Darbouxian Integrability for Polynomial Vector Fields on the 2–Dimensional Sphere [†]

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(Research paper)

1. Introduction

In 1878 Darboux [6] showed how can be constructed the first integrals of planar polynomial vector fields possessing sufficient invariant algebraic curves. In particular, he proved that if a planar polynomial vector field of degree m has at least $[m(m+1)/2] + 1$ invariant algebraic curves, then it has a first integral, which has an easy expression in function of the invariant algebraic curves, see Theorem 2(b). Jouanolou [7] in 1979 (see also [5]) shows that if the number of invariant algebraic curves of a planar polynomial vector field of degree m is at least $[m(m+1)/2] + 2$, then the vector field has a rational first integral, and consequently all its solutions are invariant algebraic curves, see Theorem 2(c). For more details and results on the Darbouxian theory of integration for planar polynomial vector fields, see [1, 3, 4, 5, 8].

In another context we must mention the good extensions of the Darbouxian method to dimension larger than 2 for differential polynomial vector fields on k^n being k a differential field of zero characteristic, see for instance [9] and the references quoted there; or extensions to algebraic Pfaff equations, see [7].

Our main goal is to extend to polynomial vector fields on the 2–dimensional sphere the Darbouxian theory of integrability for the planar polynomial vector fields. In this sense this paper can be thought as the natural continuation

of [5] where we improve the classical results of Darbouxian theory for planar polynomial vector fields, essentially these improvements are presented in Theorem 2.

The paper is organized as follows. In section 2 we present a summary of the main results of the Darbouxian theory of integrability for the planar polynomial vector fields that we will extend to polynomial vector fields on the 2–dimensional sphere. In Section 3 we give the definitions of polynomial vector field on the 2–dimensional sphere and of a first integral for such vector fields. In Sections 4 and 5 we introduce the notion of invariant algebraic curve and of exponential factor for a polynomial vector field on the 2–dimensional sphere, respectively. In Section 6 we describe the expressions of the differential equations associated to a polynomial vector field on the 2–dimensional sphere through the stereographic projection. Finally, in Section 7 we present the Darbouxian theory of integration for polynomial vector fields on the 2–dimensional sphere, see Theorem 5.

2. The method of Darboux for planar polynomial vector fields

As far as we know, the problem of integrating a polynomial vector field by using its invariant algebraic curves was started to be considered by Darboux in [6] for the planar vector fields. The version that we present here improves Darboux's original exposition essentially because here we also take into account the exponential factors (see [5] for more details and proofs) and the independent singular points (see [2]). Before stating the main results of the Darbouxian theory for planar polynomial vector fields we need some definitions.

Let

$$Y = p(x,y)\frac{\partial}{\partial x} + q(x,y)\frac{\partial}{\partial y}.$$

be a planar polynomial vector field of degree m, i.e., p and q are polynomials in the variables x and y with coefficients in the real field $\mathbb{R}$, and $m = \max\{\deg p, \deg q\}$.

Let $f \in \mathbb{R}[x,y]$, where $\mathbb{R}[x,y]$ denotes the ring of all polynomials in the variables x and y with coefficients in $\mathbb{R}$. The algebraic curve $f = 0$ is an *invariant algebraic curve* of the polynomial vector field Y if for some polynomial $K \in \mathbb{R}[x,y]$ we have

$$Yf = p\frac{\partial f}{\partial x} + q\frac{\partial f}{\partial y} = Kf.$$

The polynomial K is called the *cofactor* of the invariant algebraic curve $f = 0$.

We note that since the polynomial system has degree m, then any cofactor has at most degree $m - 1$.

Let $h, g \in \mathbb{R}[x, y]$ and assume that h and g are relatively prime in the ring $\mathbb{R}[x, y]$. Then the function $\exp(g/h)$ is called an *exponential factor* of the polynomial vector field Y if for some polynomial $K \in \mathbb{R}[x, y]$ of degree at most $m - 1$ it satisfies the equality

$$Y\big(\exp(g/h)\big) = K \exp(g/h).$$

As before we say that K is the *cofactor* of the exponential factor $\exp(g/h)$.

