EXTRACTA MATHEMATICAE Vol. **17**, Núm. 2, 239 – 246 (2002)

Moufang $H^\star$-algebras

José Antonio Cuenca Mira*

Departamento de Álgebra, Geometría y Topología, Facultad de Ciencias, Universidad de Málaga, 29080 *Málaga, Spain*

(Research paper presented by A. Rodríguez Palacios)

AMS *Subject Class.* (2000): 17A60, 17D05, 46K70 *Received July* 30, 2001

We recall that a nonassociative algebra V over $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$) is said to be an $H^\star$-algebra when V is also a Hilbert space over $\mathbb{K}$ (the inner product of which is denoted by $(\ |\ )$) and is endowed with an algebra involution $\star$ which is linear in the real case and conjugate-linear in the complex case, and satisfies

$$(xy|z) = (x|zy^\star) = (y|x^\star z)$$

for all x, y, $z \in V$. $H^\star$-algebras were introduced by W. Ambrose [2] in a complex associative setting in a slightly different way. The real associative case was studied by I. Kaplansky [39] (see also [5, 10, 18]). Today the structure theory of several particular classes of nonassociative $H^\star$-algebras is well-known [47, 48, 3, 4, 35, 49, 43, 36, 37, 25, 32, 26, 29, 7, 11, 12, 34, 8, 22, 30, 42, 33]. There is also a complete determination of a wide class of ternary $H^\star$-structures [40, 41, 18, 12, 20, 17, 16, 23, 24, 58], as well as a germinal theory for both binary and ternary arbitrary $H^\star$-structures (see [32, 34, 31, 38, 6, 9, 14, 15, 56, 50, 13, 54, 53, 45] and [19, 21, 57, 51, 52, 58], respectively). The reader is referred to Section E of [44] for a complete survey on $H^\star$-theory.

Let V be an $H^\star$-algebra. Then the left annihilator

$$\operatorname{Lann} V = \{x \in V :\ xV = 0\}$$

coincides with the right annihilator

$$\operatorname{Rann} V = \{x \in V :\ Vx = 0\}$$

* This work was partially supported by DGYCIT (PB97-1497), AECI and PAI (FQM-0125).

[32, Proposition 2]. We denote it by $\operatorname{Ann} V$. There is a decomposition of V as an orthogonal direct sum

$$V = \overline{\operatorname{Lin} V^2} \oplus \operatorname{Ann} V\,, \tag{1}$$

where $\overline{\operatorname{Lin} V^2}$ is the closed linear span of the set $\{xy : x, y \in V\}$ (see [32, Proposition 2] and [34, Theorem 1]). Moreover $\overline{\operatorname{Lin} V^2}$, under a suitable involution and the restriction of the inner product and the algebraic operations of V, is an $H^\star$-algebra with zero annihilator in itself [25, 32, 34]. On the other hand, if V has zero annihilator, then V splits as an orthogonal direct sum

$$V = \overline{\bigoplus_{\alpha\in\mathcal{A}} I_\alpha}\,, \tag{2}$$

where $\{I_\alpha\}$ is the family of all the minimal closed ideals of V, being furthermore every I_α a topologically simple $H^\star$-algebra with the structure inherited from V [32, 34, 38]. We recall that an $H^\star$-algebra W is called *topologically simple* if it has nonzero product and its unique closed two-sided ideals are 0 and W.

The decompositions given by (1) and (2) reduce the study of every class of nonassociative $H^\star$-algebras defined by identities to the determination of those $H^\star$-algebras in the class that are topologically simple. This note is devoted to the description of $H^\star$-algebras satisfying some of the Moufang identities. We recall that an *alternative algebra* is a nonassociative algebra satisfying the alternative identities given by

$$x^2y = x(xy), \qquad (yx)x = yx^2.$$

It is well known that every alternative algebra satisfies the following identities

$$y((xz)x) = ((yx)z)x \tag{3}$$
$$(xy)(zx) = (x(yz))x \tag{4}$$
$$(x(yx))z = x(y(xz)) \tag{5}$$

(see [55, 46]). These identities are called *right*, *middle*, and *left Moufang* identity, respectively.

