting as units on X need not fulfill Condition γ above. Let us fix e in S_X. For x in X, the *numerical range*, $V(X, e, x)$, of x relative to (X, e) is defined by the equality

$$V(X, e, x) := \{f(x) : f \in S_{X^*},\ f(e) = 1\}.$$

The *numerical index* of X relative to e, $n(X, e)$, is given by

$$n(X, e) := \mathrm{Inf}\{\mathrm{Sup}\{|\lambda| : \lambda \in V(X, e, x)\} : x \in S_X\}.$$

We put $H(X, e) := \{x \in X : V(X, e, x) \subseteq \mathbb{R}\}$.

Example 4.2. First we show the existence of $*$-invariant closed subspaces X of $C^{\mathbb{C}}(K)$, for a suitable choice of the Hausdorff compact topological space K, which contain the unit e of $C^{\mathbb{C}}(K)$ and such that, regarded as a pointed Banach space, (X, e) cannot be of the form $(C^{\mathbb{C}}(K'), e)$, for any Hausdorff compact K'. Such a pathology occurs for instance for the subspace

$$X := Lin\{(1, 1, 1, 1), (1, -1, 0, 0), (0, 0, 1, -1)\}$$

of the complex space ℓ_∞^4. Indeed, if (X, e) were of the form $(C^{\mathbb{C}}(K'), e)$, then we would have $(X, (1, 1, 1, 1)) = (\ell_\infty^3, (1, 1, 1))$ as pointed Banach spaces. But this is not possible because $V(X, (1, 1, 1, 1), (1, -1, i, -i))$ is a square, whereas $V(\ell_\infty^3, (1, 1, 1), y)$ is a (possibly degenerated) triangle for every y in ℓ_∞^3.

Now let K be a Hausdorff compact topological space such that $C^{\mathbb{C}}(K)$ contains a closed subspace X with the pathology explained above. Clearly e acts as a unit in X. Since X is $*$-invariant in $C^{\mathbb{C}}(K)$, we have $X = H(X, e) + iH(X, e)$. On the other hand, since $n(C^{\mathbb{C}}(K), e) = 1$, we have also $n(X, e) = 1$. Then the equality $sm(X, e) = 1$ is not possible since, otherwise, it would follow from [64, Corollary 4.6] that (X, e) would be of the form $(C^{\mathbb{C}}(K'), e)$ for some Hausdorff compact topological space K'.

We already know that a complex Hilbert space of dimension ≥ 2 has no element acting as a unit. In order to provide us with a multiplicative characterization of Hilbert spaces covering the complex ones, we introduce the following weakening of the concept of element acting as a unit. We say that an element e of a Banach space X over $\mathbb{K}$ *acts weakly as a unit* on X whenever e belongs to S_X and fulfills Condition ϵ which follows:

(ϵ) There exists a Banach space Y over $\mathbb{K}$ containing X isometrically, together with a norm-one bounded bilinear mapping $f : X \times X \to Y$, such that the equality $f(e, x) = f(x, e) = x$ holds for every x in X.

LEMMA 4.3. *For a norm-one element e of a Banach space X over $\mathbb{K}$, the following assertions are equivalent:*

1. *e acts weakly as a unit on X.*
2. *The equality $\|e \otimes x + x \otimes e\|_\pi = 2\|x\|$ holds for every x in X, where $\|.\|_\pi$ means projective tensor norm.*
3. *There exists a Banach space Y over $\mathbb{K}$, together with a norm-one bounded symmetric bilinear mapping $f : X \times X \to Y$, such that the equality $\|f(e, x)\| = \|x\|$ holds for every x in X.*

Proof. $1 \Rightarrow 2$. Let Y be a Banach space over $\mathbb{K}$ containing X isometrically and such that there exists a norm-one bounded bilinear mapping $f : X \times X \to Y$ satisfying $f(e, x) = f(x, e) = x$ for every x in X. Then there is a norm-one bounded linear mapping h from the projective tensor product $X \otimes_\pi X$ into Y such that $f(x_1, x_2) = h(x_1 \otimes x_2)$ for all x_1, x_2 in X. Therefore, for x in X we obtain

$$2\|x\| = \|f(e, x) + f(x, e)\| = \|h(e \otimes x + x \otimes e)\| \leq \|e \otimes x + x \otimes e\|_\pi.$$

$2 \Rightarrow 3$. If 2 is true, then 3 follows with Y equal to the complete projective tensor product $X \hat{\otimes}_\pi X$, and $f(x_1, x_2) := \frac{1}{2}(x_1 \otimes x_2 + x_2 \otimes x_1)$.

$3 \Rightarrow 1$. Assume that 3 holds. Then the mapping $x \to \hat{x} := f(x, e)$ from X to Y is a linear isometry, and the mapping $\hat{f} : \hat{X} \times \hat{X} \to Y$ defined by $\hat{f}(\hat{x_1}, \hat{x_2}) := f(x_1, x_2)$ is bilinear and bounded with $\|\hat{f}\| = 1$, and satisfies $\hat{f}(\hat{e}, \hat{x}) = \hat{f}(\hat{x}, \hat{e}) = \hat{x}$ for every $\hat{x}$ in $\hat{X}$. ∎

Now we can prove the announced multiplicative characterization of real or complex Hilbert spaces.

Theorem 4.4. *For a Banach space X over $\mathbb{K}$, the following assertions are equivalent:*

1. *X is a Hilbert space.*
2. *Every element of S_X acts weakly as a unit on X.*
3. *There is a dense subset of S_X consisting of elements which act weakly as unit on X.*
4. *The equality $\|x_1 \otimes x_2 + x_2 \otimes x_1\|_\pi = 2\|x_1\| \, \|x_2\|$ holds for all x_1, x_2 in X, where $\|.\|_\pi$ means projective tensor norm.*
5. *There exists a Banach space Y over $\mathbb{K}$, together with a symmetric bilinear mapping $f : X \times X \to Y$, such that the equality $\|f(x_1, x_2)\| = \|x_1\| \, \|x_2\|$ holds for all x_1, x_2 in X.*

Proof. Keeping in mind Lemma 4.3, in the chain of implications $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5 \Rightarrow 1$ only the first and last ones merit a proof.

$1 \Rightarrow 2$. Assume that 1 holds. If $\mathbb{K} = \mathbb{R}$, then we know that every element of S_X actually fulfills Condition α in the top of this section, and hence 2 is true. Assume additionally that $\mathbb{K} = \mathbb{C}$. Take a conjugation (i.e., a conjugate-linear involutive isometry) $x \to \overline{x}$ on the complex Hilbert space X, for x, z in X denote by $x \odot z$ the operator on X defined by $(x \odot z)(t) := (t|z)x$, write $(Y, \|.\|_\tau)$ for the complex Banach space of all trace class operators on X [67], and consider the mapping $f : X \times X \to Y$ given by $f(x, z) := \frac{1}{2}(x \odot \overline{z} + z \odot \overline{x})$. In view of the implication $3 \Rightarrow 1$ of Lemma 4.3, to realize that Assertion 2 in the theorem is true it is enough to show that the equality $\|f(x, z)\|_\tau = 1$ holds for every x, z in S_X. Let x, z be linearly independent elements of S_X. Let T stand for the two-dimensional operator $x \odot \overline{z} + z \odot \overline{x}$. Then we have

$$TT^* = \|y\|^2 x \odot x + (z|x) x \odot z + (x|z) z \odot x + \|x\|^2 z \odot z,$$

and hence the eigen-values of TT^* are $(1 \pm |(x|z)|)^2$. Since $\|T\|_\tau$ is nothing but the sum of the eigen-values of $(TT^*)^{\frac{1}{2}}$, we obtain $\|T\|_\tau = 2$.