It is well–known that if $F = \exp(g/h)$ is an exponential factor for the polynomial vector field Y and h is non–constant, then $h = 0$ is an invariant algebraic curve, and g satisfies the equation

$$Yg = gK_h + hK_F,$$

where K_h and K_F are the cofactors of h and F respectively. In fact, in [3] or [5] it is shown that the existence of a exponential factor $\exp(g/h)$ is due to the fact the invariant algebraic curve as solution of the vector field Y has multiplicity higher than 1.

The polynomial vector field Y is *integrable* on an open subset V of $\mathbb{R}^2$ if there exists a nonconstant analytic function $H : V \to \mathbb{R}$, called a *first integral* of the system on V, which is constant on all solution curves $(x(t), y(t))$ of the vector field Y on V; i.e., $H(x(t), y(t)) = \text{constant}$ for all values of t for which the solution $(x(t), y(t))$ is defined on V. Clearly H is a first integral of the vector field Y on V if and only if $YH = 0$ on V.

Let V be an open subset of $\mathbb{R}^2$ and let $R : V \to \mathbb{R}$ be an analytic function which is not identically zero on V. The function R is an *integrating factor* of the polynomial vector field Y on V if one of the following three equivalent conditions holds

$$\frac{\partial(Rp)}{\partial x} = -\frac{\partial(Rq)}{\partial y}, \qquad \text{div}(Rp, Rq) = 0, \qquad YR = -R \, \text{div}(p, q),$$

on V. As usual the divergence of the vector field Y is defined by

$$\text{div}(Y) = \text{div}(p, q) = \frac{\partial p}{\partial x} + \frac{\partial q}{\partial y}.$$

The first integral H associated to the integrating factor R is given by

$$H(x, y) = \int R(x, y)p(x, y)dy + h(x),$$

satisfying $\frac{\partial H}{\partial x} = -Rq$. Then

$$\dot{x} = Rp = \frac{\partial H}{\partial y}, \qquad \dot{y} = Rq = -\frac{\partial H}{\partial x}. \tag{1}$$

In order that this function H to be well defined the open set V must be simply connected.

Conversely, given a first integral H of the vector field Y we always can find an integrating factor R for which (1) holds.

LEMMA 1. *If the polynomial vector field Y has two integrating factors R_1 and R_2 on the open subset V of $\mathbb{R}^2$, then in the open set $V \setminus \{R_2 = 0\}$ the function R_1/R_2 is a first integral.*

Proof. Since R_i is an integrating factor, it satisfies that $YR_i = -R_i \operatorname{div}(p,q)$ for $i = 1,2$. Therefore, the lemma follows immediately from the next computations:

$$Y\left(\frac{R_1}{R_2}\right) = \frac{(YR_1)R_2 - R_1(YR_2)}{R_2^2} = 0. \qquad \blacksquare$$

If $S(x,y) = \sum_{i+j=0}^{m-1} a_{ij} x^i y^j$ is a polynomial of degree $m-1$ with $m(m+1)/2$ coefficients in $\mathbb{R}$, then we write $S \in \mathbb{R}_{m-1}[x,y]$. We identify the linear vector space $\mathbb{R}_{m-1}[x,y]$ with $\mathbb{R}^{m(m+1)/2}$ through the isomorphism

$$S \to (a_{00}, a_{10}, a_{01}, \ldots, a_{m-1,0}, a_{m-2,1}, \ldots, a_{0,m-1}).$$

We say that r points $(x_k, y_k) \in \mathbb{R}$, $k = 1, \ldots, r$, are *independent* with respect to $\mathbb{R}_{m-1}[x,y]$ if the intersection of the r hyperplanes

$$\sum_{i+j=0}^{m-1} x_k^i y_k^j a_{ij} = 0, \qquad k = 1, \ldots, r,$$

in $\mathbb{R}^{m(m+1)/2}$ is a linear subspace of dimension $[m(m+1)/2] - r$.

