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Stability in Banach Spaces

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1. Introduction

Many questions in Banach space theory are of the type: Let X be an infinite dimensional Banach space. Let (P) be a property. Does X contain a closed infinite dimensional subspace Y with (P)? Sometimes the question takes the form: If X has a certain property (Q), does X contain Y having (P)? Of course the answers and solutions (if known) depend upon the particular properties involved. But there is perhaps one theme that runs throughout many of these problems. One often wishes to stabilize some function on some substructure of X. This might be achieved quite simply, say by the pigeonhole principle or an analytical argument using compactness or may be more involved using infinitary combinatorics (e. g., Ramsey theory). One may require new ad hoc arguments that in turn lead to new combinatorial results. Before continuing we set some notation and give some background.

A Banach space $(X, \|\cdot\|)$ is a complete normed linear space. A norm $|\!|\!|\cdot|\!|\!|$ on X is an *equivalent norm* if for some constants $A, B > 0$ for all $x \in X$

$$A^{-1}|\!|\!|x|\!|\!| \le \|x\| \le B|\!|\!|x|\!|\!| \,.$$

The norm $\|\cdot\|$ on X is determined by the unit ball $B_X \equiv \{x \in X : \|x\| \le 1\}$ or by the unit sphere $S_X = \{x \in X \,:\, \|x\| = 1\}$. $|\!|\!|\cdot|\!|\!|$ is an equivalent norm just means that

$$B^{-1}B_{(X,|\!|\!|\cdot|\!|\!|)} \subseteq B_X \subseteq A \cdot B_{(X,|\!|\!|\cdot|\!|\!|)}$$

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for some $A, B > 0$. If X is finite dimensional all norms on X are equivalent. B_X is compact if and only if X is finite dimensional.

We shall mention some open problems from time to time and denote these by (Q1), (Q2), ... Here is the first one. It is really a problem about nonseparable Banach spaces. We do not discuss this problem below. It is fun to raise however because one only needs to know the definition of a Banach space in order to understand it.

(Q1) Let X be an infinite dimensional Banach space. Do there exist closed linear subspaces $X_1 \subsetneqq X_2 \subsetneqq X_3 \subset \cdots$ with $\overline{\bigcup_n X_n} = X$?

It can be shown [32] that this is equivalent to asking if there exists a separable infinite dimensional Banach space Y and a bounded linear operator $T : X \to Y$ which is onto. Thus some quotient space of X is isomorphic to Y. Banach spaces X and Y are *isomorphic* (denoted, $X \sim Y$) if there exists a 1–1 bounded linear operator T from X onto Y. In this case we set the "distance" between them to be

$$d(X,Y) = \inf\left\{\|T\|\cdot\|T^{-1}\| : \begin{array}{c} T : X \to Y \text{ is a bounded} \\ \text{linear 1–1 onto operator} \end{array}\right\}.$$

Actually $d(X,Y) \le d(X,Z)d(Z,Y)$ if X, Y and Z are isomorphic so to get a metric one must take "$\log d$". This gives a metric on a class of isomorphic Banach spaces where we identify X and Y if $d(X,Y) = 1$. X and Y are *isometric* (denoted, $X \cong Y$) if there exists a bounded linear operator $T : X \to Y$ which is 1–1 onto and such that $1 = \|T\| = \|T^{-1}\|$. Curiously there exist (infinite dimensional) spaces X and Y with $d(X,Y) = 1$ but X and Y are not isometric. If $\mathcal{M}_n$ denotes the class of all n-dimensional Banach spaces, $\log d(\cdot,\cdot)$ defines a complete metric on $\mathcal{M}_n$.

Henceforth we shall use $X, Y, Z, \ldots$ and sometimes E to denote real separable infinite dimensional Banach spaces and use $F, G, \ldots$ to denote finite dimensional spaces. The theorems we present generally pass to the complex setting with few changes and thus it is simpler to concentrate on the real case. Thus X will denote a real separable infinite dimensional Banach space.

A sequence $(x_i)_{i=1}^\infty$ is a *basis* for X if for all $x \in X$ there exists a unique sequence $(a_n)_{n=1}^\infty \subseteq \mathbb{R}$ so that $x = \sum_{n=1}^\infty a_n x_n$. In this case we can define basis projections $P_n : X \to X$ given by $P_n(x) = \sum_{i=1}^n a_i x_i$ if $x = \sum_{i=1}^\infty a_i x_i$. A *projection* P on X is a bounded linear operator on X with $P^2 = P$. In this case the *range* of P, $P(X)$, is said to be *complemented* in X. If (x_n) is a basis for X one can prove that $C \equiv \sup_n \|P_n\| < \infty$ and we call C

the *basis constant* of (x_n), $\mathrm{bc}(x_n)$. Thus for all $n < m$ and $(a_i)_{i=1}^m \subseteq \mathbb{R}$, $\|\sum_{i=1}^n a_i x_i\| \le C\|\sum_{i=1}^m a_i x_i\|$ and this, in fact, characterizes bases.

Proposition 1.1. *Let $(x_n)_{n=1}^\infty \subseteq X$ be a sequence of nonzero vectors satisfying: there exists $C < \infty$ so that for all $n < m$ and $(a_i)_{i=1}^m \subseteq \mathbb{R}$,*

(i) $\Big\|\sum_{i=1}^n a_i x_i\Big\| \le C\Big\|\sum_{i=1}^m a_i x_i\Big\|$,

(ii) *the* closed linear span *of (x_n), denoted $[(x_n)] = X$.*

Then (x_n) is a basis for X.

The proof is not hard but it is a bit tricky (see the list of standard references given at the end of this section).

A sequence $(x_n) \subseteq X$ is *basic* if it is a basis for $[(x_n)]$. Thus a basic sequence (x_n) is characterized by all $x_n \neq 0$ and (i).

A basis (x_n) for X is *unconditional* if for all $x \in X$ there exists a unique sequence $(a_n) \subseteq \mathbb{R}$ so that $x = \sum_{n=1}^\infty a_{\pi(n)} x_{\pi(n)}$ for all permutations π of $\mathbb{N} = \{1, 2, 3, \ldots\}$. (x_n) is *unconditional basic* if it is an unconditional basis for $[(x_n)]$.