$5 \Rightarrow 1$. Assume that 5 holds. Then, for x, z in S_X, we have

$$4 = 4\|f(x, z)\| = \|f(x + z, x + z) - f(x - z, x - z)\| \leq \|x + z\|^2 + \|x - z\|^2.$$

Therefore, by Schoenberg's theorem [69], X is a Hilbert space. ∎

We do not know if Theorem 4.4 remains true with Assertion 3 replaced by the one that there exists a non rare subset of S_X consisting of elements which act weakly as units. To conclude this section, we note that, if a Banach space X over $\mathbb{K}$ satisfies Condition 5 in Theorem 4.4 with $Y = X$, then, by the commutative Urbanik-Wright theorem [73, Theorem 3], X has dimension ≤ 2 over $\mathbb{R}$.

5. Transitivity of Banach spaces having a Jordan structure

As we have seen in Section 2, a big part of the literature dealing with transitivity conditions of the norm centers its attention in the study of such conditions on the Banach spaces of the form $C_0^{\mathbb{K}}(L)$ for some locally compact Hausdorff topological space L, and $L_1^{\mathbb{K}}(\Gamma, \mu)$ for some localizable measure space (Γ, μ). Today such classical Banach spaces have a wider understanding in the setting of C^*-algebras (or even their non associative generalizations, the JB^*-triples) and JB-algebras. Indeed, the $C_0^{\mathbb{C}}(L)$-spaces are nothing but the commutative C^*-algebras, and the $L_1^{\mathbb{C}}(\Gamma, \mu)$-spaces are precisely the preduals of commutative W^*-algebras. Analogously, the $C_0^{\mathbb{R}}(L)$-spaces and the $L_1^{\mathbb{R}}(\Gamma, \mu)$-spaces coincide with the associative JB-algebras and the preduals of associative JBW-algebras, respectively.

Motivated by the ideas in the above comment, we studied in [11] transitivity conditions on the norm of JB^*-triples, JB-algebras, and their preduals. Sometimes, in this wider setting, questions and results attain a better formulation. For instance, Wood's conjecture (Conjecture 2.20) becomes a particular case of the more ambitious one that complex Hilbert spaces are the unique almost transitive JB^*-triples (see Problem 5.1 below), and Theorem 2.11 follows from the more general fact that $\mathbb{R}$ is the unique almost transitive JB-algebra (see Corollary 5.20 below). All material we are dealt with in the present section will flow between Problem 5.1 and Corollary 5.20 just mentioned.

We recall that a complex Banach space X is said to be a *JB^*-triple* if it is equipped with a continuous triple product $\{\cdots\}$ which is conjugate-linear in the middle variable, linear and symmetric in the outer variables, and satisfies the following two conditions:

1. $D(a,b)D(x,y) - D(x,y)D(a,b) = D(D(a,b)(x),y) - D(x,D(b,a)(y))$ for all a,b,x,y in X, where the operator $D(a,b) : X \to X$ is defined by $D(a,b)(x) := \{abx\}$ for all x in X.
2. For every x in X, $D(x,x)$ is hermitian with non negative spectrum and satisfies $\| D(x,x) \| = \| x \|^2$.

JB^*-triples, introduced by W. Kaup [49], are of capital importance in complex Analysis because their open unit balls are bounded symmetric domains, and every bounded symmetric domain in a complex Banach space is biholomorphically equivalent to the open unit ball of a suitable JB^*-triple [50]. Every complex Hilbert space is a JB^*-triple under the triple product defined by $\{xyz\} := \frac{1}{2}((x \mid y)z + (z \mid y)x)$. Now, it seems reasonable to raise the following problem.

PROBLEM 5.1. If X is an almost transitive JB^*-triple, is X a Hilbert space?

Let $\mathcal{J}$ denote the category of JB^*-triples (with morphisms equal to those linear mappings which preserve triple products). It is known that $\mathcal{J}$ is closed under ultraproducts [32], and it is easily seen that it is an admissible subcategory of the category of Banach spaces. It follows from Proposition 2.18 and Cabello's Theorem 2.24 that, if the answer to Problem 5.1 is negative, then in fact there exist transitive non Hilbert JB^*-triples, as well as almost transitive separable non Hilbert JB^*-triples.

A *JBW^*-triple* is a JB^*-triple having a (complete) predual. Such a predual is unique [6] in the strongest sense of the word: two preduals of a JBW^*-triple X coincide when they are canonically regarded as subspaces of the dual X^* of X. JBW^*-triples are very abundant: the bidual of every JB^*-triple X is a JBW^*-triple under a suitable triple product which extends the one of X [32]. The fact that every complex Hilbert space is the predual of a JBW^*-triple could invite us to consider the question whether all transitive preduals of JBW^*-triples are Hilbert spaces. Contrarily to what happens in relation to Problem 5.1 (which, as far as we know, remains unanswered), it is known that, without additional assumptions, the answer to the question just raised can be negative. To explain our assertion by an example, let us recall that every C^*-algebra is a JB^*-triple under the triple product $\{xyz\} := \frac{1}{2}(xy^*z + zy^*x)$, and therefore $C_0^{\mathbb{C}}(L)$-spaces and $L_\infty^{\mathbb{C}}(\Gamma, \mu)$-spaces are JB^*-triples and JBW^*-triples, respectively. Then it is enough to invoke Example 2.3 (respectively, 2.13) to provide us with a transitive (respectively, almost transitive separable) non Hilbert predual of a JBW^*-triple.

The following lemma (which can be verified arguing as in the proof of [63, Lemma 1]) becomes a common tool to provide partial affirmative answers to both Problem 5.1 and the question above raised whether transitive preduals of JBW^*-triples are Hilbert spaces.

Lemma 5.2. ([11, Lemma 2.3]) *Let X be a JB^*-triple such that for all x in X the equality $\{xxx\} = \| x \|^2 x$ holds. Then X is a Hilbert space.*

With the aide of the above lemma and some basic results in the theory of JB^*-triples taken from [49] and [38], we obtained in [11] several characterizations of complex Hilbert spaces, which are collected in the next proposition. We recall that an element e in a JB^*-triple is called a *tripotent* if the equality $\{eee\} = e$ holds.

