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On the $p-$Drop Theorem, $1 \leq p \leq \infty$

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1. Introduction

Let $(X, \|\cdot\|)$ be a Banach space and B be the closed unit ball of X. By a drop $D(x,B)$ determined by a point $x \in X \setminus B$ we shall mean the convex hull of the set $\{x\} \cup B$. If a nonvoid closed set S of a Banach space $(X, \|\cdot\|)$ having a positive distance from the unit ball B is given, then there exists a point $a \in S$ such that $D(a,B) \cap S = \{a\}$, which is the so-called Daneš drop theorem [1].

In [4] these notions were considered in the context of quasi-Banach spaces. More precisely: Let X be a p–Banach space, $0 < p < 1$. Let A be a non-empty, closed subset of X; and let B be a closed, bounded and p–convex subset of X so that $d(A,B) > 0$. Then, there exists a point $a \in A$ such that $D_p(a,B) \cap A = \{a\}$.

In this paper, we shall consider the notion of p–drop for $1 \leq p \leq \infty$. Let C be a closed convex subset of X, and $a \in X$. The set

$$D_p(a,C) := \{\alpha a + \beta y : \alpha, \beta \in [0,1] \text{ with } \alpha^p + \beta^p = 1,\, y \in C\}$$

is called the p–drop of center a defined by C. For $p = \infty$ we put $D_\infty(a,C) = conv\,(C \cup \{a + C\})$. We give some properties of p–convex sets (for $1 \leq p \leq \infty$). We study also the p–drop theorem for $1 \leq p \leq \infty$.

It is clear that a drop $D(x,B)$ is never smooth. A smooth drop theorem of Daneš type for spaces with smooth norms was shown in [7] (see also [3]). A closed convex set D is called a smooth drop if 0 is in the interior of D and the Minkowski functional of D, $\rho(x) = \inf\{\lambda > 0 : x\lambda^{-1} \in D\}$, is smooth.

We show that the p–drop is Fréchet-smooth (resp. Gâteaux-smooth) whenever the dual norm is locally uniformly convex (resp. strictly convex).

2. Preliminaries

Definition 2.1. Let $(X, \|\cdot\|)$ be a Banach space. Let C be a closed and convex subset of X and let $p \geq 1$. For $x \in X$, the set

$$D_p(x, C) := \left\{ tx + (1 - t^p)^{\frac{1}{p}}\, y : \, y \in C, \, t \in [0,1] \right\}$$

is called the p–drop of center x defined by C. For $p = \infty$, $D_\infty(x, C)$ is the set *conv* $(\{x + C\} \cup C)$.

Proposition 2.2. *Let $(X, \|\cdot\|)$ be a Banach space, $a \in X \setminus \{0\}$ and $1 < p < \infty$. Then we have the following:*

(i) $\|x\| \leq \left(\|a\|^{\frac{p}{p-1}} + 1 \right)^{\frac{p-1}{p}}$ *for all $x \in D_p(a, B)$,*

(ii) $D_p(a, B) \subset$ *conv* $(B \cup \{a + B\})$.

Proof. Let $x \in D_p(a, B)$. By the definition of the p–drop, there exist $\alpha \in [0,1]$ and $y \in B$ such that $x = \alpha a + (1 - \alpha^p)^{\frac{1}{p}}\, y$. Therefore,

$$\|x\| \leq \alpha \|a\| + (1 - \alpha^p)^{\frac{1}{p}}.$$

Let $g : [0,1] \to \mathbb{R}$ be such that $g(t) = t\|a\| + (1 - t^p)^{\frac{1}{p}}$. Then, $g'(t) = \|a\| - t^{p-1}(1 - t^p)^{\frac{1-p}{p}}$, and there is a unique $t_0 \in [0,1]$ such that $g'(t_0) = 0$. Which means that

$$t_0 = \left(\frac{\|a\|^{\frac{p}{p-1}}}{1 + \|a\|^{\frac{p}{p-1}}} \right)^{\frac{1}{p}}.$$

Moreover, g attains its maximum at t_0. Therefore, we deduce that

$$\|x\| \leq g(t_0) = \left[\|a\|^{\frac{p}{p-1}} + 1 \right]^{\frac{p-1}{p}}$$

and the first property is proved.