Proposition 1.2. *Let $(x_n) \subseteq X$ be a sequence of nonzero vectors. (x_n) is unconditional basic if either of the following hold.*

(i) *There exists $C < \infty$ so that for all $n \in \mathbb{N}$, $(a_i)_{i=1}^n \subseteq \mathbb{R}$ and $F \subseteq \{1, \ldots, n\}$,*
$$\Big\|\sum_{i\in F} a_i x_i\Big\| \le C\Big\|\sum_{i=1}^n a_i x_i\Big\|.$$

(ii) *There exists $D < \infty$ so that for all $n \in \mathbb{N}$, $(a_i)_{i=1}^n \subseteq \mathbb{R}$ and $\varepsilon_i = \pm 1$ for $i \le n$,*
$$\Big\|\sum_{i=1}^n \varepsilon_i a_i x_i\Big\| \le D\Big\|\sum_{i=1}^n a_i x_i\Big\|.$$

The smallest C satisfying (i) is called the *suppression unconditional basis constant* of (x_n), s-ubc(x_n)

$$C = \sup\Big\{\|P_F\| : F \subseteq \mathbb{N},\ F \text{ finite}\Big\}$$

where $P_F : [(x_i)] \to [(x_i)]$ is given by

$$P_F\Big(\sum a_i x_i\Big) = \sum_{i\in F} a_i x_i\,.$$

The smallest D satisfying (ii) is called the *unconditional basis constant* of (x_n), $\text{ubc}(x_n)$. If we fix $\varepsilon_i = \pm 1$ and set $Q_{(\varepsilon_i)}(\sum a_i x_i) = \sum \varepsilon_i a_i x_i$ then $D = \sup\{\|Q_{(\varepsilon_i)}\| : \varepsilon_i = \pm 1\}$. One has $\text{s-ubc}(x_n) \le \text{ubc}(x_n) \le 2\,\text{s-ubc}(x_n)$.

If (x_n) is a basis for X the *biorthogonal functionals* (x_n^*) of (x_n) are defined by $x_n^*(\sum_{i=1}^\infty a_i x_i) = a_n$. Since $\|a_n x_n\| = (P_n - P_{n-1}) \sum_{i=1}^\infty a_i x_i$ one has $\|x_n^*\| \le \frac{2\,\text{bc}(x_n)}{\|x_n\|}$ and so if (x_n) is *normalized* (i.e., $\|x_n\| = 1$ for all n) then $\|x_n^*\| \le 2\,\text{bc}(x_n)$. In particular for all $x = \sum_{i=1}^\infty a_i x_i \in X$, $\|x\| \ge \frac{1}{2\,\text{bc}(x_n)} \sup_n |a_n|$. Also it is not hard to check that (x_n^*) is a basic sequence in X^*, the dual space of X, with

$$\text{bc}(x_n^*) = \sup_n \|P_n^*|_{[(x_n^*)]}\| \le \sup_n \|P_n^*\| = \sup_n \|P_n\| = \text{bc}(x_n)\,;$$

$\text{bc}(x_n^*)$ can be smaller than $\text{bc}(x_n)$ [13].

Any theory requires examples. One has of course to start with the classical Banach spaces. Let $1 \le p < \infty$. ℓ_p is the Banach space of all sequences $x = (a_n)_{n=1}^\infty$ of reals so that $\|x\|_p \equiv (\sum_{n=1}^\infty |a_n|^p)^{1/p} < \infty$. c_0 is the Banach space of all null sequences (a_n) under

$$\|(a_n)\|_\infty = \sup_n |a_n| = \max_n |a_n|\,.$$

From our above remarks, if (x_n) is a normalized basis for X then

$$\Big\| \sum a_i x_i \Big\| \ge \frac{1}{2\,\text{bc}(x_i)} \|(a_i)\|_\infty\,,$$

i.e., we have an automatic *lower c_0 estimate*. Also by the triangle inequality $\|\sum a_i x_i\| \le \sum |a_i| = \|(a_i)\|_1$ so we also have an *upper ℓ_1 estimate*. Thus for a space with a normalized basis (we should note that if (x_n) is a basis for X, $\left(\frac{x_n}{\|x_n\|}\right)$ is also a basis with the same basis constant) then the norm sits between the ℓ_1 (largest) and c_0 (smallest) norms.

The unit vector basis $(e_n)_{n=1}^\infty$ is defined by $e_n = (0, \ldots, 0, 1, 0, \ldots)$, the 1 occurring in the n^{th} coordinate. It is easy to see that (e_n) is a normalized unconditional basis for ℓ_p $(1 \le p < \infty)$ or c_0 with $\text{ubc}(e_n) = 1$ in all cases.

It is not hard to show that $\ell_p^* \cong \ell_q$ for $1 \le p < \infty$ where $\frac{1}{p} + \frac{1}{q} = 1$ and $c_0^* \cong \ell_1$, all in a canonical way: if $x^* = (b_i) \in \ell_q$ and $x = (a_i) \in \ell_p$ then

$$x^*(x) = \sum_{i=1}^\infty a_i b_i\,.$$

The (nonseparable) Banach space ℓ_∞ is the space of all bounded sequences (a_n) under $\|(a_n)\|_\infty = \sup_n |a_n|$. Being nonseparable it cannot have a basis.

One can also define a basis for a finite dimensional space as just a linearly independent spanning set (or Hamel basis) and the basis constant is defined as above. It is perhaps worth noting that a Hamel basis for an infinite dimensional Banach space X is uncountable (in fact if X is separable it has the cardinality of the continuum [34]).