In this note we prove that $H^\star$-algebras satisfying some of the Moufang identities are in fact alternative. A similar result on alternativeness of middle Moufang algebras has been recently obtained in [27] and [28] for division and composition middle Moufang algebras. Applying among other results a deep theorem of M. Slater [55, Theorem 9, p. 194], alternative $H^\star$-algebras can be completely described [44, p. 148] (see also [43]). It follows that the result in this note concludes the structure theory of Moufang $H^\star$-algebras.

1. The results

Let A be a nonassociative algebra over a field F and $g : A \times A \longrightarrow F$ a symmetric bilinear form. We recall that g is said to be *associative* if

$$g(xy, z) = g(x, yz)$$

for all x, y, $z \in A$.

Lemma 1.1. *Let A be a nonassociative algebra over a field F such that there exists a nondegenerate associative symmetric bilinear form on A. Then the following assertions are equivalent:*

(i) *A satisfies the right Moufang identity.*
(ii) *A satisfies the middle Moufang identity.*
(iii) *For all x, z, $u \in A$ the equality $((xz)x)u = x(z(xu))$ holds.*

Proof. Let $g : A \times A \longrightarrow F$ be a nondegenerate associative symmetric bilinear form, and let x, y, z, u be in A. Then we have

$$\begin{aligned} g\Big(y\big((xz)x\big) - \big((yx)z\big)x, u\Big) &= g\big((xz)x, uy\big) - g\big(z, (xu)(yx)\big) \\ &= g\Big(z, \big(x(uy)\big)x - (xu)(yx)\Big)\,. \end{aligned} \tag{6}$$

On the other hand,

$$\begin{aligned} g\Big(z, \big(x(uy)\big)x - (xu)(yx)\Big) &= g\big(xz, x(uy)\big) - g\big(z(xu), yx\big) \\ &= g\Big(\big((xz)x\big)u - x\big(z(xu)\big), y\Big)\,. \end{aligned} \tag{7}$$

Since g is nondegenerate, (6) and (7) yield the equivalence of Assertions (i), (ii), and (iii). ∎

Theorem 1.2. *Let V be an $H^\star$-algebra. Then the following assertions are equivalent:*

(i) *V satisfies the right Moufang identity.*
(ii) *V satisfies the middle Moufang identity.*
(iii) *For all x, z, $u \in V$ the equality $((xz)x)u = x(z(xu))$ holds.*
(iv) *V satisfies the left Moufang identity.*
(v) *V is an alternative algebra.*

Proof. Let $g : V \times V \longrightarrow \mathbb{K}$ be the symmetric bilinear form defined by $g(x,y) = (x|y^\star) + (y|x^\star)$ for all x, $y \in V$. A routine calculation shows that g is associative and nondegenerate. Now Lemma 1.1 yields the equivalence of (i), (ii), and (iii).

Assume that the equivalent Conditions (i) and (iii) are fulfilled. By (i), for x, y, z, $t \in V$ we have

$$\begin{aligned}((xy)x - x(yx) \,|\, zt) &= \Big(z^\star\big((xy)x\big) \,|\, t\Big) - \Big(z^\star\big(x(yx)\big) \,|\, t\Big) \\ &= \Big(\big((z^\star x)y\big)x \,|\, t\Big) - \Big(z^\star \,|\, t\big(x(yx)\big)^\star\Big) \\ &= \Big(z^\star \,|\, \big((tx^\star)y^\star\big)x^\star\Big) - \Big(z^\star \,|\, \big((tx^\star)y^\star\big)x^\star\Big) = 0\,.\end{aligned}$$

Without loss of generality we can assume that $\operatorname{Ann} V = \{0\}$. Then, since the linear span of $\{zt : z,\, t \in V\}$ is dense in V (see (1)), and $(\,\cdot\,|\,\cdot\,)$ is nondegenerate, we obtain that $(xy)x = x(yx)$ for all x, $y \in V$. In this way we have proved that V is a flexible algebra. By (iii), V is left Moufang.

Now assume that Condition (iv) is fulfilled. Since the opposite algebras of left Moufang $H^\star$-algebras are right Moufang $H^\star$-algebras, and these last algebras are flexible, we obtain that V is also flexible. Then, applying again that V is left Moufang, we realize that Assertion (iii) holds for V.