Let $x \in D_p(a, B)$. Then there exist $t \in [0,1]$ and $y \in B$ such that $x = ta + (1 - t^p)^{\frac{1}{p}}\, y$. Let $z = (1 - t^p)^{\frac{1}{p}}\, y$. Then $z \in B$. Moreover, we have $x = t(a + z) + (1 - t)z \in$ *conv* $(\{a + B\} \cup B)$. And thus we have the second property. ∎

Let $a \in X$ and let

$$b = \frac{\|a\|^{\frac{1}{p-1}}}{(1+\|a\|^{\frac{p}{p-1}})^{\frac{1}{p}}}\, a + \left(1 - \frac{\|a\|^{\frac{p}{p-1}}}{1+\|a\|^{\frac{p}{p-1}}}\right)^{\frac{1}{p}} \frac{a}{\|a\|}.$$

It is clear that $b \in D_p(a,B)$. Moreover, we have

$$b = \left(1+\|a\|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \|a\|^{-1} a \quad \text{and} \quad \|b\| = \left(\|a\|^{\frac{p}{p-1}} + 1\right)^{\frac{p-1}{p}}.$$

The element b is called the vertex of $D_p(a,B)$.

It is clear that the drop $D(x,B) := \{tx + (1-t)b : \ t \in [0,1], b \in B\}$ is a closed subset. The following proposition shows that a p–drop is also a closed subset.

Proposition 2.3. *Let $(X, \|\cdot\|)$ be a Banach space. Let C be a convex, closed, and bounded subset of X, $1 \le p \le \infty$ and $x \in X$, $x \neq 0$. Then $D_p(x,C)$ is closed.*

Proof. In the case $p = \infty$ see Proposition 3-2 of [3]. Assume now $1 \le p < \infty$. Let (y_n) be a sequence in $D_p(x,C)$ such that $y_n \to y$. Then there is a sequence (t_n) in $[0,1]$ and a sequence (c_n) in C such that:

$$y_n = t_n x + (1-t_n^p)^{\frac{1}{p}}\, c_n \longrightarrow y.$$

Without loss of generality, we assume that $t_n \to t_0$.

Case 1. $t_0 = 1$. Since C is bounded, then $y_n \to x$. Therefore $y = x \in D_p(x,C)$.

Case 2. $t_0 \in [0,1)$. For n large enough, we can assume that $(1-t_n^p)^{\frac{1}{p}} \neq 0$. Therefore we can write:

$$c_n = \frac{t_n x + (1-t_n^p)^{\frac{1}{p}} c_n}{(1-t_n^p)^{\frac{1}{p}}} - \frac{t_n x}{(1-t_n^p)^{\frac{1}{p}}} \longrightarrow \frac{y}{(1-t_0^p)^{\frac{1}{p}}} - \frac{t_0 x}{(1-t_0^p)^{\frac{1}{p}}}.$$

Since $(c_n) \subset C$ and C is closed, then $c := (y - t_0 x)/(1-t_0^p)^{\frac{1}{p}} \in C$. Thus, $y = (1-t_0^p)^{\frac{1}{p}} c + t_0 x \in D_p(x,C)$, and the proof for $1 \le p < \infty$ is complete. ∎

The drop $D(x,B)$ is by definition a convex subset. The following lemma shows that the p–drop is also a convex subset.

Lemma 2.4. *Let $(X, \|\cdot\|)$ be a Banach space and let $a \in X$ and $1 \le p \le \infty$. Then $D_p(a, B)$ is a convex subset.*

Proof. By definition $D_\infty(a, B) = conv\,(a \cup \{a + B\})$, then $D_\infty(a, B)$ is a convex subset. Moreover, if $p = 1$, $D_1(a, B) = D(a, B) = conv\,(\{a\} \cup B)$, which is a convex subset. Assume now that $1 < p < \infty$.

If $a = 0$, then $D_p(a, B) = B$. In this case there is nothing to prove. Then assume that $a \neq 0$.

Let $x, y \in D_p(a, B)$ and $\lambda \in [0, 1]$. Hence we can write:

$$x = ta + (1 - t^p)^{\frac{1}{p}}\, b \quad \text{and} \quad y = \alpha a + (1 - \alpha^p)^{\frac{1}{p}}\, c,$$

for some $b, c \in B$ and $t, \alpha \in [0, 1]$. Without loss of generality we assume that $\alpha \le t$.