The finite dimensional versions of the above spaces are denoted by ℓ_p^n or ℓ_∞^n, $n \in \mathbb{N}$. For $1 \le p < \infty$,

$$L_p[0,1] = \left\{ f : [0,1] \to \mathbb{R} \ : \ \begin{array}{c} \text{is Lebesgue measurable and} \\ \|f\|_p \equiv \left(\int_0^1 |f(t)|^p \, dm(t) \right)^{1/p} < \infty \end{array} \right\}.$$

The *Haar functions* $(h_n)_{n=0}^\infty$ form a *monotone* basis for L_p (i.e., $\mathrm{bc}(h_n) = 1$) and an unconditional basis if $1 < p < \infty$. To prove they form a monotone basis is not difficult while the unconditionality result is quite difficult. They are defined by $h_0 \equiv 1$, $h_1 = 1$ on $[0, 1/2]$ and -1 on $(1/2, 1]$, $h_2 = 1$ on $[0, 1/4]$, -1 on $(1/4, 1/2]$ and 0 elsewhere, $h_3 = 1$ on $[1/2, 3/4]$ and -1 on $(3/4, 1]$ and 0 elsewhere, $h_4 = 1$ on $[0, 1/8]$, -1 on $(1/8, 1/4]$ and 0 elsewhere and so on.

If K is compact metric, $C(K)$ is the Banach space of all continuous $f : K \to \mathbb{R}$ under $\|f\| = \sup\{|f(t)| : t \in K\}$. The prime example is $C[0,1]$. The *Schauder basis* $(f_n)_{n=0}^\infty$ is a monotone basis (but not unconditional) for $C[0,1]$. $f_0 \equiv 1$, $f_1(t) = t$, f_2 takes values 0 at 0 and 1 and 1 at 1/2 and is linear in between these. f_3 looks like f_2 but shrunken: $f_3(0) = f_3(\frac{1}{2}) = f_3(1) = 0$, $f_3(\frac{1}{4}) = 1$ and it is linear on the intervals in between and so on. Banach [3] showed that $C[0,1]$ is *universal* for all separable Banach spaces X, i.e. every such X is isometric to a subspace of $C[0,1]$. The Hahn Banach theorem yields easily that ℓ_∞ is also universal for such X: let (x_n) be dense in X and for all n choose $f_n \in X^*$ with $\|f_n\| = 1$, $f(x_n) = \|x_n\|$. Define $T : X \to \ell_\infty$ by $Tx = (f_n(x))$. Then T is an *into isometry*, i.e., $T : X \to TX$ is an isometry. ℓ_1 is also universal for such X in a different way. X must be isometric to a quotient space of ℓ_1. To see this choose (x_n) dense in S_X and define $T : \ell_1 \to X$ by $T(a_n) = \sum a_n x_n$.

Not every X has a basis. This is a difficult result due to Enflo [14]. However it is relatively easy to see every X must contain a basic sequence (x_n). And given $\varepsilon > 0$ one can obtain (x_n) with $\mathrm{bc}(x_n) < 1 + \varepsilon$. To do this let $x_1 \in S_X$. Assume $x_1, \ldots, x_n$ have been chosen. Let $F_n = \mathrm{span}(x_n, \ldots, x_n)$. Since B_{F_n} is compact there exists a finite set $G_n \subseteq S_{X^*}$ which $(1+\varepsilon)$-*norms* F_n. By this we mean that for $x \in F_n$, $(1+\varepsilon)\sup\big\{|g(x)| : g \in G_n\big\} \ge \|x\|$. We may also assume $G_n \subseteq G_m$ if $n < m$. $\bigcap_{g\in G_n} \operatorname{Ker} g$ is a finite codimensional subspace of X so there exists $x_{n+1} \in S_X$ with $g(x_{n+1}) = 0$ for all $g \in G_n$. If $n < m$ and $(a_i)_1^m \subseteq \mathbb{R}$ choose $g \in G_n$ with

$$\Big\|\sum_1^n a_i x_i\Big\| \le (1+\varepsilon) g\Big(\sum_1^n a_i x_i\Big) = (1+\varepsilon) g\Big(\sum_1^m a_i x_i\Big) \le (1+\varepsilon)\Big\|\sum_1^m a_i x_i\Big\|.$$

Thus (x_n) is basis with $\operatorname{bc}(x_n) \le 1+\varepsilon$. We have proved (a) below and (b) can be proved by a slight variation.

PROPOSITION 1.3. *Let X be an infinite dimensional Banach space.*

(a) *For all $\varepsilon > 0$ there exists a basic sequence $(x_n) \subseteq X$ with $\operatorname{bc}(x_n) < 1+\varepsilon$.*

(b) *Let $(x_n) \subseteq S_X$ with $x_n \to 0$ weakly (i.e. for all $f \in X^*$, $\lim_n f(x_n) = 0$).*

Then given $\varepsilon > 0$ there exists a basic subsequence (y_n) of (x_n) with $\operatorname{bc}(y_n) < 1+\varepsilon$.

Notice that the proof of (a) involved stabilization: given $\mathcal{G}_n$ we choose $x_{n+1}, x_{n+2}, \dots$ so that each $g \in \mathcal{G}_n$ takes the same value, namely 0, on these elements.

In the type of problem described earlier, given X find a subspace Y with (P), Proposition 1.3 allows us to assume that X has a basis which gives us some structure, namely a coordinate system, with which to work, In constructing a basic sequence we only had to choose the x_n's so as to control the norm of the basis projection P_n. To find an unconditional basic sequence by a similar method would not work in that the combinatorial difficulties in controlling $\|P_F\|$ over all finite $F \subseteq \mathbb{N}$ would be insurmountable. In fact there exist spaces X not containing any unconditional basic sequence [23].