Proposition 5.3. *For a complex Banach space X the following assertions are equivalent:*

1. *X is a Hilbert space.*
2. *X is a smooth JB^*-triple.*
3. *X is a smooth predual of a JBW^*-triple.*
4. *X is an almost transitive JB^*-triple and has a non-zero tripotent.*
5. *X is an almost transitive JBW^*-triple.*

We note that the implication $2 \Rightarrow 1$ in the above proposition is nothing but Tarasov's theorem [71] already quoted in Section 2. It is also worth mentioning that, as a bi-product of the implication $3 \Rightarrow 1$ in Proposition 5.3, we obtain that *transitive separable preduals of JBW^*-triples are Hilbert spaces* [11, Corollary 2.5], thus answering affirmatively Problem 2.2 in the class of preduals of JBW^*-triples.

Now we are dealing with deeper characterizations of complex Hilbert spaces in terms of JB^*-triples. Let Y be a JBW^*-triple Y (with predual denoted by Y_*). Each element x in Y_* has an associated tripotent $s(x)$ in Y, called the "support" of x, which is determined by rather technical conditions (see [38, p. 75] for details). In any case, since the definition of the support depends only on the norm and the triple product, and surjective linear isometries between JB^*-triples preserve triple products [49], for x in Y_* and F in $\mathcal{G}(Y_*)$ we have $s(F(x)) = (F^*)^{-1}(s(x))$. We recall that Y is said to be *atomic* (respectively, *purely atomic*) if B_{Y_*} has some extreme point (respectively, Y_* is the closed linear hull of the set of all extreme points of B_{Y_*}).

Theorem 5.4. *Let X be the predual of a purely atomic JBW^*-triple. If the norm of X is strongly maximal, then X is a Hilbert space.*

Proof. By [38, Lemma 2.11], there exists a contractive conjugate-linear mapping $\pi : X \to X^*$ sending each extreme point e of B_X to its support $s(e)$.

Moreover, the mapping $(x, y) \to (x|y) := \pi(y)(x)$ from $X \times X$ to $\mathbb{C}$ becomes a (continuous) inner product on X [5, Pag. 270]. On the other hand, for F in $\mathcal{G}$, the equality $s(F(x)) = (F^*)^{-1}(s(x))$ $(x \in X)$ pointed out above leads to $\pi \circ F = (F^*)^{-1} \circ \pi$. Therefore $(\cdot|\cdot)$ actually is an invariant inner product on X. Finally, if the norm of X is strongly maximal, then, by Proposition 2.39, X is a Hilbert space.

Recalling that convex transitivity implies strong maximality of the norm, we straightforwardly deduce from Theorem 5.4 the following corollary.

COROLLARY 5.5. ([11, Theorem 3.1]) *Let X be a convex–transitive predual of an atomic JBW^*-triple. Then X is a Hilbert space.*

The characterizations of complex Hilbert spaces collected in the next proposition are more or less difficult consequences of Corollary 5.5 above. For their proofs the reader is referred to [11, Theorem 3.2 and Corollaries 3.4, 3.5, and 3.9].

PROPOSITION 5.6. *For a complex Banach space X the following assertions are equivalent:*

1. *X is a Hilbert space.*
2. *X is the predual of a JBW^*-triple and there is some non rare set in S_X consisting only of extreme points of B_X.*
3. *X is a convex-transitive atomic JBW^*-triple.*
4. *X is smooth and X^{**} is a JB^*-triple.*
5. *X is transitive, X^{**} is a JB^*-triple, and every element in $B_{X^{**}}$ is the w^*-limit of a sequence of elements of B_X.*

The implication $4 \Rightarrow 1$ in the above proposition refines Tarasov's theorem [71], and provides us with a new partial affirmative answer to the Banach-Mazur rotation problem, namely *transitive separable complex Banach spaces whose biduals are JB^*-triples are in fact Hilbert spaces* [11, Corollary 3.6]. This generalizes Corollary 2.5 because the class of complex Banach spaces whose biduals are JB^*-triples is strictly larger than that of JB^*-triples [11, Example 3.10].

Now we prove some new characterizations of complex Hilbert spaces among JB^*-triples in terms of some kind of transitivity. Let X be a Banach space over $\mathbb{K}$, and u an element in X. We say that u is a *big point* of X if $u \in S_X$ and $\overline{co}(\mathcal{G}(u)) = B_X$. When u lies in S_X, we consider the set $D(X, u)$ of *states* of X relative to u, given by $D(X, u) := \{f \in B_{X^*} : f(u) = 1\}$.

LEMMA 5.7. *Let X be a Banach space, and u an element of S_X. Then the following assertions are equivalent:*

1. *u is a big point of X.*
2. *For every positive number δ, the set*

$$\Delta_\delta(u) := \{T^*(f) : f \in D(X,x), x \in S_X, \|x-u\| \leq \delta, T \in \mathcal{G}\}$$

is dense in S_{X^*}.

Proof. 1 ⇒ 2. We fix $\delta > 0$ and $g \in S_{X^*}$, and take $0 < \varepsilon < 1$. By the assumption 1, there exists T in $\mathcal{G}$ such that $|g(T^{-1}(u)) - 1| < \frac{\varepsilon'^2}{4}$, where $\varepsilon' := \min\{\varepsilon, \delta\}$. By the Bishop-Phelps-Bollobás theorem [18, Theorem 16.1], there is x in S_X and f in $D(X,x)$ satisfying $\|u-x\| < \varepsilon' \leq \delta$ and $\|g \circ T^{-1} - f\| < \varepsilon' \leq \varepsilon$. This shows that $g \in \overline{\Delta_\delta(u)}$.

2 ⇒ 1. Assume that 1 is not true, so that there is z in $B_X \setminus \overline{co}(\mathcal{G}(u))$. Then, by the Hahn-Banach theorem, there exists f in S_{X^*} such that $1 \geq f(z) > \sup\{f(a) : a \in \overline{co}(\mathcal{G}(u))\}$. By the assumption 2, for n in $\mathbb{N}$, the set $\Delta_{\frac{1}{n}}(u)$ is dense in S_{X^*}, and hence there are x_n in S_X with $\|u - x_n\| \leq 1/n$, g_n in $D(X, x_n)$, and T_n in $\mathcal{G}$ such that $\|f - g_n \circ T_n\| \leq 1/n$. In this way we obtain

$$|f(T_n^{-1}(u)) - 1| \leq |f(T_n^{-1}(u)) - g_n(u)| + |g_n(u) - g_n(x_n)| \leq \frac{2}{n},$$

which implies $\sup\{f(a) : a \in \mathcal{G}(u)\} = 1$, contrarily to the choice of f. ∎

Let X be a Banach space over $\mathbb{K}$. An element f in X^* is said to be a *w^*-superbig point* of X^* if f belongs to S_{X^*} and the convex hull of $\{F^*(f) : F \in \mathcal{G}\}$ is w^*-dense in B_{X^*}. Minor changes in the proof of Lemma 5.7 allows us to establish the following result.