Remarking that $0 \le \lambda t + (1 - \lambda)\alpha \le 1$. Let $\beta := \lambda t + (1 - \lambda)\,\alpha$.

Case 1. If $0 < \lambda < 1$, then $t = 1$ and $\alpha = 1$. Consequently, $x = a$ and $y = a$. This implies that $\lambda x + (1 - \lambda)y = a$, which is in $D_p(a, B)$.

Case 2. If $\lambda = 0$ or $\lambda = 1$, then it is direct since x, y are in the drop.

Case 3. $\beta \in [0, 1)$. We can write:

$$\begin{aligned}
\lambda x + (1 - \lambda)y &= \lambda\big[ta + (1 - t^p)^{\frac{1}{p}}\, b\big] + (1 - \lambda)\big[\alpha a + (1 - \alpha^p)^{\frac{1}{p}}\, c\big] \\
&= a\big[\lambda t + (1 - \lambda)\alpha\big] + \lambda(1 - t^p)^{\frac{1}{p}}\, b + (1 - \lambda)(1 - \alpha^p)^{\frac{1}{p}}\, c \\
&= \big[1 - (\lambda t + (1 - \lambda)\alpha)^p\big]^{\frac{1}{p}}\ \frac{\lambda(1 - t^p)^{\frac{1}{p}}\, b + (1 - \lambda)(1 - \alpha^p)^{\frac{1}{p}}\, c}{\big[1 - (\lambda t + (1 - \lambda)\alpha)^p\big]^{\frac{1}{p}}} \\
&\quad + a\big[\lambda t + (1 - \lambda)\alpha\big].
\end{aligned}$$

Put:

$$Y = \frac{\lambda(1 - t^p)^{\frac{1}{p}}\, b + (1 - \lambda)(1 - \alpha^p)^{\frac{1}{p}}\, c}{(1 - \beta^p)^{\frac{1}{p}}}.$$

Then we have $\lambda x + (1 - \lambda)y = \beta a + (1 - \beta^p)^{1/p}\, Y$. We affirm that $Y \in B$. Indeed,

$$\|Y\| \le \frac{\lambda\,(1 - t^p)^{\frac{1}{p}}}{(1 - \beta^p)^{\frac{1}{p}}} + \frac{(1 - \lambda)\,(1 - \alpha^p)^{\frac{1}{p}}}{(1 - \beta^p)^{\frac{1}{p}}}.$$

Let $h(\lambda) := \lambda(1-t^p)^{\frac{1}{p}} + (1-\lambda)(1-\alpha^p)^{\frac{1}{p}} - (1-\beta^p)^{\frac{1}{p}}$. Therefore:

$$\begin{aligned} h'(\lambda) &= (1-t^p)^{\frac{1}{p}} - (1-\alpha^p)^{\frac{1}{p}} \\ &\quad + [1-(\lambda t + (1-\lambda)\alpha)^p]^{\frac{1-p}{p}} (\lambda t + (1-\lambda)\alpha)^{p-1}(t-\alpha), \\ h''(\lambda) &= [1-(\lambda t + (1-\lambda)\alpha)^p]^{\frac{1-p}{p}} (p-1)(\lambda t + (1-\lambda)\alpha)^{p-2}(t-\alpha) \\ &\quad + (p-1)[1-(\lambda t + (1-\lambda)\alpha)^p]^{\frac{1-2p}{p}} (\lambda t + (1-\lambda)\alpha)^{2(p-1)}(t-\alpha)^2. \end{aligned}$$

Since we have assumed that $t \geq \alpha$, then $h''(\lambda) \geq 0$, for all $\lambda \in [0,1]$. Which implies that the function h is convex in $[0,1]$, and we have $h(0) = h(1) = 0$. Thus $h(\lambda) \leq 0$ for all $\lambda \in [0,1]$. This implies that $\|Y\| \leq 1$. Then $D_p(a,B)$ is a convex subset. ∎

The following lemma shows that the p–drop $D_p(a,B)$ can give a nice equivalent norm.