Finally, we assume that V satisfies the equivalent Assertions (i)-(iv), and show that V is alternative. In view of (1), (2), and [9, Theorem 1], we can assume that V is topologically simple and complex. We already know that V is flexible, and that, consequently, for x in V the equality $x^2x = xx^2$ holds. On the other hand, by Assertion (ii), we have $x^2x^2 = (xx^2)x$ for every $x \in V$. Since there exists a nondegenerate associative symmetric bilinear form on V, it follows from [1] (see also [25, p. 39]) that V is a noncommutative Jordan $H^\star$-algebra. Then, by [32, Theorems 2 and 4] and [2], either V is anticommutative or there exists an approximate unit for V. Assume that V is anticommutative. Then, by Assertion (i), we have $R_x^3 = 0$ for every x in V, where R_x stands for the operator of right multiplication by x on V. Therefore, if x is in V and satisfies $x^\star = \varepsilon x$ for $\varepsilon \in \{+,-\}$, then for every y in V we have $0 = (R_x^4(y)|y) = \|R_x^2(y)\|^2$, and hence $0 = (R_x^2(y)|y) = \varepsilon\|R_x(y)\|^2$, so that $x \in \operatorname{Ann} V = \{0\}$. Since $V = \operatorname{Sym}(V,\star) \oplus \operatorname{Sk}(V,\star)$, where $\operatorname{Sym}(V,\star)$ (respectively, $\operatorname{Sk}(V,\star)$) means the real vector subspace of all self-adjoint (respectively, skew-adjoint) elements of V, we deduce $V = 0$, which contradicts that V is topologically simple. Now assume that V has an approximate unit (say $\{e_\lambda\}_{\lambda\in\Lambda}$). Then, making $z = e_\lambda$ in (3) and $y = e_\lambda$ in (5), and taking limits in λ, the alternativeness of V follows. ∎

As we said in the introduction, alternative $H^\star$-algebras are well-understood. Indeed, *the topologically simple alternative nonassociative $H^\star$-algebras are: the algebra of complex octonions, in the complex case; and the same algebra regarded as a real algebra, together with the two real octonions algebras, in the real case* (see [44, p. 148]). On the other hand *every topologically simple associative $H^\star$-algebra is isomorphic to the $H^\star$-algebra $\mathcal{HS}(H)$ of all Hilbert-Schmidt operators on a suitable Hilbert space H, which is complex in the complex case, and either real, complex, or quaternionic in the real case* (see [2] and [39]).

References

[1] Albert, A.A., Power associative rings, *Trans. Amer. Math. Soc.*, **64** (1948), 552 – 593.

[2] Ambrose, W., Structure theorems for a special class of Banach algebras, *Trans. Amer. Math. Soc.*, **57** (1945), 364 – 386.

[3] Balachandran, V.K., Simple $L^\star$-algebras of classical type, *Math. Ann.*, **180** (1969), 205 – 219.

[4] Balachandran, V.K., Real $L^\star$-algebras, *Indian J. Pure Appl. Math.*, **3** (1972), 1224 – 1246.

[5] Balachandran, V.K., Swaminathan, N., Real $H^\star$-algebras, *J. Funct. Anal.*, **65** (1986), 64 – 75.

[6] Bensebah, A., JV-algebras et $JH^\star$-algèbres, *Canad. Math. Bull.*, **34** (1991), 447 – 455.

[7] Cabrera, M., Marrackchi, A. El, Martínez, J., Rodríguez,A., An Allison-Kantor-Koecher-Tits construction for Lie $H^\star$-algebras, *J. Algebra*, **164** (2) (1994), 361 – 408.

[8] Cabrera, M., Martínez, J., Rodríguez, A., Malcev $H^\star$-algebras, *Math. Proc. Camb. Phil. Soc.*, **103** (1988), 463 – 471.

[9] Cabrera, M., Martínez, J., Rodríguez, A., Nonassociative real $H^\star$-algebras, *Publicacions Matematiques*, **32** (1988), 267 – 274.

[10] Cabrera, M., Martínez, J., Rodríguez, A., A note on real $H^\star$-algebras, *Math. Proc. Camb. Phil. Soc.*, **105** (1989), 131 – 132.

[11] Cabrera, M., Martínez, J., Rodríguez, A., Structurable $H^\star$-algebras, *J. Algebra*, **147** (1992), 19 – 62.

[12] Cabrera, M., Martínez, J., Rodríguez, A., Hilbert modules over $H^\star$-algebras in relation with Hilbert ternary rings, in "Nonassociative Algebraic Models", S. González and H. Ch. Myung (editors), Nova Science Publishers, New York, 1992, 33 – 44.