LEMMA 5.8. *Let X be a Banach space, and f an element of S_{X^*}. Then the following assertions are equivalent:*

1. *f is a w^*-superbig point of X^*.*
2. *For every positive number δ, the set*

$$\Delta^*_\delta(f) := \{T(x) : x \in D(X^*, g) \cap X, g \in S_{X^*}, \|f - g\| \leq \delta, T \in \mathcal{G}\}$$

is dense in S_X.

Since by Definition 2.26 (respectively, by Proposition 2.28), a Banach space X is convex-transitive if and only if every element of S_X is a big point of X (respectively, every element of S_{X^*} is a w^*-superbig point of X^*), the following corollary follows from Lemma 5.7 (respectively, 5.8).

COROLLARY 5.9. *For a Banach space X the following assertions are equivalent:*

1. *X is convex-transitive.*
2. *For every u in S_X and every $\delta > 0$, the set $\Delta_\delta(u)$ is dense in S_{X^*}.*
3. *For every f in S_{X^*} and every $\delta > 0$, the set $\Delta_\delta^*(f)$ is dense in S_X.*

Concerning our present interest of obtaining new characterizations of complex Hilbert spaces among JB^*-triples, the above corollary is not needed. By the contrary, the next result becomes crucial. Given a Banach space X, we denote by $Sm(X)$ the set of all elements $x \in S_X$ such that X is smooth at x.

COROLLARY 5.10. *For a Banach space X we have:*

1. *If there is some big point of X in the interior of $Sm(X)$ relative to S_X, then the set of all extreme points of B_{X^*} is dense in S_{X^*}.*
2. *If there is some w^*-superbig point of X^* in the interior of $Sm(X^*)$ relative to S_{X^*}, then the set of all extreme points of B_X is dense in S_X.*

Proof. Assertion 1 (respectively, 2) follows from Lemma 5.7 (respectively, 5.8) and the standard fact that the unique state relative to a smooth point u of a Banach space E is an extreme point of B_{E^*}. ∎

PROPOSITION 5.11. *Let X be a JB^*-triple. Then X is a Hilbert space whenever some of the following conditions are fulfilled:*

1. *There is some big point of X in the interior of $Sm(X)$ relative to S_X.*
2. *There is some w^*-superbig point of X^* in the interior of $Sm(X^*)$ relative to S_{X^*}.*

Proof. Assume that Condition 1 is fulfilled. Then, by Assertion 1 in Corollary 5.10 and the implication $2 \Rightarrow 1$ in Proposition 5.6, X is a Hilbert space. Now assume that Condition 2 is satisfied. Then, since the extreme points of B_X are tripotents [51, Proposition 3.5], Assertion 2 in Corollary 5.10 and Lemma 5.2 apply, giving that X is a Hilbert space. ∎

Now we pass to deal with transitivity conditions on the norm of C^*-algebras. First, we notice that all results for JB^*-triples collected above automatically get a stronger form when they are applied to C^*-algebras. The reason lies in the folklore fact that $\mathbb{C}$ is the unique C^*-algebra whose C^*-norm derives from an inner product. (Indeed, from the continuous functional calculus for a single self-adjoint element of a C^*-algebra, it follows that, if X is a smooth C^*-algebra, then every norm-one element e in the self-adjoint part X_{sa} of X satisfies either $e^2 = e$ or $e^2 = -e$, which implies that $S(X_{sa})$ is disconnected, and hence the real Banach space X_{sa} is one-dimensional.) By the folklore result just mentioned, an affirmative answer to Problem 5.1 would imply the verification of Wood's conjecture (Conjecture 2.20). Actually, if Problem 5.1 had an affirmative answer, then the natural conjecture that $\mathbb{C}$ is the unique (non necessarily commutative) almost transitive C^*-algebra would be right. We note also that the subcategory of Banach spaces consisting of C^*-algebras is admissible and closed under ultraproducts. Let us say that a C^*-algebra is proper whenever it is different from $\mathbb{C}$. It follows from Proposition 2.18 and Theorem 2.24 that, if there is some almost transitive proper C^*-algebra, then actually there exist transitive proper C^*-algebras, as well as almost transitive separable proper C^*-algebras. Accordingly to previously reviewed results on JB^*-triples, a transitive proper C^*-algebra must be non separable, and an almost transitive proper C^*-algebra cannot have non-zero self-adjoint idempotents. For background about C^*-algebras the reader is referred to [33] and [59].

Let X be a C^*-algebra, and let $M(X)$ denote the C^*-algebra of multipliers of X. The so called Jordan $*$-automorphisms of X, as well as the operators of left multiplication on X by unitary elements in $M(X)$, become distinguished examples of surjective linear isometries on X. *Jordan $*$-automorphisms* of X are defined as those linear bijections from X to X preserving the C^*-involution and the squares. Consequently, if $Pos(X)$ denotes the set of all positive elements in X, and if F is a Jordan $*$-automorphism of X, then we have $F(S_X \cap Pos(X)) = S_X \cap Pos(X)$. Let us denote by U_X the set of all unitary elements of $M(X)$, and by $\mathcal{G}^+$ the group of all Jordan $*$-automorphisms of X. The Kadison-Paterson-Sinclair [58] theorem asserts that every surjective linear isometry on X is the composition of an element of $\mathcal{G}^+$ with the operator of left multiplication by an element of U_X. The modulus $\mid x \mid$ of an element x of X is defined as the unique positive square root of x^*x. The next theorem characterizes transitive C^*-algebras in purely algebraic terms, and becomes the non-commutative generalization of Greim-Rajalopagan Theorem 2.4.

THEOREM 5.12. ([11, Proposition 4.1]) *Let X be a C^*-algebra. Then the following assertions are equivalent:*

1. *X is transitive.*
2. *$\mathcal{G}^+$ acts transitively on $S_X \cap Pos(X)$, and every element x in X has a "polar decomposition" $x = u \mid x \mid$, where u is in U_X.*

In [11] we also obtained a characterization of convex-transitive C^*-algebras, which generalizes Wood's Theorem 2.29. The precise formulation of such a characterization reads as follows.

THEOREM 5.13. ([11, Theorem 4.3]) *Let X be a C^*-algebra. Then X is convex-transitive if and only if, for every pure state g of X and every norm-one positive linear functional f on X, g belongs to the w^*-closure in X^* of the set $\{F^*(f) : F \in \mathcal{G}^+\}$.*

Let H be a complex Hilbert space. We denote by $L(H)$ the C^*-algebra of all bounded linear operators on H, and by $K(H)$ the closed ideal of $L(H)$ consisting of all compact operators on H. For x in $L(H)$, we put $\| x \|_{ess} := \| x + K(H) \|$. The next result determines the big points of $L(H)$ in the case that H is infinite-dimensional and separable.