We summarize in the next theorem the main results of the Darbouxian theory of integrability for planar polynomial vector fields.

THEOREM 2. *Suppose that the planar polynomial vector field Y of degree m admits p invariant algebraic curves $f_i = 0$ with cofactors K_i for $i = 1, \ldots, p$,*

q exponential factors $\exp(g_j/h_j)$ with cofactors L_j for $j = 1, \ldots, q$, and r independent singular points $(x_k, y_k) \in \mathbb{R}^2$ such that $f_i(x_k, y_k) \neq 0$ for $i = 1, \ldots, p$ and for $k = 1, \ldots, r$. Of course, every h_j is equal to some f_i except if h_j is constant. Then the following statements hold.

(a) There exist $\lambda_i, \mu_j \in \mathbb{R}$ not all zero such that $\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = 0$, if and only if the function

$$|f_1|^{\lambda_1} \cdots |f_p|^{\lambda_p} \big(\exp(g_1/h_1) \big)^{\mu_1} \cdots \big(\exp(g_q/h_q) \big)^{\mu_q} \tag{2}$$

   is a first integral of the vector field Y.

(b) If $p + q + r = [m(m+1)/2] + 1$, then there exist $\lambda_i, \mu_j \in \mathbb{R}$ not all zero such that $\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = 0$.

(c) The vector field Y has a rational first integral if and only if $p + q + r \geq [m(m+1)/2] + 2$. Moreover, all trajectories of Y are contained in invariant algebraic curves.

3. Polynomial vector fields on $\mathbb{S}^2$ and first integrals

A *polynomial vector field* X in $\mathbb{R}^3$ is a vector field of the form

$$X = P(x, y, z)\frac{\partial}{\partial x} + Q(x, y, z)\frac{\partial}{\partial y} + R(x, y, z)\frac{\partial}{\partial z},$$

where P, Q and R are polynomials in the variables x, y and z with real coefficients. In all this paper $m = \max\{\deg P, \deg Q, \deg R\}$ will denote the *degree* of the polynomial vector field X. In what follows X will denote the above polynomial vector field.

Let $\mathbb{S}^2$ be the 2–dimensional sphere $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$. A *polynomial vector field X on $\mathbb{S}^2$* is a polynomial vector in $\mathbb{R}^3$ such that restricted to the sphere $\mathbb{S}^2$ defines a vector field on $\mathbb{S}^2$, i.e., it must satisfy the following equality

$$xP(x, y, z) + yQ(x, y, z) + zR(x, y, z) = 0,$$

for all the points (x, y, z) of the sphere $\mathbb{S}^2$.

We denote by $\mathbb{R}_m[x, y, z]$ the real linear vector space of all polynomials in the variables x, y, z with real coefficients and of degree at most m. It is easy to see that the dimension of $\mathbb{R}_m[x, y, z]$ is $(m+2)(m+1)m/6$.

By $\mathbb{R}_m[x, y, z = \sqrt{1 - x^2 - y^2}]$ we denote the real linear vector space of all the functions obtained from all polynomial of $\mathbb{R}_m[x, y, z]$ after substituting z by $\sqrt{1 - x^2 - y^2}$. Since this vector space can be obtained adding to a polynomial of $\mathbb{R}_m[x, y]$ another polynomial of $\mathbb{R}_{m-1}[x, y]$ multiplied by $\sqrt{1 - x^2 - y^2}$, it follows that a basis for the vector space $\mathbb{R}_m[x, y, z = \sqrt{1 - x^2 - y^2}]$ can be obtained adding to the basis $\{1, x, y, z, \ldots, z^m\}$ of $\mathbb{R}_m[x, y]$ the following independent vectors

$$\sqrt{1 - x^2 - y^2}, \ x\sqrt{1 - x^2 - y^2}, \ y\sqrt{1 - x^2 - y^2},$$
$$z\sqrt{1 - x^2 - y^2}, \ \ldots, \ z^m\sqrt{1 - x^2 - y^2}.$$