Lemma 2.5. *Let $(X, \|\cdot\|)$ be a Banach space. Let $0 \neq a \in X$ and $p > 1$. Then $D := D_p(a,B) \cup D_p(-a,B)$ is a convex subset.*

Proof. Let $x, y \in D$ and $\alpha \in [0,1]$. By Lemma 2.4, it suffices to show the case $x \in D_p(a,B)$ and $y \in D_p(-a,B)$. By the definition of p–drop, there exist $t, \lambda \in [0,1]$ and $b, c \in B$ such that:

$$x = ta + (1-t^p)^{\frac{1}{p}}\, b \quad \text{and} \quad y = -\lambda a + (1-\lambda^p)^{\frac{1}{p}}\, c.$$

Let $\alpha \in [0,1]$. Consider $\alpha x + (1-\alpha)\, y$ and we like to prove that it is in $D_p(a,B)$, or in $D_p(-a,B)$.

$$\begin{aligned} \alpha x + (1-\alpha)\, y &= \alpha \left[ta + (1-t^p)^{\frac{1}{p}}\, b\right] + (1-\alpha)\left[(1-\lambda^p)^{\frac{1}{p}}\, c - \lambda a\right] \\ &= a\left[\alpha t - \lambda(1-\alpha)\right] + \alpha(1-t^p)^{\frac{1}{p}}\, b + (1-\alpha)(1-\lambda^p)^{\frac{1}{p}}\, c. \end{aligned}$$

For t, λ fixed in [0, 1], we consider the function f defined in [0, 1] by $f(\alpha) := \alpha t - \lambda(1-\alpha)$. Therefore $f'(\alpha) = t + \lambda \geq 0$. Then, the function f is increasing and we have,

$$-1 \leq -\lambda = f(0) \leq f(\alpha) \leq f(1) = t \leq 1,$$

for all α in [0, 1]. Put $\beta := \alpha t - \lambda(1-\alpha)$, then $\beta \in [-1,1]$.

Case 1. $\beta = 1$. In this case, it is easy to show that $\alpha = 1$ and $t = 1$. Then, $\alpha x + (1-\alpha)y = a \in D_p(a,B) \subset D$.

Case 2. $\beta = -1$. In this case, necessarily $\alpha = 0$ and $\lambda = 1$. Then, $\alpha x + (1-\alpha)y = -a \in D_p(-a, B) \subset D$.

Case 3. $\beta \in [0,1)$. In this case we can write:

$$\alpha x + (1-\alpha)y = \beta a + (1-\beta^p)^{\frac{1}{p}} \left[\frac{\alpha\,(1-t^p)^{\frac{1}{p}}\, b}{(1-\beta^p)^{\frac{1}{p}}} + \frac{(1-\alpha)\,(1-\lambda^p)^{\frac{1}{p}}\, c}{(1-\beta^p)^{\frac{1}{p}}} \right].$$

Put

$$Y := \frac{\alpha\,(1-t^p)^{\frac{1}{p}}\, b + (1-\alpha)\,(1-\lambda^p)^{\frac{1}{p}}\, c}{(1-\beta^p)^{\frac{1}{p}}}.$$

The same techniques used in the proof of Lemma 2.4, show that $\|Y\| \leq 1$. Then, $\alpha x + (1-\alpha)y \in D_p(a, B)$.

Case 4. $\beta \in (-1, 0]$. This is equivalent to $-\beta \in [0,1)$ and we have:

$$\alpha x + (1-\alpha)y = -a(-\beta) + (1-(-\beta)^p)^{\frac{1}{p}} \left[\frac{\alpha\,(1-t^p)^{\frac{1}{p}}\, b}{(1-(-\beta)^p)^{\frac{1}{p}}} + \frac{(1-\alpha)\,(1-\lambda^p)^{\frac{1}{p}}\, c}{(1-(-\beta)^p)^{\frac{1}{p}}} \right].$$

Let

$$Y_1 := \frac{\alpha\,(1-t^p)^{\frac{1}{p}}\, b + (1-\alpha)\,(1-\lambda^p)^{\frac{1}{p}}\, c}{(1-(-\beta)^p)^{\frac{1}{p}}}.$$

By the same techniques we prove that $\|Y_1\| \leq 1$. Then $\alpha x + (1-\alpha)y \in D_p(-a, B)$.

Conclusion, D is a convex subset. ∎

3. p–DROP THEOREM

Recall that the norm $\|\cdot\|$ has the *Kadeč-Klee property* if for all $\|x\| = \|x_n\| = 1$ such that the sequence (x_n) converges weakly to x, then the sequence $(\|x_n - x\|)$ converges to 0.