THEOREM 5.14. ([11, Theorem 4.5]) *Let H be an infinite-dimensional separable complex Hilbert space. Then the big points of $L(H)$ are precisely those elements x in $S_{L(H)}$ satisfying $\| x \|_{ess} = 1$.*

We recall that the Calkin algebra [26] is defined as the quotient $L(H)/K(H)$, where H is an infinite-dimensional separable complex Hilbert space. The following corollary is an easy consequence of Theorem 5.14 above, and provides us with the first known example of a convex-transitive non commutative C*-algebra.

COROLLARY 5.15. ([11, Corollary 4.6]) *The Calkin algebra is convex-transitive.*

We devote the remaining part of this section to study transitivity conditions on the norm of JB-algebras. *JB-algebras* are defined as those Jordan-Banach real algebras X satisfying $\| x \|^2 \leq \| x^2 + y^2 \|$ for all x, y in X. The basic reference for the the theory of JB-algebras is [43]. A natural example of a JB-algebra is the Banach space X_{sa}, where X is a C^*-algebra, whenever we define the Jordan product $x \cdot y$ of elements x, y in X_{sa} as $x \cdot y := \frac{1}{2}(xy + yx)$.

In particular the spaces $C_0^{\mathbb{R}}(L)$, with L a locally compact Hausdorff topological space, become JB-algebras (*actually these are the unique associative JB-algebras* [43, 3.2.2]). JB-algebras are closely related to JB^*-triples. Indeed, if X is a JB-algebra, and if we define a triple product on X by $\{xyz\} := (x \cdot y) \cdot z + (y \cdot z) \cdot x - (x \cdot z) \cdot y$, then $(X, \{\cdots\})$ can be regarded as a closed real subtriple of a suitable JB^*-triple (cf. [43, 3.3.9], [76], and [20]).

Let X be a JB-algebra with a unit $\mathbf{1}$. If u is an element in X satisfying $u^2 = \mathbf{1}$, then we say that u is a *symmetry* in X. Central symmetries in X are characterized as the isolated points of the set of all extreme points of B_X [47, Proposition 1.3]. Such a characterization implies the following result.

Proposition 5.16. ([11, Proposition 5.1]) *Let X be a JB–algebra with a unit $\mathbf{1}$. If the linear hull of $\mathcal{G}(\mathbf{1})$ is dense in X (for instance, if X is convex-transitive), then X is associative.*

JBW-algebras (see [43, 4.1.1] for a definition) can actually be characterized as those JB-algebras which are Banach dual spaces [43, 4.4.16]. If X is a JBW-algebra, then X has a unit [43, 4.1.7], and the product of X is separately w^*-continuous [43, 4.4.16 and 4.1.6]. Applying again the geometric characterization in [47] of central symmetries of unital JB-algebras, the next result is easily obtained.

Proposition 5.17. ([11, Proposition 5.2]) *Let X be the predual of a JBW-algebra. If X has no non trivial $\mathcal{G}$-invariant closed subspaces (for instance, if X is convex-transitive), then X^* is associative.*

Another not difficult result in the line of the above proposition is the following. Note that, if X is a unital JB-algebra, then $\mathcal{G}(\mathbf{1})$ coincides with the set of all central symmetries of X. Indeed, we already know that $\mathcal{G}(\mathbf{1})$ consists of central symmetries of X, and multiplications by central symmetries of X are elements of $\mathcal{G}$.

Proposition 5.18. *Let X be a JBW-algebra. Then the centre of X is the norm-closed linear hull of $\mathcal{G}(\mathbf{1})$. Therefore, the following assertions are equivalent for X:*

1. *The linear hull of $\mathcal{G}(\mathbf{1})$ is w^*-dense in X.*
2. *X is associative.*
3. *The linear hull of $\mathcal{G}(\mathbf{1})$ is norm-dense in X.*

Proof. Keeping in mind that the centre of X is a w^*-closed in X (and hence, a JBW-algebra) and the comment immediately before the proposition, the first assertion follows from [43, 4.2.3] and the fact that $\mathbf{1}-2e$ is a symmetry whenever e is an idempotent in X. Now, the consequence follows from the first assertion. ∎

Let X be a JB-algebra. Then the bidual of X is a JBW-algebra containing X as a subalgebra [43, 4.4.3], and the set

$$M(X) := \{z \in X^{**} : z \cdot X \subseteq X\}$$

is a subalgebra of X^{**} [35] called the multiplier algebra of X. According to the Kadison type theorem in [47], every surjective linear isometry on X is the composition of an algebra automorphism of X with the operator of multiplication by a central symmetry in $M(X)$. This result is one of the key tools in the proof of the theorem which follows. We recall that an element x in X is said to be *positive* if there exists y in X such that $y^2 = x$.

THEOREM 5.19. ([11, Theorem 5.3]) *Let X be a JB-algebra. If some positive element in X is a big point of X (for instance, if X is convex-transitive), then X is associative.*

Now the following corollary follows from the above theorem and Greim-Rajalopagan Theorem 2.11.

COROLLARY 5.20. ([11, Corollary 5.4]) $\mathbb{R}$ *is the unique almost transitive JB-algebra.*

It follows from Theorem 5.19 and Proposition 5.17 that the question of convex transitivity for JB-algebras and preduals of JBW-algebras reduces to the consideration of a similar question on the classical Banach spaces $\mathcal{C}_0^{\mathbb{R}}(L)$ and $L_1^{\mathbb{R}}(\Gamma, \mu)$, respectively. The reader is referred to Wood's theorem 2.30 for the $C_0^{\mathbb{R}}(L)$ case. As far as we know, the convex transitivity of the norm for $L_1^{\mathbb{R}}(\Gamma, \mu)$ spaces has not been systematically studied. For the particular case of the almost transitivity of the norm on such spaces, the reader is referred to [40]. We note that a C^*-algebra X is commutative if and only if the JB-algebra X_{sa} is associative [72]. Therefore Proposition 5.16, Proposition 5.17, and Theorem 5.19 provide us with transitivity conditions implying the commutativity of unital C^*-algebras, W^*-algebras, and (possibly non unital) C^*-algebras, respectively.

We conclude this section by proving a refinement Theorem 5.19. *Order unit spaces* (see [43, 1.2.1] for a definition) can be characterized as those real normed spaces X with a distinguished norm-one element e, called the *order unit* of X, satisfying $n(X, e) = 1$ [43, 1.2.2 and 1.2.5]. (The numerical index $n(X, e)$ of a normed space X relative to an element e in S_X was defined before Example 4.2 in the case that X is complete and complex, but the definition works verbatim in the non complete and/or real case.) The following lemma is folklore in the theory of order unit spaces.

LEMMA 5.21. *Let (X, e) be an order unit space, and u an element of S_X satisfying $\|e - u\| < 2$. Then we have $D(X, u) \subseteq D(X, e)$.*

Proof. Let f be in $D(X, u)$. By [43, 1.2.2 and 1.2.6], there exist g, h in $D(X, e)$ and $0 \leq \alpha \leq 1$ such that $f = \alpha g - (1 - \alpha)h$. Then, to realize that f lies in $D(X, e)$ it is enough to show that $\alpha = 1$. But, it this were not the case, then we would have

$$1 = f(u) = \alpha g(u) - (1 - \alpha)h(u) = \alpha g(u) + (1 - \alpha)(h(e - u) - 1)$$

$$\leq \alpha + (1 - \alpha)(\|e - u\| - 1) < \alpha + (1 - \alpha)(2 - 1) = 1,$$

a contradiction. ∎

The following corollary follows from Lemmas 5.7, 5.8, and 5.21.