Therefore, the dimension of the linear vector space $\mathbb{R}_m[x, y, z = \sqrt{1 - x^2 - y^2}]$ is

$$\frac{(m + 2)(m + 1)}{2} + \frac{(m + 1)m}{2} = (m + 1)^2.$$

PROPOSITION 3. *For each point of $\mathbb{R}^d$ with*

$$d = \frac{(m + 3)(m + 2)(m + 1)}{2} - (m + 2)^2 = \frac{(m + 2)(m^2 + 2m - 1)}{2},$$

there is a different polynomial vector field X on the sphere $\mathbb{S}^2$ of degree m.

Proof. If P, Q and R are polynomials of degree m, then the identity

$$xP(x, y, z) + yQ(x, y, z) + zR(x, y, z)|_{z = \sqrt{1 - x^2 - y^2}} \equiv 0,$$

defines an element of the linear vector space $\mathbb{R}_{m+1}[x, y, z = \sqrt{1 - x^2 - y^2}]$. So this identity implies $(m + 2)^2$ relations between the coefficients of the polynomials P, Q and R.

Since each polynomial P, Q and R has $(m+3)(m+2)(m+1)/6$ coefficients, and every polynomial vector field X on $\mathbb{S}^2$ must satisfy the above identity, the statement of the proposition follows. ∎

Since for each point of $\mathbb{R}^{(m+2)(m+1)}$ there is a planar polynomial vector field of degree m, from Proposition 3, it follows that for $m > 1$ there are more polynomial vector fields on the sphere $\mathbb{S}^2$ of degree m than planar polynomial vector fields of degree m.

Let U be an open subset of $\mathbb{R}^3$. A polynomial vector field X on the sphere $\mathbb{S}^2$ is *integrable* on the open subset $U \cap \mathbb{S}^2$ if there exists a nonconstant analytic function $H : U \to \mathbb{R}^3$, called a *first integral* of X on $U \cap \mathbb{S}^2$, which

is constant on all solution curves $(x(t), y(t), z(t))$ of the vector field X on $U \cap \mathbb{S}^2$; i.e., $H(x(t), y(t), z(t)) = $ constant for all values of t for which the solution $(x(t), y(t), z(t))$ is defined on $U \cap \mathbb{S}^2$. Clearly H is a first integral of the polynomial vector field X on $U \cap \mathbb{S}^2$ if and only if $XH = 0$ on all the points (x, y, z) of $U \cap \mathbb{S}^2$.

4. Invariant algebraic curves

Let $f \in \mathbb{R}[x, y, z]$, where as it is usual $\mathbb{R}[x, y, z]$ denotes the ring of the polynomials in the variables x, y and z with real coefficients. The algebraic surface $f = 0$ defines an *invariant algebraic curve* $\{f = 0\} \cap \mathbb{S}^2$ of the polynomial vector field X on the sphere $\mathbb{S}^2$ if

(i) for some polynomial $K \in \mathbb{R}[x, y, z]$ we have

$$X f = P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} + R\frac{\partial f}{\partial z} = Kf, \tag{3}$$

on all the points (x, y, z) of the sphere $\mathbb{S}^2$;

(ii) the intersection of the two surfaces $f = 0$ and $\mathbb{S}^2$ is transversal; i.e., for all the points $(x, y, z) \in \{f = 0\} \cap \mathbb{S}^2$ we have that

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} \neq 0.$$

The polynomial K is called the *cofactor* of the invariant algebraic curve $\{f = 0\} \cap \mathbb{S}^2$. We note that since the polynomial vector field has degree m, then any cofactor has at most degree $m - 1$.