We have described a few of the classical Banach spaces X. You could say we have described all of them if you consider as well all subspaces of $C[0,1]$ (or quotients of ℓ_1) but such a description rarely gives any useful information. Sometimes in order to solve problems one needs to create new Banach spaces and one way this is done follows the following scheme. Let c_{00} denote the linear space of all finitely supported sequences of reals. One constructs a norm $\|\cdot\|$ on c_{00} that makes the unit vector basis (e_n) into a normalized basis for the completion of $\big(c_{00}, \|\cdot\|\big)$. For example we can think of ℓ_p as being constructed this way. As a new example we have the *Schreier space*: S_C where

$$\|(a_n)\| = \sup\Big\{\sum_{i=1}^m |a_{n_i}| : m \in \mathbb{N}, m \le n_1 < \cdots < n_m\Big\}.$$

Sometimes the norm is not defined directly by a formula as above but one constructs a certain family $\mathcal{F} \subseteq B_{\ell_\infty}$ and sets

$$\|x\| = \sup\left\{ \left| \sum_1^\infty f(n)x(n) \right| : f = \big(f(n)\big) \in \mathcal{F} \right\}$$

for $x = \big(x(n)\big) \in c_{00}$. If for example $e_n \in \mathcal{F}$ for all n and $f \in \mathcal{F}$ implies $P_n f \in \mathcal{F}$, where $P_n f$ is the restriction of f to the first n coordinates, then (e_n) will be a normalized monotone basis for the space constructed. For example S_C would be generated in this manner by taking

$$\mathcal{F} = \left\{ f = \big(f(n)\big) : \begin{array}{l} \text{there exists } m \in \mathbb{N},\ m \le n_1 < \cdots < n_m, \text{ with } f(n) = \\ 0 \text{ if } n \notin \{n_1, \dots, n_m\} \text{ and } f(n_i) = \pm 1 \text{ for } i \le m \end{array} \right\}.$$

If X has a basis (x_n) then some of the nice subspaces of X are those subspaces Y which have a basis (y_n) which is a block basis of (x_n). (y_n) is a *block basis* of (x_n) if there exist $(a_i)_1^\infty \subseteq \mathbb{R}$ and integers $k_0 = 0 < k_1 < k_2 < \cdots$ so that for all n,

$$y_n = \sum_{i=k_{n-1}+1}^{k_n} a_i x_i \qquad \text{and} \qquad y_n \ne 0\,.$$

In this case (y_n) is basic with $\mathrm{bc}(y_n) \le \mathrm{bc}(x_n)$. Basic sequences (x_n) and (y_n) will be called *C-equivalent* if for all $(a_n) \subseteq \mathbb{R}$

$$C^{-1}\Big\| \sum a_n y_n \Big\| \le \Big\| \sum a_n x_n \Big\| \le C \Big\| \sum a_n y_n \Big\|\,.$$

This just says that the operator $T : \big[(x_n)\big] \to \big[(y_n)\big]$ given by $Tx_n = y_n$ for all n is an isomorphism with $\|T\| \le C$ and $\|T^{-1}\| \le C$. Sometimes we wish to perturb things a bit.

PROPOSITION 1.4. (a) *Let (x_n) be a normalized basic sequence in X and let $\varepsilon > 0$. Then there exists $\varepsilon_n \downarrow 0$ (depending only on $\mathrm{bc}(x_n)$) so that if $(y_n) \subseteq X$ with $\|y_n - x_n\| < \varepsilon_n$ for all n, then (y_n) is basic and $(1+\varepsilon)$-equivalent to (x_n).*

(b) *Let X have a basis (x_n). Then for all $Y \subseteq X$ (by which we mean Y is a closed infinite dimensional subspace of X), $\varepsilon > 0$ and $\varepsilon_n \downarrow 0$ there exists a normalized basic sequence $(y_n) \subseteq Y$ and a normalized block basis (z_n) of (x_n) so that $\|y_n - z_n\| < \varepsilon_n$ for all n and (y_n) is $(1+\varepsilon)$-equivalent to (z_n).*

(c) *Let X have a basis (x_n). Let $(y'_n) \subseteq X$ be normalized and satisfy for all m, $\lim_{n\to\infty} x^*_m(y'_n) = 0$. Then given $\varepsilon > 0$ and $\varepsilon_n \downarrow 0$ there exists a subsequence (y_n) of (y'_n) which satisfies the conclusion of* (b).

Thus for the purpose of many of our searches for "Y with (P)" it will usually be sufficient to focus on Y's generated by block bases of (x_n). The next proposition illustrates this. Proposition 1.4 (a) is a standard perturbation result (see the references given at the end of the section) and (b) and (c) are proved using (a) much like the proof of Proposition 1.3.

PROPOSITION 1.5. *Let X have a basis (x_n) and let $Y \subseteq X$ be isomorphic to ℓ_p for some $1 \le p < \infty$ or c_0. Then some block basis of (x_n) is equivalent to the unit vector basis of ℓ_p (or c_0).*

Proof. Suppose that $Y \sim \ell_p$ $(1 < p < \infty)$ or c_0 and let $(y_n) \subseteq Y$ be equivalent to the unit vector basis of ℓ_p/c_0. Then $y_n \to 0$ weakly. Since any subsequence of (y_n) is still equivalent to the unit vector basis of ℓ_p/c_0, Proposition 1.4 (c) yields the result. If $Y \sim \ell_1$ we let $(y'_n) \subseteq Y$ be equivalent to the unit vector basis of ℓ_1. By passing to a subsequence $(y_n) \subseteq (y'_n)$ we may assume (stabilization again!) that for all m, $\lim_{n\to\infty} x^*_m(y_n)$ exists. But then the sequence

$$\left(\frac{y_1 - y_2}{\|y_1 - y_2\|}, \frac{y_3 - y_4}{\|y_3 - y_4\|}, \dots\right)$$

is coordinatewise null in X and still equivalent to the unit vector basis of ℓ_1 (easily checked) so the same argument applies. ∎

Remark. If Y is isomorphic to ℓ_p for some $1 \le p < \infty$ or c_0 then for all $Z \subseteq Y$ some subspace of Z is isomorphic to Y, i. e. Y is *minimal*. This follows from Propositions 1.3 and the easily verified fact that a normalized block basis of a basis equivalent to the unit vector basis of ℓ_p/c_0 is also equivalent to the same basis.

A basis (x_n) is *C-subsymmetric* if it is C-equivalent to each of its subsequences. It is *C-symmetric* if it is C-equivalent to $(x_{\pi(n)})$ for all permutations π of $\mathbb{N}$. The unit vector basis for ℓ_p $(1 \le p < \infty)$ or c_0 is 1-symmetric. c_0 possesses another 1-subsymmetric basis, (s_n), defined by $s_n = e_1 + \cdots + e_n$. Note that

$$\Big\|\sum_i a_i s_i\Big\| = \sup_n \Big|\sum_{i=n}^{\infty} a_i\Big|$$

and thus (s_n) is called the *summing basis* for c_0. Also $\| \sum_{i=1}^n s_i\| = n$ but $\| \sum_{i=1}^n (-1)^n s_i\| = 1$, and so (s_n) is not unconditional.