COROLLARY 5.22. *Let (X, e) be a complete order unit space. Then e is a big point of X if and only if the set*

$$\Delta_0(e) := \{T^*(f) : f \in D(X, e), T \in \mathcal{G}\}$$

is dense in S_{X^*}. If moreover X is a dual space (with predual X_* say), then e is a w^*-superbig point of X if and only if the set

$$\Delta_0^*(e) := \{T(y) : y \in D(X, e) \cap X_*, T \in \mathcal{G}(X_*)\}$$

is dense in S_{X_*}.

PROPOSITION 5.23. *Let (X, e) be a complete (respectively, dual) order unit space such that there exists some big (respectively, w^*-superbig) point u in X satisfying $\|e - u\| < 2$. Then e is a big (respectively, w^*-superbig) point of X.*

Proof. Take $0 < \delta < 2 - \|e - u\|$. Then we have $\|x - e\| < 2$ whenever $x \in S_X$ and $\|x - u\| < \delta$. Therefore, by Lemma 5.21, we have $\Delta_\delta(u) \subseteq \Delta_0(e)$ (respectively, $\Delta^*_\delta(u) \subseteq \Delta^*_0(e)$). Since u is a big (respectively, w^*-superbig) point of X, it follows from Lemma 5.7 (respectively 5.8) and Corollary 5.22 that e is a big (respectively, w^*-superbig) point of X. ∎

Now the next variant of Theorem 5.19 follows from Propositions 5.23 and 5.16, and the fact that, if X is a JB-algebra with unit $\mathbf{1}$, then $(X, \mathbf{1})$ is an order unit space [43, 3.3.10].

Proposition 5.24. *Let X be a JB-algebra with a unit $\mathbf{1}$. If there is some big point u of X satisfying $\|\mathbf{1} - u\| < 2$, then X is associative.*

To realize that Proposition 5.24 is a variant of Theorem 5.19, note that, if X is a JB-algebra with unit $\mathbf{1}$, and if p is a norm-one element in X, then p is positive if and only if $\|\mathbf{1} - p\| \leq 1$. We conclude this section with the refinement of Theorem 5.19 announced above.

Theorem 5.25. *Let X be a JB-algebra such that there are a big point u of X and a positive element p in $2B_X$ satisfying $\|p - u\| < 1$. Then X is associative.*

Proof. We know that X^{**} is a unital JB-algebra containing X as a subalgebra, and that, consequently $(X^{**}, \mathbf{1})$ is an order unit space. On the other hand, since p is a positive element in $2B_X$, we have $\|\mathbf{1} - p\| \leq 1$, and hence $\|\mathbf{1} - u\| < 2$ (because $\|p - u\| < 1$). Moreover, u is a w^*-superbig point of X^{**} (because u is a big point of X). It follows from Proposition 5.23 that $\mathbf{1}$ is a w^*-superbig point of X^{**}. Finally, by Proposition 5.18, X^{**} (and hence X) is associative. ∎

When we straightforwardly derive Theorem 5.19 from Theorem 5.25, we are provided with a completely new proof of Theorem 5.25.

6. The geometry of convex-transitive Banach spaces

The "leit motiv" of this section can be roughly summarized by saying that a convex-transitive Banach space fulfilling some "minor" isometric or isomorphic condition is in fact almost transitive and superreflexive. Sometimes, in results of such a kind, the requirement of convex transitivity can be relaxed to the mere existence of a big point. In this way the present section could have been subtitled "the magic of big points".

The results in the line of the above comment appear in the papers of the authors [12] and [13], and will be reviewed in detail a little later. For the moment, as an aperitive, we prove a new result concerning faces of the closed unit ball of a convex-transitive Banach space.

Let X be a Banach space over $\mathbb{K}$. We recall that proper faces of B_X are contained in S_X, and that, consequently, if $\mathbb{K} = \mathbb{C}$, then no proper face of B_X has interior points relative to S_X.

Proposition 6.1. *Let X be a real Banach space such that there exists a proper face C of B_X, together with a big point u of X which lies in the interior of C relative to S_X. Then X is one-dimensional.*

Proof. Take $0 < \delta < 1$ such that $\frac{u+y}{\|u+y\|}$ lies in C whenever y is in X with $\|y\| \leq \delta$. Then, for y in X with $\|y\| \leq \delta$, we have

$$2(\|u+y\| + \|u-y\|)^{-1}u =$$
$$\frac{\|u+y\|}{\|u+y\|+\|u-y\|}\frac{u+y}{\|u+y\|} + \frac{\|u-y\|}{\|u+y\|+\|u-y\|}\frac{u-y}{\|u-y\|} \in C.$$

Since C is contained in S_X, for such an y we obtain $\|u-y\| + \|u+y\| = 2$. Now the set

$$\{x \in X : \|x-y\| + \|x+y\| \leq 2, \ \forall y \in X \text{ with } \|y\| \leq \delta\}$$

is convex, closed, and $\mathcal{G}$-invariante, and contains u. It follows from the bigness of u that $\|x-y\|+\|x+y\| = 2$ whenever x is in S_X and y is in X with $\|y\| \leq \delta$. Now note that this last condition remains true when X is replaced by any of its subspaces, and that to prove the proposition it is enough to show that every non-zero finite-dimensional subspace M of X is one-dimensional. But, if M is such a subspace, then we can choose an extreme point x of B_M, so that, for every y in M with $\|y\| \leq \delta$ we have

$$x = \frac{\|x+y\|}{2}\frac{x+y}{\|x+y\|} + \frac{\|x-y\|}{2}\frac{x-y}{\|x-y\|}$$

and $\frac{\|x+y\|}{2} + \frac{\|x-y\|}{2} = 1$, which implies $M = \mathbb{R}x$. ∎

Recall that a subspace M of a Banach space X is said to be an *M-summand* of X if there is a linear projection π from X onto M satisfying $\|x\| = \max\{\|\pi(x)\|, \|x-\pi(x)\|\}$. It follows from Proposition 6.1 above that, *if X is a real Banach space, and if there is some big point u in X such that $\mathbb{R}u$ is an M-summand of X, then X is one-dimensional* (compare Corollary 6.10 below).

An element f in the dual X^* of a Banach space X is said to be a *w^*-big point* of X^* if f belongs to S_{X^*} and the convex hull of $\{F(f) : F \in \mathcal{G}(X^*)\}$ is w^*-dense in B_{X^*}. Minor changes in the proof of Proposition 6.1 allows us to establish the following result.