Since on the points of the algebraic curve $\{f = 0\} \cap \mathbb{S}^2$ the gradient $(\partial f/\partial x, \partial f/\partial y, \partial f/\partial z)$ of the surface $f = 0$ is orthogonal to the polynomial vector field $X = (P, Q, R)$ (see (3)), and the vector field X is tangent to the sphere $\mathbb{S}^2$, it follows that the vector field X is tangent to the curve $\{f = 0\} \cap \mathbb{S}^2$. Hence, the curve $\{f = 0\} \cap \mathbb{S}^2$ is formed by trajectories of the vector field X. This justifies the name of invariant algebraic curve given to the algebraic curve $\{f = 0\} \cap \mathbb{S}^2$ satisfying (3) for some polynomial K, invariant under the flow defined by X.

5. Exponential factors

In this section we introduce the notion of exponential factor due to Christopher [3], see also [5]. We will see that each exponential factor will play the

same role that an invariant algebraic curve in order to obtain a first integral of the polynomial vector field X on the sphere $\mathbb{S}^2$.

Let h and g polynomials of $\mathbb{R}[x, y, z]$ and assume that

$$h(x, y, z = \sqrt{1 - x^2 - y^2}), \qquad g(x, y, z = \sqrt{1 - x^2 - y^2})$$

are relatively prime in the ring $\mathbb{R}[x, y, \sqrt{1 - x^2 - y^2}]$. Then the function $\exp(g/h)$ is called an *exponential factor* of the polynomial vector field X on $\mathbb{S}^2$ if for some polynomial $K \in \mathbb{R}[x, y, z]$ of degree at most $m - 1$ it satisfies the following equality

$$X\big(\exp(g/h)\big) = K \exp(g/h), \tag{4}$$

on all the points (x, y, z) of $\mathbb{S}^2$. As before we say that K is the *cofactor* of the exponential factor $\exp(g/h)$.

As we will see from the point of view of the integrability of polynomial vector fields on $\mathbb{S}^2$ the importance of the exponential factors is double. On one hand, they verify equation (4), and on the other hand, their cofactors are polynomials of degree at most $m - 1$. These two facts will allow that they play the same role that the invariant algebraic curves in the integrability of a polynomial vector field X on $\mathbb{S}^2$. We note that the exponential factors do not define invariant curves of the flow of the vector field X on $\mathbb{S}^2$.

PROPOSITION 4. *If $F = \exp(g/h)$ is an exponential factor for the polynomial vector field X on the sphere $\mathbb{S}^2$, then $\{h = 0\} \cap \mathbb{S}^2$ is an invariant algebraic curve of X, and g satisfies the equation*

$$Xg = gK_h + hK_F,$$

on the points (x, y, z) of the sphere $\mathbb{S}^2$, where K_h and K_F are the cofactors of h and F respectively.

Proof. Since $F = \exp(g/h)$ is an exponential factor with cofactor K_F, we have

$$K_F \exp(g/h) = X\big(\exp(g/h)\big),$$

on the points of $\mathbb{S}^2$; and since

$$X\big(\exp(g/h)\big) = \exp(g/h) X(g/h) = \exp(g/h) \frac{(Xg)h - g(Xh)}{h^2},$$

we obtain that

$$(Xg)h - g(Xh) = h^2 K_F,$$

on the points of $\mathbb{S}^2$. So, since h and g are relatively prime in the ring $\mathbb{R}[x, y, \sqrt{1 - x^2 - y^2}]$, we obtain that h divides Xh on the points $(x, y, z = \pm\sqrt{1 - x^2 - y^2}$ of the sphere $\mathbb{S}^2$. So $\{h = 0\} \cap \mathbb{S}^2$ is an invariant algebraic curve with cofactor $K_h = Xh/h$ for the vector field X on $\mathbb{S}^2$. Now substituting Xh by $K_h h$ in the last equality, we have that $Xg = gK_h + hK_F$ on the points of $\mathbb{S}^2$. ∎