[13] Cabrera, M., Martínez, J., Rodríguez, A., Bounded differential operators on Hilbert modules and derivations of structurable $H^\star$-algebras, *Comm. Algebra*, **21** (1993), 2905 – 2945.

[14] Cabrera, M., Rodríguez, A., Extended centroid and central closure of semiprime normed algebras: a first approach, *Comm. Algebra*, **18** (1990), 2293 – 2326.

[15] Cabrera, M., Rodríguez, A., Nonassociative ultraprime normed algebras, *Quart. J. Math. Oxford*, **43** (1992), 1 – 7.
[16] Castellón, A., Cuenca, J.A., Compatibility in Jordan $H^\star$-triple systems, *Bolletino U.M.I.*, **4** (B) (1990), 433 – 447.
[17] Castellón, A., Cuenca, J.A., Alternative $H^\star$-triple systems, *Comm. Algebra*, **11** (20) (1992), 3191 – 3260.
[18] Castellón, A., Cuenca, J.A., Associative $H^\star$-triple systems, in "Nonassociative Algebraic Models", S. González and H. Ch. Myung (editors), Nova Science Publishers, New York, 1992, 45 – 67.
[19] Castellón, A., Cuenca, J.A., Isomorphisms of $H^\star$-triple systems, *Annali Della Scuola Normale Superiore di Pisa. Ser. IV*, **XIX** (4) (1992), 507 – 514.
[20] Castellón, A., Cuenca, J.A., Martín, C., Ternary $H^\star$-algebras, *Bolletino U.M.I.*, **6-B** (7) (1992), 217 – 228.
[21] Castellón, A., Cuenca, J.A., Centroid and metacentroid of an $H^\star$-triple system, *Bulletin de la Societé Mathematique de Belgique*, **45** (fascicules 1-2 serie A) (1993), 85 – 91.
[22] Castellón, A., Cuenca, J.A., Martín, C., Applications of ternary $H^\star$-algebras to associative $H^\star$-superalgebras, *Algebras, Groups and Geometries*, **10** (1993), 181 – 190.
[23] Castellón, A., Cuenca, J.A., Martín, C., Jordan $H^\star$-triple systems, in "Nonassociative Algebras and its Applications", S. González (editor), Kluwer, Dordrecht-Boston-New York, 1994, 66 – 72.
[24] Castellón, A., Cuenca, J.A., Martín, C., Special Jordan $H^\star$-triple systems, *Comm. Algebra*, **28** (10) (2000), 4699 – 4706.
[25] Cuenca, J.A., Sobre $H^\star$-álgebras no asociativas. Teoría de estructura de las $H^\star$-álgebras de Jordan no conmutativas semisimples, Secretariado de Publicaciones de la Universidad de Málaga, Málaga, 1984.
[26] Cuenca, J.A., Sur la theorie de structure des $H^\star$-algèbres de Jordan non commutatives, *Ann. Sci. Univ. Clermont-Ferrand II Math.*, **27** (1991), 143 – 152.
[27] Cuenca, J.A., On division algebras satisfying Moufang identities, preprint.
[28] Cuenca, J.A., Composition algebras satisfying Moufang identities, preprint.
[29] Cuenca, J.A., García, A., Martín, C., Structure theory for $L^\star$-algebras, *Math. Proc. Camb. Phil. Soc.*, **107** (1990), 361 – 365.
[30] Cuenca, J.A., Martín, C., Jordan two-graded $H^\star$-álgebras,in "Nonassociative Algebraic Models", S. González and H. Ch. Myung (editors), Nova Science Publishers, New York, 1992, 91 – 100.
[31] Cuenca, J.A., Rodríguez, A., Isomorphisms of $H^\star$-algebras, *Math. Proc. Camb. Phil. Soc.*, **97** (1985), 93 – 99.
[32] Cuenca, J.A., Rodríguez, A., Structure theory for noncommutative Jordan $H^\star$-algebras, *J. Algebra*, **106** (1987), 1 – 14.
[33] Cuenca, J.A., Rodríguez, A., Absolute values on $H^\star$-algebras, *Comm. Algebra*, **23** (5) (1995), 2595 – 2610.
[34] Cuenca, J.A., Sánchez, A., Structure theory for real noncommutative Jordan $H^\star$-algebras, *J. Algebra*, **164** (1994), 481 – 499.