Proposition 6.2. *Let X be a real Banach space such that there exists a proper face C of B_{X^*}, together with a w^*-big point of X^* which lies in the interior of C relative to S_{X^*}. Then X is one-dimensional.*

Keeping in mind Proposition 2.28, the next corollary follows from Propositions 6.1 and 6.2 above.

Corollary 6.3. *Let X be a convex-transitive real Banach space of dimension > 1. Then all proper faces of B_X and B_{X^*} have empty interior relative to S_X and S_{X^*}, respectively. As a consequence, for f in S_{X^*} (respectively, x in S_X), $D(X^*, f) \cap X$ (respectively, $D(X, x)$) has empty interior relative to S_X (respectively, S_{X^*}).*

Now recall that a subspace M of a Banach space X is said to be an *L-summand* of X if there is a linear projection π from X onto M satisfying $\|x\| = \|\pi(x)\| + \|x - \pi(x)\|$. It follows from Corollary 6.3 that, *if X is a convex-transitive real Banach space, and if there is some one-dimensional L-summand of X, then X is one-dimensional* (compare Corollary 6.11 below).

Now we retake the fundamental line of this section. *From now on $\mathcal{J}$ will denote the class of almost transitive superreflexive Banach spaces.* We already reviewed at the conclusion of Section 2 the result of C. Finet [36] that *every member of $\mathcal{J}$ is uniformly smooth and uniformly convex*, as well as those of F. Cabello [24] that *for an almost transitive Banach space, superreflexivity is equivalent to reflexivity (and even to either enjoy the Radon-Nikodym property or be Asplund)*, and that *for a superreflexive Banach space, almost transitivity is equivalent to convex transitivity*. In [12] and [13] we refined Cabello's results in several directions, and now we are surveying such refinements in some detail. The key tools in our argument are Propositions 6.4, 6.5, 6.6 and 6.7 below.

Let X be a Banach space, and u a norm-one element in X. For x in X, the number $\lim_{\alpha \to 0^+} \frac{\|u+\alpha x\|-1}{\alpha}$ (which always exists because the mapping

$\alpha \to \|u + \alpha x\|$ from $\mathbb{R}$ to $\mathbb{R}$ is convex) is usually denoted by $\tau(u, x)$. We say that the norm of X is *strongly subdifferentiable* at u if

$$\lim_{\alpha \to 0^+} \frac{\|u + \alpha x\| - 1}{\alpha} \;=\; \tau(u, x) \text{ uniformly for } x \in B_X.$$

The reader is referred to [1] and [37] for a comprehensive view of the usefulness of the strong subdifferentiability of the norm in the theory of Banach spaces. We note that the Fréchet differentiability of the norm of X at u is nothing but the strong subdifferentiability of the norm of X at u together with the smoothness of X at u.

PROPOSITION 6.4. ([13, Lemma 3]) *Let X be a Banach space, and u a big point of X such that the norm of X is strongly subdifferentiable at u. Then the set*

$$\{T^*(f) \;:\; f \in D(X, u),\; T \in \mathcal{G}\}$$

is norm-dense in S_{X^*}.

New proof. Let us fix $\varepsilon > 0$ and g in S_{X^*}. Since the norm of X is strongly subdifferentiable at u, by [37, Theorem 1.2], there exists $\delta > 0$ such that $D(X, x) \subseteq D(X, u) + \frac{\varepsilon}{2} B_{X^*}$ whenever $x \in S_X$ and $\|u - x\| \leq \delta$. But, since u is a big point of X, Lemma 5.7 applies, giving suitable y in S_X with $\|u - y\| \leq \delta$, h in $D(X, y)$, and T in $\mathcal{G}$ satisfying $\|g - T^*(h)\| \leq \frac{\varepsilon}{2}$. By choosing f in $D(X, u)$ such that $\|f - h\| \leq \frac{\varepsilon}{2}$ we finally obtain $\|g - T^*(f)\| \leq \varepsilon$. ∎

Let X be a Banach space. For u in S_X, we put

$$\eta(X, u) := \limsup_{\|h\| \to 0} \frac{\|u + h\| + \|u - h\| - 2}{\|h\|}.$$

We say that X is *uniformly non-square* if there exists $0 < a < 1$ such that $\|x - y\| < 2a$ whenever x, y are in B_X satisfying $\|x + y\| \geq 2a$. We remark that, thanks to a celebrated theorem of R.C. James (see for instance [29, Theorem VII.4.4]), uniformly non-square Banach spaces are superreflexive.

PROPOSITION 6.5. ([13, Lemma 1]) *Let X be a Banach space such that there exists a big point u in X satisfying $\eta(X, u) < 2$. Then X is uniformly non-square.*

Given $\epsilon > 0$, the Banach space X is said to be *ϵ-rough* if, for every u in S_X, we have $\eta(X, u) \geq \epsilon$. We say that X is *rough* whenever it is ϵ-rough for some $\epsilon > 0$, and *extremely rough* whenever it is 2-rough. Since, for u in S_X, the Fréchet differentiability of the norm of X at u can be characterized by the equality $\eta(X,u) = 0$ [30, Lemma I.1.3], it follows that the roughness of X can be seen as a uniform non Fréchet-differentiability of the norm, and hence becomes the extremely opposite situation to that of the uniform smoothness.

PROPOSITION 6.6. ([12, Lemma 2.5]) *Let X be a Banach space, and u, v big points of X. If the norm of X^* is non rough, then u belongs to the closure of $\mathcal{G}(v)$ in S_X.*

Let X be a Banach space, and u an element in X. We say that u is a *denting* (respectively, *quasi-denting*) *point* of B_X if u belongs to B_X and, for every $\varepsilon > 0$, u does not belongs to $\overline{co}(B_X \setminus (u + \varepsilon B_X))$ (respectively, $\overline{co}(B_X \setminus (u + \varepsilon B_X))$ is not equal to B_X). The next result is a refinement of [12, Lemma 2.7].

PROPOSITION 6.7. *Let X be a Banach space. If u is a quasi-denting point of B_X, and if v is a big point of X, then u belongs to the closure of $\mathcal{G}(v)$. Therefore, if there are big points of X and quasi-denting points of B_X, then big points of X and quasi-denting points of B_X coincide. Moreover, if there are big points of X and denting points of B_X, then the set of all quasi-denting points of B_X is the closure of the set of all denting points of B_X.*

Proof. Let u be a quasi-denting point of B_X, and v a big point of X. If u does not belong to $\overline{\mathcal{G}(v)}$, then there exists $\varepsilon > 0$ satisfying $\|u - T(v)\| > \varepsilon$ for every T in $\mathcal{G}$ (i.e., the inclusion $\mathcal{G}(v) \subseteq B_X \setminus (u + \varepsilon B_X)$ holds), and hence

$$B_X = \overline{co}\mathcal{G}(v) \subseteq \overline{co}(B_X \setminus (u + \varepsilon B_X)) \subset B_X,$$

a contradiction. The consequences asserted in the statement follow from the assertion just proved and the facts that the relation $x \in \overline{\mathcal{G}(y)}$ is symmetric, and that the set of all big points of X, as well as that of all quasi-denting points of B_X, is closed (concerning big points, see the proof of [13, Corollary 1]). ∎