6. STEREOGRAPHIC PROJECTION

We identify $\mathbb{R}^2$ as the tangent plane to the sphere $\mathbb{S}^2$ at the point $(0, 0, -1)$, and we denote the points of $\mathbb{R}^2$ as $(u, v) = (u, v, -1)$. Let $\pi : \mathbb{R}^2 \to \mathbb{S}^2 \setminus \{(0, 0, 1)\}$ be the diffeomorphism given by

$$\pi(u, v) = \left(x = \frac{2u}{1 + u^2 + v^2}, \, y = \frac{2v}{1 + u^2 + v^2}, \, z = \frac{u^2 + v^2 - 1}{1 + u^2 + v^2} \right).$$

That is, π is the inverse map of the stereographic projection $\pi^{-1} : \mathbb{S}^2 \setminus \{(0, 0, 1)\} \to \mathbb{R}^2$ defined by

$$\pi^{-1}(x, y, z) = \left(u = \frac{x}{1 - z}, v = \frac{y}{1 - z} \right).$$

The polynomial differential system on the sphere $\mathbb{S}^2$

$$\dot{x} = P(x, y, z), \qquad \dot{y} = Q(x, y, z), \qquad \dot{z} = R(x, y, z),$$

associated to the vector field X becomes, through the stereographic projection π^{-1}, the following rational differential system

$$\dot{u} = \frac{1 + u^2 + v^2}{2}(\bar{P} + u\bar{R}), \qquad \dot{v} = \frac{1 + u^2 + v^2}{2}(\bar{Q} + v\bar{R}), \qquad (5)$$

on the plane $\mathbb{R}^2$. Here

$$\bar{F} = F\left(\frac{2u}{1 + u^2 + v^2}, \frac{2v}{1 + u^2 + v^2}, \frac{u^2 + v^2 - 1}{1 + u^2 + v^2} \right).$$

It t denotes the independent variable in the above differential system, then that system becomes polynomial introducing the new independent variable s through $ds = (1 + u^2 + v^2)^{m-1}dt$.

7. The method of Darboux for polynomial vector fields on $\mathbb{S}^2$

In this section we will extend the results of the Darbouxian theory of integration for planar polynomial vector fields (i.e. Theorem 2) to polynomial vector fields on the sphere $\mathbb{S}^2$.

We say that r points $(x_l, y_l, z_l) \in \mathbb{S}^2$, for $l = 1, \ldots, r$, are *independent* with respect to the linear vector space $\mathbb{R}_{m-1}[x, y, z = \sqrt{1 - x^2 - y^2}]$ if the intersection of its r hyperplanes

$$\sum_{i+j+k=0}^{m-1} x_l^i y_l^j \left(\sqrt{1 - x_l^2 - y_l^2}\right)^k x^i y^j \left(\sqrt{1 - x^2 - y^2}\right)^k = 0, \qquad l = 1, \ldots, r,$$

written in the basis of $\mathbb{R}_{m-1}[x, y, z = \sqrt{1 - x^2 - y^2}]$ given in Section 3, define a linear subspace of dimension $m^2 - r$.

THEOREM 5. *Suppose that the polynomial vector field X on the sphere $\mathbb{S}^2$ of degree m admits p invariant algebraic curves $\{f_i = 0\} \cap \mathbb{S}^2$ with cofactors K_i for $i = 1, \ldots, p$, q exponential factors $\exp(g_j/h_j)$ with cofactors L_j for $j = 1, \ldots, q$, and r independent singular points (x_k, y_k, z_k) of X in $\mathbb{S}^2$ such that $f_i(x_k, y_k, z_k) \neq 0$ for $i = 1, \ldots, p$ and for $k = 1, \ldots, r$. Of course, every h_j is equal to some f_i except if h_j is constant. Then the following statements hold.*

(a) *There exist $\lambda_i, \mu_j \in \mathbb{R}$ not all zero such that $\sum_{i=1}^p \lambda_i K_i + \sum_{j=1}^q \mu_j L_j = 0$ on all the points (x, y, z) of the sphere $\mathbb{S}^2$, if and only if the function*

$$|f_1|^{\lambda_1} \cdots |f_p|^{\lambda_p} \left(\exp(g_1/h_1)\right)^{\mu_1} \cdots \left(\exp(g_q/h_q)\right)^{\mu_q} \qquad (6)$$

is a first integral of the vector field X on $\mathbb{S}^2$.