Recall that the norm $\|\cdot\|$ is said to be *strictly convex* (s.c. for short), if, for all $\|x\| = \|y\| = 1$ such that $\|x+y\| = 2$, we have $x = y$.

THEOREM 3.1. *Let $(X, \|\cdot\|)$ be a reflexive Banach space. Assume that the norm is strictly convex and has the Kadeč-Klee property. Let S be a closed subset at positive distance to B. Let $1 < p < \infty$. Then, there exist $a, a' \in X$, $\delta > 0$, such that:*

(i) *$D_\infty(a, B) \cap S$ is a singleton.*

(ii) $B \subset B[a, 1+\delta]$ *and* $D_p(a', B[a, 1+\delta]) \cap S$ *is a singleton.*

Proof. Let $\varepsilon := \operatorname{dist}(S, B) > 0$. By hypothesis the space X is reflexive and the norm is strictly convex and has the Kadeč-Klee property. By Lau theorem [5], [6], there is a $\mathcal{G}_\delta$ dense subset Γ of $X \setminus S$ such that for all $x \in \Gamma$, there is an unique $s \in S$ such that $\|x - s\| = \operatorname{dist}(x, S)$. Therefore we choose a in $\partial \delta B$, where $0 < \delta < \varepsilon/2$, such that there exists z_0 in ∂S, satisfying that $\|a - z_0\| = \operatorname{dist}(a, S)$. We have:

$$\|a - z_0\| = \operatorname{dist}(a, S) \geq 1 + \varepsilon - \delta > 1 + \frac{\varepsilon}{2} > 1.$$

Then there exists a' in the segment $[a, z_0]$ such that $\|a' - z_0\| = 1$. Consequently:

$$\{z_0\} \subset \operatorname{conv}\big(B[a', 1] \cup B\big) \cap S \subset B(a, \|a - z_0\|) \cap S = \{z_0\},$$

then we have (i).

Let $x \in B$. Then, $\|x - a\| \leq \|x\| + \|a\| \leq 1 + \delta$. Which means that $B \subset B[a, 1+\delta]$. Let x in $D_p[a', B[a, K]]$ with $K = 1 + \delta$. Then,

$$x = ta' + (1 - t^p)^{\frac{1}{p}} b \quad \text{for some} \quad t \in [0,1] \quad \text{and} \quad b \in B[a, K].$$

Therefore,

$$x - a = t\left(a' - a\right) + \alpha(t)(b - a) + a\left(\alpha(t) + t - 1\right),$$

with $\alpha(t) = (1 - t^p)^{\frac{1}{p}}$. Hence,

$$\|x - a\| \leq t\|a' - a\| + \alpha(t)K + \|a\|\left[\alpha(t) + t - 1\right] =: h(t).$$

The maximum of $h(t)$ is attained at

$$t_0 = [1 + D]^{\frac{-1}{p}}, \quad \text{where} \quad D := \left[\frac{\|a' - a\| + \|a\|}{K + \|a\|}\right]^{\frac{p}{1-p}},$$

and we have,

$$h(t_0) = \frac{1}{(1 + D)^{\frac{1}{p}}}\left[\|a' - a\| + D^{\frac{1}{p}} K + \|a\|\left(1 + D^{\frac{1}{p}} - (1 + D)^{\frac{1}{p}}\right)\right].$$

An easy calculation shows that,

$$h(t_0) \leq \|a' - a\| + 1.$$

We know that $\operatorname{dist}(S,B) = \|a'-a\|+1$. Then, we have the p–drop $D_p\left(a', B\left[a, 1+\delta\right]\right)$ defined by $B\left[a,1+\delta\right]$ and of vertex z_0 is contained in $B\left[a, \operatorname{dist}\left(a,S\right)\right]$. Then

$$D_p\left(a', B\left[a, 1+\delta\right]\right) \cap S = \left\{z_0\right\}.$$

The proof of our theorem is complete. ∎

Recall that, the norm $\|\cdot\|$ is said to be *locally uniformly convex* (l.u.c. in short), if for all $\|x\| = \|x_n\| = 1$, such that $\|x+x_n\| \to 2$, we have $\|x-x_n\| \to 0$.