Let X be a Banach space. We say that X^* is *w^*-convex-transitive* if every element of S_{X^*} is a w^*-big point of X^*. An element f of X^* is called a *w^*-denting point* of B_{X^*} if f belongs to B_{X^*} and, for every $\varepsilon > 0$, f does not belongs to the w^*-closure of $co(B_{X^*} \setminus (f + \varepsilon B_{X^*}))$. We say that X has *Mazur's intersection property* whenever every bounded closed convex subset of X can be represented as an intersection of closed balls in X. Analogously, we say that X^* has *Mazur's w^*-intersection property* whenever every bounded w^*-closed convex subset of X^* can be expressed as an intersection of closed balls in X^*. According to [39, Theorem 2.1] (respectively, [39, Theorem 3.1]), *X (respectively, X^*) has Mazur's intersection (respectively, w^*-intersection) property if and only if the set of all w^*-denting (respectively, denting) points of B_{X^*} (respectively, B_X) is dense in S_{X^*} (respectively, S_X).*

Now, putting together Propositions 6.4, 6.5, 6.6 and 6.7, as well as the dual variants of the three last ones (see [13, Lemma 2], [12, Lemma 2.6], and [12, Lemma 2.8], respectively), the following theorem is easily obtained (see [12, Theorems 3.2 and 3.4] and [13, Theorem 1 and Remark 2], for details).

THEOREM 6.8. *For a Banach space X, the following assertions are equivalent:*

1. *X is a member of $\mathcal{J}$.*
2. *There exists a big point u in X such that the norm of X is Fréchet differentiable at u.*
3. *There exists a w^*-big point f in X^* such that the norm of X^* is Fréchet differentiable at f.*
4. *X is convex transitive and the norm of X is not extremely rough.*
5. *X^* is convex w^*-transitive and the norm of X^* is not extremely rough.*
6. *The set of all big points of X is non-rare in S_X, and the norm of X^* is non rough.*
7. *The set of all w^*-big points of X^* is non-rare in S_{X^*}, and the norm of X is non rough.*
8. *The set of all denting points of B_X is non rare in S_X, and there exists a big point in X.*
9. *The set of all w^*-denting points of B_{X^*} is non rare in S_{X^*}, and there exists a w^*-big point in X^*.*
10. *The norm of X^* is Fréchet differentiable at every point of S_{X^*}, and there exist a big point in X.*
11. *The norm of X is Fréchet differentiable at every point of S_X, and there exist a w^*-big point in X^*.*
12. *X^* has Mazur's w^*-intersection property, and there is some big point in X.*
13. *X has Mazur's intersection property, and there is some w^*-big point in X^*.*

The non-roughness of the norm, required in Assertions 4, 5, 6, and 7 of the above theorem, is a condition much weaker than the one of uniform smoothness enjoyed by the members of $\mathcal{J}$ and their duals. On the other hand, the remaining requirements in such assertions also are ostensibly weaker than the one of almost transitivy (also enjoyed by the members of $\mathcal{J}$ and their duals). Therefore we can obtain many other intermediate characterizations of almost transitive super-reflexive Banach spaces in terms of convex transitivity and related conditions. We leave to the taste of the reader the codification of the more relevant such characterizations involving isometric conditions. As a hint, we recall that the existence of points in S_{X^*} where the norm of X^* is Fréchet differentiable implies the existence of strongly exposed points in B_X, that strongly exposed points are denting points, and that the existence of denting points in B_X implies that the norm of X^* is non-rough. Analogously, the existence of points in S_X of Fréchet differentiability of the norm of X implies the existence of strongly w^*-exposed (and hence w^*-denting) points of B_{X^*}, and this last fact implies that the norm of X is non-rough.

Concerning characterizations of members of $\mathcal{J}$ in terms of convex transitivity and isomorphic conditions, we have the next corollary (see [12, Corollary 3.3] and [13, Corollary 1]. The reader is referred to [19], [30], and [31] for background on Asplund spaces and the Radon-Nikodym property.

COROLLARY 6.9. *For a Banach space X, the following assertions are equivalent:*

1. *X is a member of $\mathcal{J}$.*
2. *X is convex-transitive and has the Radon-Nikodym property.*
3. *X^* is w^*-convex-transitive, and X is Asplund.*
4. *X is convex-transitive and Asplund.*
5. *X^* is w^*-convex-transitive, and X has the Radon-Nikodym property.*
6. *There exists a non rare subset of S_X consisting of big points of X, and X has the Radon-Nikodym property.*
7. *There exists a non rare subset of S_{X^*} consisting of w^*-big points of X^*, and X is Asplund.*
8. *There exists a non rare subset of S_X consisting of big points of X, and X is Asplund.*
9. *There exists a non rare subset of S_{X^*} consisting of w^*-big points of X^*, and X has the Radon-Nikodym property.*

Now, the difference between convex transitivity and the less restrictive notion of maximality of the norm becomes very big. Indeed, c_0 *has strongly maximal norm* (by Proposition 2.41) *but has no convex-transitive equivalent renorming* (by the above corollary). We do not know if the assumption on a Banach space X that the set of its big points is non rare in S_X is strictly weaker than that of convex transitivity. Since we know that the set of all big points X is closed, the actual question is if the requirement that the set of all big points of X has non-empty interior in S_X implies that X is convex-transitive. In any case, our results provide us with some non-trivial information about this question. For instance, thanks to Theorem 6.8 and Corollary 2.42, we have that *if X is a finite-dimensional Banach space, and if the set of all big points of X has non-empty interior in S_X, then X is a Hilbert space.*

To conclude the paper, let us deal with other minor consequences of Theorem 6.8, whose proofs also involve Theorems 2.16 and 2.33. Given $1 \leq p \leq \infty$, a subspace M of the Banach space X is said to be an *L^p-summand* of X if there is a linear projection π from X onto M such that, for every x in X, we have

$$\|x\|^p = \|\pi(x)\|^p + \|x - \pi(x)\|^p \quad (1 \leq p < \infty),$$
$$\|x\| = \max\{\|\pi(x)\|, \|x - \pi(x)\|\} \quad (p = \infty).$$

Note that L^1-summands (respectively, L^∞-summands) are nothing but L-summands (respectively, M-summands) previously introduced.

Corollary 6.10. ([13, Corollary 2]) *Let X be a Banach space over $\mathbb{K}$ such that there exists a big point u in X satisfying that $\mathbb{K}u$ is an L^p-summand of X for some $1 < p \leq \infty$. Then X is a Hilbert space. If in addition $p \neq 2$, then X is one-dimensional.*

Corollary 6.11. ([12, Corollary 3.5]) *Let X be a convex-transitive Banach space having a one-dimensional L^p-summand for some $1 \leq p \leq \infty$. Then X is a Hilbert space. If in addition $p \neq 2$, then X is one-dimensional.*

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