The reader who is not at least partially familiar with the material presented above may wish to consult some books such as [35], [25], [12], or [15]. Another source for an interesting collection of more advanced results is [10]. Other suggested books are [5], [61], and [11]. We note that the forthcoming volumes [30], [31] will prove to be a valuable encyclopedic reference on Banach space theory.

2. Ramsey Theory and applications

Ramsey theory has developed into a fairly large area in combinatorics (see e. g. [24]). Its theorems are of the sort: given a finitely valued function f on certain objects one can find a substructure of some type on which f is constant. An example (as given by D. Kleitman) is amongst three ordinary people at least two have the same sex. We begin with the original theorem [54]. Let $[\mathbb{N}]$ denote all infinite subsequences of $\mathbb{N}$. For $n \in \mathbb{N}$, $[\mathbb{N}]^n$ denotes all finite subsequences of $\mathbb{N}$ of length n. If $M \in [\mathbb{N}]$ we use similar notation, $[M]$ and $[M]^n$ to denote all subsequences (or all length n subsequences) of M.

Theorem 2.1. (Ramsey's Theorem) *Let $n, m \in \mathbb{N}$. Let $f : [\mathbb{N}]^n \to \{1, 2, \ldots, m\}$. Then for all $L \in [\mathbb{N}]$ there exists $N \in [L]$ and $i \in \{1, \cdots, m\}$ so that for all $M \in [N]^n$, $f(M) = i$.*

In other words if we finitely color the length n subsequences of $\mathbb{N}$ then we can find N (inside of any given L) so that $(m_1, \ldots, m_n)$ is monochromatic for all $m_1 < \cdots < m_n$ in N.

Proof. It is enough to prove this if $m = 2$ and $L = \mathbb{N}$. Suppose $n = 2$. We shall use the coloring language and write for $n < m$, $(n, m) \in R$ if it is colored red and $(n, m) \in B$ if it is colored blue. By the pigeonhole principle (the simplest Ramsey theorem) there exists $L_1 \in [\mathbb{N}]$ so that $1 < n$ for all $n \in L_1$ and: (i) $(1, n) \in R$ for all $n \in L_1$, or (ii) $(1, n) \in B$ for all $n \in L_1$. Let $n_1 = 1$ and $n_2 = \min(L_1)$. We repeat the argument finding $L_2 \in [L_1]$, $n_2 < n$ for all $n \in L_2$ with either: (i) $(n_2, n) \in R$ for all $n \in L_2$, or (ii) $(n_2, n) \in B$ for all $n \in L_2$. We continue inductively thus defining $(n_i)_{i=1}^\infty \in [\mathbb{N}]$. Now for each n_i we had either alternative (i) or (ii). And one of these, say (i), occurs infinitely often. Let (m_i) be the subsequence for which (i) holds. Then for all $i < j$, $(m_i, m_j) \in R$. We leave the general case ($n > 2$) as an exercise. ∎

There is a nice application of this theorem to Banach spaces in terms of spreading models.

Theorem 2.2. *Let (x_n) be a normalized basic sequence in a Banach space X. Let $\varepsilon_n \downarrow 0$. Then there exists a subsequence (y_n) of (x_n) with the following property. For all integers $n \in \mathbb{N}$ and $n \le i_1 < \cdots < i_n$, $n \le j_1 < \cdots < j_n$ and $(a_n)_{i=1}^n \in [-1,1]^n$,*

$$\left| \left\| \sum_{k=1}^n a_k y_{i_k} \right\| - \left\| \sum_{k=1}^n a_k y_{j_k} \right\| \right| < \varepsilon_n \,. \tag{2.1}$$

In particular we can define a norm on c_{00} by

$$\left\| \sum_{k=1}^n a_k e_k \right\| \equiv \lim_{i_1 \to \infty} \cdots \lim_{i_n \to \infty} \left\| \sum_{k=1}^n a_k y_{i_k} \right\| ,$$

which makes (e_i) a normalized basis for the completion E of $(c_{00}, \|\cdot\|)$. This is easy to check. (e_i) or E is called a *spreading model* of X or of (y_n).

Proof. By standard diagonalization it suffices to, for a fixed $n \in \mathbb{N}$, obtain $(y_i) \subseteq (x_i)$ satisfying (2.1) for all integers $i_1 < \cdots < i_n$, $j_1 < \cdots < j_n$. First let $(a_i)_{i=1}^n \in [-1,1]^n$ be fixed. Note that

$$0 \le \left\| \sum_{k=1}^n a_k x_{i_k} \right\| \le n \qquad \text{for } i_1 < \cdots < i_k \,.$$

Let $I_1, I_2, \ldots, I_m$ be a partition of $[0,n]$ into intervals of length each $< \varepsilon_n/2$. Define

$$f(i_1, \ldots, i_n) = r \qquad \text{if } \left\| \sum_{k=1}^n a_k x_{i_k} \right\| \in I_r \,.$$

By Ramsey's theorem we can find $(y_i) \subseteq (x_i)$ so that (2.1) holds for this choice of $(a_i)_1^k$ with ε_n replaced by $\varepsilon_n/2$. We repeat this finitely often for each choice of $(a_i)_1^k$ in some finite $\frac{\varepsilon_n}{4}$-net A in $(B_{\ell_\infty^n}, \|\cdot\|_1)$ (a *δ-net* for a metric space K is a subset $A \subseteq K$ with for all $x \in K$, $d(x,a) < \delta$ for some $a \in A$). In this case the metric space is $B_{\ell_\infty^n}$ under the metric defined by the ℓ_1 norm. Given $(b_i)_1^n \in B_{\ell_\infty^n}$ choose $(a_i)_1^n \in A$ with $\sum_1^n |a_i - b_i| < \frac{\varepsilon_n}{4}$. Then for $i_1 < \cdots < i_n$,