[35] DE LA HARPE, P., Classification des $L^\star$-algebres semi-simples réelles separables, *C.R. Acad. Sci*, Ser A, **272** (1971), 1559 – 1561.

[36] DEVAPAKKIAM, C. VIOLA, Hilbert space methods in the theory of Jordan algebras I, *Math. Proc. Camb. Phil. Soc.*, **78** (1975), 293 – 300.

[37] DEVAPAKKIAM, C. VIOLA, REMA, P.S., Hilbert space methods in the theory of Jordan algebras II, *Math. Proc. Camb. Phil. Soc.*, **79** (1976), 307 – 319.

[38] FERNÁNDEZ, A., RODRÍGUEZ, A., A Wedderburn theorem for nonassociative complete normed algebras, *J. London Math. Soc.*, **33** (1986), 328 – 338.

[39] KAPLANSKY, I., Dual rings, *Ann. of Math.*, **49** (3) (1948), 689 – 701.

[40] KAUP, W., Über die klassifikation der symmetrischen hermiteschen mannigfaltigkeiten unendlicher dimension II, *Math. Ann.*, **262** (1983), 57 – 75.

[41] NEHER, E., "Jordan Triple Systems by the Grid Approach", Springer-Verlag, Berlin-Heidelberg-New York, 1987.

[42] NEHER, E., Generators and relations for 3-graded Lie algebras, *J. Algebra*, **155** (1) (1993), 1 – 35.

[43] PÉREZ DE GUZMÁN, I., Structure theorems for alternative $H^\star$-algebras, *Math. Proc. Camb. Phil. Soc.*, **94** (1983), 437 – 446.

[44] RODRÍGUEZ, A., Jordan structures in Analysis, in "Jordan Algebras", K. McCrimmon, W. Kaup, and H. P. Petersson (editors), Walter de Gruyter, Berlin, 1994, 97 – 186.

[45] RODRÍGUEZ, A., Continuity of densely valued homomorphisms into $H^\star$-algebras, *Quart. J. Math. Oxford*, **46** (1995), 107 – 118.

[46] SCHAFER, R.D., "An Introduction to Nonassociative Algebras", Academic Press, New York-San Francisco-London, 1966.

[47] SCHUE, J.R., Hilbert space methods in the theory of Lie algebras, *Trans. Amer. Math. Soc.*, **95** (1960), 68 – 80.

[48] SCHUE, J.R., Cartan decompositions for $L^\star$-algebras, *Trans. Amer. Math. Soc.*, **98** (1961), 334 – 349.

[49] UNSAIN, I., Classification of the simple separable real $L^\star$- algebras, *J. Differential Geom.*, **7** (106) (1972), 423 – 451.

[50] VILLENA, A., Continuity of derivations on $H^\star$-algebras, *Proc. Amer. Math. Soc.*, **122** (1994), 821 – 826.

[51] VILLENA, A.R., ZALAR, B., ZOHRY, M., Continuity of derivations pairs on Hilbert triple systems, *Bollettino U.M.I.*, **7** (9-B) (1995), 459 – 477.

[52] VILLENA, A., ZHORY, M., Continuity of homomorphism pairs into $H^\star$-triple systems, *Proc. (Math. Sci.) I. A. Sc.*, **106** (1996), 79 – 90.

[53] VILLENA, A., ZHORY, M., Rings isomorphisms of $H^\star$-algebras, *J. Algebra*, **181** (1996), 329-346.

[54] VILLENA, A., ZHORY, M., Rings derivations on $H^\star$-algebras, *Bollettino U.M.I.*, **11-A** (1997), 129 – 138.

[55] ZHEVLAKOV, K.A., SLIN'KO, A.M., SHESTAKOV, I.P., SHIRSHOV, A.I., "Rings that are Nearly Associative", Academic Press, New York, 1982.

[56] ZALAR, B., Continuity of derivations on Malcev $H^\star$-algebras, *Math. Proc. Camb. Phil. Soc.*, **110** (1991), 455 – 459.

[57] Zalar, B., Continuity of derivations pairs on alternative and Jordan $H^\star$-triples, *Boll. Un. Mat. Ital.*, **7** (B) (1993), 215 – 230.
[58] Zalar, B., Theory of Hilbert triple systems, *Yokohama Math. J.*, **41** (1994), 95 – 126.