Lemma 3.2. *Let $(X, \|\cdot\|)$ be a Banach space. Let $a \in X$ and $p > 1$. Assume that the dual norm $\|\cdot\|_*$ is locally uniformly convex (resp. strictly convex). Then, the norm $\|\cdot\|_1$ whose unit ball is $D := \operatorname{conv}(D_p(a,B) \cup D_p(-a,B))$, is an equivalent norm in X such that its dual norm is also locally uniformly convex (resp. strictly convex).*

Proof. Assume that the norm $\|\cdot\|$ is such that its dual norm is locally uniformly convex. Let

$$D := \operatorname{conv}(D_p(a,B) \cup D_p(-a,B)).$$

By Lemma 2.5 and Proposition 2.3, D is convex, symmetric, closed and containing the unit ball. Hence, D is a ball for an equivalent norm $\|\cdot\|_1$. Let D^0 be the polar of D,

$$D^0 := \{x^* \in X : x^*(x) \le 1 \text{ for all } x \in D\}.$$

Let $\|\cdot\|_1^*$ the Minkowski functional associated to D^0. We claim that

$$\|x^*\|_1^* = \left[\|x^*\|_*^q + |x^*(a)|^q\right]^{\frac{1}{q}},$$

where q is such that $1/p+1/q=1$. For this, let x^* in D^0. By definition, $x^*(x) \le 1$ for all x in D. This implies that $x^*\left(\pm ta + (1-t^p)^{1/p}\, b\right) \le 1$ for all $t \in [0,1]$ and $b \in B$. Since x^* is linear, $tx^*(\pm a) + (1-t^p)^{1/p}\, x^*(b) \le 1$ for all $t \in [0,1]$ and $b \in B$. We deduce that $t|x^*(a)| + (1-t^p)^{1/p}\, \|x^*\|_* \le 1$ for all t in $[0,1]$. Letting $f(t) := t|x^*(a)| + (1-t^p)^{1/p}\, \|x^*\|_*$. A simple verification shows that $\sup\{f(t) : t \in [0,1]\} = (\|x^*\|^q + |x^*(a)|^q)^{1/q}$. Thus we have $(\|x^*\|^q + |x^*(a)|^q)^{1/q} \le 1$ for all $x^* \in D^0$.

Conversely, let $x^* \in X^*$ such that $(\|x^*\|^q + |x^*(a)|^q)^{1/q} \leq 1$. Let $x \in D$. Assume $x \in D_p(a, B)$. Therefore, there exist $t \in [0,1]$ and $b \in B$ such that $x = ta + (1-t^p)^{1/p}\, b$. Then, we have:

$$\begin{aligned} x^*(x) &= x^*\big(ta + (1-t^p)^{\frac{1}{p}}\, b\big) = tx^*(a) + (1-t^p)^{\frac{1}{p}} x^*(b) \\ &\leq t|x^*(a)| + (1-t^p)^{\frac{1}{p}} \|x^*\|_* = f(t) \\ &\leq \sup\{f(t) : \, t \in [0,1]\} = \big(\|x^*\|_*^q + |x^*(a)|^q\big)^{\frac{1}{q}} \leq 1. \end{aligned}$$

Thus, x^* is in D^0.

We have proved that $D^0 = \big\{x^* \in X^* : \, \big(\|x^*\|^q + |x^*(a)|^q\big)^{1/q} \leq 1\big\}$.

We affirm that $\|x^*\|_1^* = (\|x^*\|_*^q + |x^*(a)|^q)^{1/q}$ is locally uniformly convex in X^*. Indeed, let $x^* \in X^*$ and $(x_n^*) \subset X^*$ be such that $\|x^*\|_1^* = \|x_n^*\|_1^* = 1$ and $\|x^* + x_n^*\|_1^* \to 2$. Put $s := (\|x^*\|_*, |x^*(a)|) \in \mathbb{R}^2$ and $s_n := (\|x_n^*\|_*, |x_n^*(a)|) \in \mathbb{R}^2$. We know that the norm $\|\cdot\|_q$ defined by $\|(x,y)\|_q = (|x|^q + |y|^q)^{1/q}$ is locally uniformly convex in $\mathbb{R}^2$ for $1 < q < \infty$. Moreover, $\|x^*\|_1^* = \|x_n^*\|_1^* = 1$, which is the same as to say that $\|s\|_q = \|s_n\|_q = 1$. First we prove that $\|x_n^*\|_* \to \|x^*\|_*$. We have:

$$\begin{aligned} \|x^* + x_n^*\|_1^* &= \big[\|x^* + x_n^*\|_*^q + |\,(x^* + x_n^*)\,(a)|^q\big]^{\frac{1}{q}} \\ &\leq \big[\,(\|x^*\|_* + \|x_n^*\|_*)^q + (|x^*(a)| + |x_n^*(a)|)^q\,\big]^{\frac{1}{q}} \\ &= \|s + s_n\|_q \leq \|s\|_q + \|s_n\|_q = 2. \end{aligned}$$

So we deduce that $\|s + s_n\|_q \to 2$. Since $\|\cdot\|_q$ is locally uniformly convex in $\mathbb{R}^2$, $\|s - s_n\|_q \to 0$. Then, we conclude that $\|x_n^*\|_* \to \|x^*\|_*$ and $|x_n^*(a)| \to |x^*(a)|$.

Finally, we prove that $\|x^* + x_n^*\|_* \to 2\|x^*\|_*$. We have:

$$\begin{aligned} \|x^* + x_n^*\|_1^* &= \big[\|x^* + x_n^*\|_*^q + |(x^* + x_n^*)(a)|^q\big]^{\frac{1}{q}} \\ &\leq \big[(\|x^*\|_* + \|x_n^*\|_*)^q + (|x^*(a)| + |x_n^*(a)|)^q\big]^{\frac{1}{q}} \\ &\to \big[(2\|x^*\|_*)^q + (2|x^*(a)|_*)^q\big]^{\frac{1}{q}} = 2\|x^*\|_1^* = 2, \end{aligned}$$

and we know that $\|x^* + x_n^*\|_1^*/2 \leq [\|x^*\|_1^* + \|x_n^*\|_1^*]/2 \to \|x^*\|_1^*$, and $\|x^* + x_n^*\|/2 \to 1 = (\|x^*\|_*^q + |x^*(a)|^q)^{1/q}$. Since $\|x_n^* + x^*\|_* \leq \|x_n^*\|_* + \|x^*\|_* \to 2\|x^*\|_*$ and $|x^*(a) + x_n^*(a)| \leq |x^*(a)| + |x_n^*(a)| \to 2|x^*(a)|$. We deduce that $\|x^* + x_n^*\|_* \to 2\|x^*\|_*$. By hypothesis, $\|\cdot\|_*$ is locally uniformly convex, then, $\|x^* - x_n^*\|_* \to 0$.

In the case where the norm $\|\cdot\|$ is such that its dual norm is strictly convex, the same proof shows that $\|\cdot\|_1$ is an equivalent norm in X such that its dual norm is strictly convex.

Therefore, the proof of our lemma is complete. ∎

It is well known that if the norm $\|\cdot\|$ is such that its dual norm is locally uniformly convex (resp. strictly convex) in X^*, then the norm $\|\cdot\|$ is Fréchet-differentiable (resp. Gâteaux-differentiable) in $X \setminus \{0\}$, (see [2]).

Recall that a convex subset C (0 is in the interior of C) is said to be *Fréchet-smooth* (resp. *Gâteaux-smooth*) if the Minkowski functional of C is Fréchet-differentiable (resp. Gâteaux-differentiable) in $X \setminus \{0\}$.

In [7], it was shown that: Let $(X, \|\cdot\|)$ be a Banach space such that its dual norm is l.u.c. (resp. s.c.). Let S be a closed subset at positive distance from the unit ball. Then there exist a Fréchet-smooth (resp. Gâteaux-smooth) drop D such that $D \cap S$ is a singleton.

Combining Theorem 3.1 and Lemma 3.2, one can give this version of the smooth drop theorem.

Corollary 3.3. *Let $(X, \|\cdot\|)$ be a reflexive Banach space where the norm is strictly convex and have the Kadeč-Klee property. Assume that the dual norm is locally uniformly convex (resp. strictly convex) in X^*. Let S be a closed subset at positive distance from the unit ball. Then there exists a Fréchet-smooth (resp. Gâteaux-smooth) drop D such that $D \cap S$ is a singleton.*

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