$j_1 < \cdots < j_n$, by the triangle inequality

$$\begin{aligned}\left|\left\|\sum_{k=1}^n b_k y_{i_k}\right\| - \left\|\sum_{k=1}^n b_k y_{j_k}\right\|\right| &= \left|\left\|\sum_{k=1}^n b_k y_{i_k} - \sum_{k=1}^n a_k y_{i_k} + \sum_{k=1}^n a_k y_{i_k}\right\|\right. \\ &\qquad \left. - \left\|\sum_{k=1}^n b_k y_{j_k} - \sum_{k=1}^n a_k y_{j_k} + \sum_{k=1}^n a_k y_{j_k}\right\|\right| \\ &< 2\frac{\varepsilon_n}{4} + \left|\left\|\sum_{k=1}^n a_k y_{i_k}\right\| - \left\|\sum_{k=1}^n a_k y_{j_k}\right\|\right| < \varepsilon_n\,.\end{aligned}$$

∎

We gather some facts about spreading models.

Proposition 2.3. *Let (y_n) be a normalized basis having a spreading model (e_n). Then if $E = [(e_n)]$,*

(a) *(e_n) is a normalized basis for E which is 1-subsymmetric;*
(b) *if (y_n) is weakly null then (e_n) is unconditional with s-ubc$(e_n) = 1$;*
(c) *E is finitely representable in $[(y_n)]$. (This means that if F is a finite dimensional subspace of E and $\varepsilon > 0$ there exists a finite dimensional subspace G of $[(y_n)]$ with $d(F, G) < 1 + \varepsilon$.)*

Proof. The reader should check (a) and (c) to see that these are easy and to begin to understand spreading models. (b) can be proved using Mazur's theorem but we present a different argument that uses

Lemma 2.4. *Let (y_n) be a normalized weakly null sequence in a Banach space X. Let $\varepsilon > 0$ and $n \in \mathbb{N}$. There exists $m > n$ with the following property. Let $f \in B_{X^*}$. Then there exists $i \in \mathbb{N}$, $n < i \le m$, with $|f(y_i)| < \varepsilon$.*

Proof. If not there exists $\varepsilon > 0$ and $n \in \mathbb{N}$ so that for all $m > n$ there exists $f_m \in B_{X^*}$ with $|f_m(y_i)| \ge \varepsilon$ for $n < i \le m$. B_{X^*} is weak* compact and metrizable in the weak* topology so (f_m) has a weak* convergent subsequence to some f. But then $|f(y_i)| \ge \varepsilon$ for all $i > n$ so (y_i) is not weakly null. ∎

Returning to (b) we may assume that (y_n) satisfies (2.1) and let $(a_i)_1^k \in [-1,1]^k$ and let $i_0 \le k$. It suffices to show that for $\varepsilon > 0$

$$\left\|\sum_{i=1,\, i\ne i_0}^k a_i e_i\right\| \le \left\|\sum_{i=1}^k a_i e_i\right\| + \varepsilon\,.$$

Let $k \le \bar{k}$ and choose $\bar{k} \le j_1 < \cdots < j_k$ so that $j_{i_0} > m = m(j_{i_0 - 1})$ as determined by the lemma for $\varepsilon_{\bar{k}}$ and (y_i), Let $f \in S_{X^*}$ with

$$\Big\| \sum_{i=1,\, i\neq i_0}^{k} a_i y_{j_i} \Big\| = f\bigg(\sum_{i=1,\, i\neq i_0}^{k} a_i y_{j_i} \bigg).$$

Choose $\bar{j} \in (j_{i_0-1}, j_{i_0+1})$ with $|f(y_{\bar{j}})| < \varepsilon_{\bar{k}}$ then

$$\begin{aligned} \Big\| \sum_{i=1,\, i\neq i_0}^{k} a_i e_i \Big\| &\le \Big\| \sum_{i=1,\, i\neq i_0}^{k} a_i y_{j_i} \Big\| + \varepsilon_{\bar{k}} \le f\bigg(\sum_{i=1,\, i\neq i_0}^{k} a_i y_{j_i} + a_{i_0} y_{\bar{j}} \bigg) + 2\varepsilon_{\bar{k}} \\ &\le \Big\| \sum_{i=1}^{k} a_i e_i \Big\| + 3\varepsilon_{\bar{k}} \end{aligned}$$

This proves (b) since $\bar{k}$ is arbitrary. ∎

Thus the theory of spreading models yields, at least in some instances, a nice structure – a space with an unconditional basis. But the price paid is that E is not necessarily a subspace of X but only lives locally inside X, albeit in an asymptotic manner. Let's move back in time to the 1960's and consider the situation as it was known then. The search was still on for a very nice subspace of a general space X. It was conjectured that every X contains an isomorph of ℓ_p for some $1 \le p < \infty$ or c_0. This was proved false in 1974 by Tsirelson [60]. Or at least every X should contain an unconditional sequence. This was proved false by Gowers and Maurey in 1993 [23]. Or at least every X contains either a reflexive space Y or an isomorph of c_0 or ℓ_1. This was proved false by Gowers in 1994 [21]. That the last conjecture is weaker than the previous one follows from a classical theorem of R. C. James [29].

THEOREM 2.5. *Let X have an unconditional basis. Then X is either reflexive or contains an isomorph of c_0 or ℓ_1.*

Recall that X is *reflexive* if a certain natural isometry of X into X^{**} is onto. This mapping is $\widehat{\ }: X \to X^{**}$ given by $\hat{x}(x^*) = x^*(x)$. A note of caution is in order. Reflexive means more than $X \cong X^{**}$ as witnessed by an example of James [29] (see also [35]) who constructed a space J with $J \cong J^{**}$ but $J^{**}/\hat{J}$ is one dimensional. A standard theorem in functional analysis courses is that X is reflexive if and only if B_X is weakly compact (i. e. compact in the weak topology on X generated by X^*). And by the Eberlein-Smulian theorem, B_X

is weakly compact if and only if every sequence $(x_n) \subseteq B_X$ admits a weakly convergent subsequence (y_n) (i. e. there exists $y \in B_X$ so that $x^*(y_n) \to x^*(y)$ for all $x^* \in X^*$).