(b) *If $p + q + r = m^2 + 1$, then there exist $\lambda_i, \mu_j \in \mathbb{R}$ not all zero such that $\sum_{i=1}^p \lambda_i K_i + \sum_{j=1}^q \mu_j L_j = 0$ on all the points (x, y, z) of the sphere $\mathbb{S}^2$.*

(c) *The vector field X has a rational first integral if and only if $p + q + r \geq m^2 + 2$. Moreover, all trajectories of X are contained in invariant algebraic curves.*

Proof. We denote $F_j = \exp(g_j/h_j)$. By hypothesis we have p invariant algebraic curves $\{f_i = 0\} \cap \mathbb{S}^2$ with cofactors K_i, and q exponential factors F_j with cofactors L_j. That is, the polynomials f_i's satisfy $Xf_i = K_i f_i$, and the F_j's satisfy $XF_j = L_j F_j$, on all the points (x, y, z) of the sphere $\mathbb{S}^2$.

(a) We have that

$$X\left(f_1^{\lambda_1} \cdots f_p^{\lambda_p} F_1^{\mu_1} \cdots F_q^{\mu_q}\right) = \left(f_1^{\lambda_1} \cdots f_p^{\lambda_p} F_1^{\mu_1} \cdots F_q^{\mu_q}\right)$$
$$\cdot \left(\sum_{i=1}^{p} \lambda_i \frac{Xf_i}{f_i} + \sum_{j=1}^{q} \mu_j \frac{XF_j}{F_j}\right).$$

Then, from this equality and the equality

$$\sum_{i=1}^{p} \lambda_i \frac{Xf_i}{f_i} + \sum_{j=1}^{q} \mu_j \frac{XF_j}{F_j} = \sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = 0.$$

on all the points (x, y, z) of the sphere $\mathbb{S}^2$, statement (a) follows.

(b) Since (x_k, y_k, z_k) is a singular point of the vector field X on the sphere $\mathbb{S}^2$, $P(x_k, y_k, z_k) = Q(x_k, y_k, z_k) = R(x_k, y_k, z_k) = 0$. Then, since $Xf_i = P(\partial f_i/\partial x) + Q(\partial f_i/\partial y) + R(\partial f_i/\partial z) = K_i f_i$ on all the points (x, y, z) of the sphere $\mathbb{S}^2$, it follows that $K_i(x_k, y_k, z_k) f_i(x_k, y_k, z_k) = 0$. By assumption $f_i(x_k, y_k, z_k) \neq 0$, therefore $K_i(x_k, y_k, z_k) = 0$ for $i = 1, \ldots, p$. Again, since $XF_j = P(\partial F_j/\partial x) + Q(\partial F_j/\partial y) + R(\partial F_j/\partial z) = L_j F_j$ on all the points (x, y, z) of the sphere $\mathbb{S}^2$, it follows that $L_j(x_k, y_k, z_k) F_j(x_k, y_k, z_k) = 0$. Since $F_j = \exp(g_j/h_j)$ does not vanish, $L_j(x_k, y_k, z_k) = 0$ for $j = 1, \ldots, q$. Consequently, since the r singular points are independent, all the vectors $K_i(x, y, z = \sqrt{1 - x^2 - y^2})$ and $L_j(x, y, z = \sqrt{1 - x^2 - y^2})$ belong to a linear vector subspace S of $\mathbb{R}_{m-1}[x, y, z = \sqrt{1 - x^2 - y^2}]$ of dimension $m^2 - r$. We have $p + q$ vectors $K_i(x, y, z = \sqrt{1 - x^2 - y^2})$ and $L_j(x, y, z = \sqrt{1 - x^2 - y^2})$, and since from the assumptions $p + q > m^2 - r$, we obtain that these $p + q$ vectors must be linearly dependent on S. So, there are $\lambda_i, \mu_j \in \mathbb{R}$ not all zero such that

$$\sum_{i=1}^{p} \lambda_i K_i(x, y, z = \sqrt{1 - x^2 - y^2}) + \sum_{j=1}^{q} \mu_j L_j(x, y, z = \sqrt{1 - x^2 - y^2}) = 0.$$

Hence statement (b) is proved.