Thus a nonreflexive space X is such because either

(A) there exists $(x_n) \subseteq S_X$ so that no subsequence of (x_n) is *weak Cauchy* (i. e. for all $(y_n) \subseteq (x_n)$ there exists $x^* \in X^*$ so that $\big(x^*(y_n)\big)_{n=1}^\infty$ is a divergent sequence of reals), or

(B) there exists a weak Cauchy sequence $(x_n) \subseteq S_X$ with no weak limit (and so $\hat{x}_n \to x^{**}$ weak* in X^{**} for some $x^{**} \in X^{**} \backslash \hat{X}$).

The prime example of (A) is the unit vector basis (e_n) for ℓ_1. Indeed given $(n_i) \in [\mathbb{N}]$ let $x^* \in \ell_\infty = (\ell_1)^*$ be given by $x^*(n_i) = (-1)^i$ and $x^*(n) = 0$ if $n \notin (n_i)$. Then $x^*(e_{n_j}) = (-1)^j$ diverges. The prime example of (B) is the summing basis for c_0, (s_n), discussed in Section 1. If $x^* = (a_n) \in \ell_1 = (c_0)^*$ then $x^*(s_n) \to \sum_1^\infty a_n$. Theorem 2.5 is proved by showing that in case (A), X contains an isomorph of ℓ_1, while in case (B) X contains an isomorph of c_0. To see (B) suppose that X has an unconditional basis (b_n) and (x_n) is as in (B). Let $a_n = \lim_{m\to\infty} b_n^*(x_m)$. Then it follows that

$$\sup_n \Big\| \sum_{i=1}^n a_n b_n \Big\| < \infty$$

but the series $\sum_1^\infty a_n b_n$ diverges. From this we obtain a *seminormalized* block basis of the form

$$y_n = \sum_{i=p_{n-1}+1}^{p_n} a_i b_i$$

for some $0 = p_0 < p_1 < \cdots$ (seminormalized means $0 < \inf\|y_n\| \le \sup\|y_n\| < \infty$) so that, using the unconditionality of (b_n),

$$\sup \left\{ \Big\| \sum_{i=1}^n \varepsilon_i y_i \Big\| \ : \ n \in \mathbb{N},\ \varepsilon_i = \pm 1 \right\} < \infty$$

which easily yields (a worthwhile exercise if new to the reader; see also the proof of Theorem 2.10 below) that (y_i) is equivalent to the unit vector basis of c_0.

Remarkably, H. Rosenthal in 1974 [55] proved that case (A) yields ℓ_1 under no unconditionality restrictions on X.

THEOREM 2.6. *Let $(x_n) \subseteq S_X$ be such that no subsequence of (x_n) is weak Cauchy. Then X contains a isomorph of ℓ_1. In fact a subsequence of (x_n) is equivalent to the unit vector basis of ℓ_1.*

This is one of the most beautiful theorems in Banach space theory and we shall discuss the proof. The proof (as modified by Farahat [16]) uses a stronger version of Ramsey's theorem than the one we presented above. The situation is as follows. Suppose we can color, using colors R and B, all infinite subsequences of $\mathbb{N}$. Thus $M \in [\mathbb{N}]$ implies $M \in R$ or $M \in B$. Does there exist $M \in [\mathbb{N}]$ so that either for all $N \in [M]$, $N \in R$ or for all $N \in [M]$, $N \in B$? The answer is no in general but yes if R is a "reasonable" set. To be more precise we topologize $[\mathbb{N}]$ by the product topology; thus a basic open set in $[\mathbb{N}]$ is of the form

$$\mathcal{O}(n_1, \ldots, n_k) = \Big\{ M = (m_i) \in [\mathbb{N}] \,:\, m_i = n_i \text{ for } i \leq k \Big\}$$

where $n_1 < \cdots < n_k$ is arbitrary.

THEOREM 2.7. *Let $\mathcal{A} \subseteq [\mathbb{N}]$ be a Borel set for the topology described above. Then for all $M \in [\mathbb{N}]$ there exists $L \in [\mathbb{N}]$ so that either $L \subseteq \mathcal{A}$ or $[L] \subseteq [\mathbb{N}] \backslash \mathcal{A}$.*

In this form the theorem is due to Galvin and Prikry [19] but many others contributed to both weaker and stronger forms (see [47]).

The theorem is used in Banach space theory as follows. Given a "reasonable" property (P) and a subsequence $(x_n) \subseteq X$ so that every subsequence admits a further subsequence having (P), then some subsequence has all of its subsequences having (P). This sort of stability can prove fruitful as we shall illustrate when we discuss the proof of Theorem 2.6. But it only gives information about the subsequences of some sequence and to handle more general problems we have to consider all block bases. Thus we should like to have a "block Ramsey" theorem. We talk about this later. We won't prove Theorem 2.7 here but will later present a proof that has the same general flavor.

Proof of Theorem 2.6. Let $(x_n) \subseteq S_X$ have no weak Cauchy subsequence. Thus $\hat{x}_n : B_{X^*} \to \mathbb{R}$ is a sequence of uniformly bounded functions with no pointwise convergent subsequence. A good model to keep in mind of such an instance are the Rademacher functions $r_n : [0,1] \to \{-1, 1\}$ where $r_1 = h_1$, $r_2 = h_2 + h_3$, $r_3 = h_4 + h_5 + h_6 + h_7, \ldots$ and (h_n) are the Haar functions described in Section 1. They have the property that for all n and $\varepsilon_i = \pm 1$,