(c) Under the assumptions of statement (c) we can apply statement (b) to two subsets $\{M_1, M_3, \ldots, M_{m^2+2}\}$ and $\{M_2, M_3, \ldots, M_{m^2+2}\}$ of $m^2 + 1$ functions defining invariant algebraic curves or exponential factors. Therefore, we get two linear dependencies between the corresponding cofactors of the following form

$$M_1 + \alpha_3 M_3 + \ldots + \alpha_{m^2+2} M_{m^2+2} = 0, \qquad M_2 + \beta_3 M_3 + \ldots + \beta_{m^2+2} M_{m^2+2} = 0,$$

where M_l are the cofactors K_i and L_j, and the α_l and β_l are real numbers. Of course, these two equalities must be satisfied only on all the points (x, y, z) of $\mathbb{S}^2$. Then, by statement (a), it follows that the two functions

$$|G_1||G_3|^{\alpha_3} \cdots |G_{m^2+2}|^{\alpha_{m^2+2}}, \qquad |G_2||G_3|^{\beta_3} \cdots |G_{m^2+2}|^{\beta_{m^2+2}},$$

are first integrals of the vector field X on $\mathbb{S}^2$, where G_l is the polynomial defining an invariant algebraic curve or the exponential factor having cofactor M_l for $l = 1, \ldots, m^2 + 2$. Then, taking logarithms to the above two first integrals, we obtain that

$$
\begin{aligned}
H_1 &= \log|G_1| + \alpha_3 \log|G_3| + \cdots + \alpha_{m^2+2} \log|G_{m^2+2}|, \\
H_2 &= \log|G_2| + \beta_3 \log|G_3| + \cdots + \beta_{m^2+2} \log|G_{m^2+2}|,
\end{aligned}
$$

are first integrals of the vector field X on $\mathbb{S}^2$.

Now we consider the expression (5) of the differential system associated to the vector field X on $\mathbb{S}^2$ in the tangent plane $\mathbb{R}^2$ to the sphere $\mathbb{S}^2$ at the point $(0, 0, -1)$. Using the notation introduced in Section 6, we denote by $\bar{H}_1$ and $\bar{H}_2$ the expressions of the first integrals H_1 and H_2 in the coordinates (u, v) of the tangent plane $\mathbb{R}^2$.

$$\dot{u} = \frac{1 + u^2 + v^2}{2}(\bar{P} + u\bar{R}), \qquad \dot{v} = \frac{1 + u^2 + v^2}{2}(\bar{Q} + v\bar{R}), \qquad (7)$$

Each first integral $\bar{H}_i$ provides an integrating factor R_i for system (7) such that

$$\frac{1 + u^2 + v^2}{2}(\bar{P} + u\bar{R})R_i = \frac{\partial \bar{H}_i}{\partial v}, \qquad \frac{1 + u^2 + v^2}{2}(\bar{Q} + v\bar{R})R_i = -\frac{\partial \bar{H}_i}{\partial u}.$$

Therefore, we obtain that

$$\frac{R_1}{R_2} = \frac{\partial \bar{H}_1/\partial v}{\partial \bar{H}_2/\partial v}.$$

Since the functions G_l are polynomials or exponentials of a quotient of polynomials and the change of variables through the stereographic projection is given by rational functions, it follows that the functions $\partial \bar{H}_i/\partial x$ are rational for $i = 1, 2$. So, from the last equality, we get that the quotient between the two integrating factors R_1/R_2 is a rational function. From Lemma 1 it follows that this quotient is a rational first integral of system (5). Again, since the change of variables through the stereographic projection is given by rational functions, we get a rational first integral of the polynomial vector field X on $\mathbb{S}^2$. In short, we have proved statement (c). ∎

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