$\bigcap_{i=1}^n [r_i = \varepsilon_i] \neq \emptyset$. It follows that in the ∞ or "sup" norm (r_n) is 1-equivalent to the unit vector basis of ℓ_1,

$$\Big\| \sum_{i=1}^n a_i r_i \Big\|_\infty = \sum_{i=1}^n |a_i| \,.$$

Lemma 2.8. *Let S be a set and for $n \in \mathbb{N}$ let A_n and B_n be subsets of S with $A_n \cap B_n = \emptyset$. Assume that for all $M \in [\mathbb{N}]$ there exists $s \in S$ so that $s \in A_n$ for infinitely many $n \in M$ and $s \in B_n$ for infinitely many $n \in M$. Then there exists $M = (m_i) \in [\mathbb{N}]$ so that $(A_{m_i}, B_{m_i})_{i=1}^\infty$ is Boolean independent, i.e. for all finite disjoint $F, G \subseteq \mathbb{N}$,*

$$\Big(\bigcap_{i \in F} A_{m_i} \Big) \cap \Big(\bigcap_{i \in G} B_{m_i} \Big) \neq \emptyset \,.$$

Proof. We use the notation $-A_n \equiv B_n$. Let

$$\mathcal{A}_k = \left\{ m = (m_i) \in [\mathbb{N}] \,:\, \bigcap_{i=1}^k (-1)^i A_{m_i} \neq \emptyset \right\}.$$

Let $\mathcal{A} = \bigcap_{k=1}^\infty \mathcal{A}_k$. Then $\mathcal{A}$ is Borel, in fact clopen, and hence satisfies Theorem 2.7. By that theorem and the hypothesis we obtain $M_0 \in [\mathbb{N}]$ with $[M_0] \subseteq \mathcal{A}$. If $M_0 = (n_i)$ then $M = (m_i)$ satisfies the conclusion of the lemma where $m_i = n_{2i}$ for $i \in \mathbb{N}$. ∎

Returning to 2.6, we claim that there exist disjoint closed intervals I_1, I_2 with rational endpoints and a subsequence (y_n) of (x_n) so that setting $A_n = \{x^* \in B_{X^*} : x^*(y_n) \in I_1\}$ and $B_n = \{x^* \in B_{X^*} : x^*(y_n) \in I_2\}$ then (A_n, B_n) satisfy the hypothesis of the lemma. Indeed if not by considering all such intervals, of which there are countably many and diagonalizing we could produce $(y_n) \subseteq (x_n)$ so that no such pair (I_1, I_2) satisfies the hypothesis of the lemma for (y_n). But there exists $x^* \in B_{X^*}$ so that $\overline{\lim}\, x^*(y_n) > \underline{\lim}\, x^*(y_n)$ and if I_1 and I_2 are disjoint closed intervals with rational endpoints each containing one of these numbers, we get a contradiction.

Let $r_1 = \inf I_1$ and $r_2 = \sup I_2$ with $r_1 > r_2$ as we may suppose. Let $\sum_1^n |a_i| = 1$ and let $F = \{i \leq n : a_i \geq 0\}$ and $G = \{i \leq n : a_i < 0\}$. Then let

$x_1^* \in \big(\bigcap_{i\in F} A_n\big) \cap \big(\bigcap_{i\in G} B_n\big)$ and $x_2^* \in \big(\bigcap_{i\in G} A_n\big) \cap \big(\bigcap_{i\in F} B_n\big)$,

$$\Big\|\sum_{i=1}^n a_i y_i\Big\| \geq \frac{x_1^* - x_2^*}{2}\Big(\sum_{i=1}^n a_i y_i\Big) = \frac{1}{2}\Big(\sum_{i=1}^n a_i (x_1^* - x_2^*)(y_i)\Big)$$
$$\geq \frac{1}{2}\Big[\sum_{i\in F} a_i(r_1 - r_2) - \sum_{i\in G} a_i(r_2 - r_1)\Big] = \frac{1}{2}(r_2 - r_1) > 0\,.$$

Thus for all $(a_i)_1^n$

$$\frac{(r_1 - r_2)}{2}\sum_1^n |a_i| \leq \Big\|\sum_1^n a_i y_i\Big\| \leq \sum_1^n |a_i|$$

and so (y_i) is equivalent to the unit vector basis of ℓ_1. ∎

If $(x_n) \subseteq X$ is weak Cauchy then $(x_1 - x_2, x_3 - x_4, \ldots)$ is weakly null. Thus by Rosenthal's theorem, every X contains either an isomorph of ℓ_1 or a normalized weakly null sequence. As a corollary of this and Theorem 2.2, Proposition 2.3, and Theorem 2.5 we have

COROLLARY 2.9. *Every X admits a spreading model E with an unconditional basis (e_i). E is either reflexive or contains an isomorph of c_0 or ℓ_1.*

One may wonder if there is a way to use Ramsey's theorem to characterize when a normalized basic sequence (x_n) admits a subsequence equivalent to the unit vector basis of c_0. (Of course (x_n) would have to be weakly null.) There is such a result due to W. B. Johnson.

THEOREM 2.10. *Let (x_n) be a normalized weakly null basic sequence. Then either*

(a) *there exists $L = (\ell_i) \in [\mathbb{N}]$ so that (x_{ℓ_i}) is equivalent to the unit vector basis of c_0 or*
(b) *there exists $L \in [\mathbb{N}]$ so that for all $M = (m_i) \in [L]$*

$$\sup_n \Big\|\sum_{i=1}^n x_{m_i}\Big\| = \infty\,.$$

Proof. For $k \in \mathbb{N}$ let

$$\mathcal{A}_k = \left\{ L = (\ell_i) \in [\mathbb{N}] \,:\, \sup_n \left\| \sum_{i=1}^n x_{\ell_i} \right\| \leq k \right\}.$$

$\mathcal{A}_k$ is closed in $[\mathbb{N}]$ so we can inductively choose $M_1 \supseteq M_2 \supseteq \cdots$ so that either $[M_k] \subseteq \mathcal{A}_k$ for some k or else $[M_k] \subseteq [\mathbb{N}] \backslash \mathcal{A}_k$ for each k. In the latter case let (m_i) be a diagonal sequence of the M_k's, i.e. $m_1 < m_2 < \cdots$; $m_i \in M_i$ for all i. It follows that then (b) holds. If $[M_k] \subseteq \mathcal{A}_k$ then letting $M_k = (\ell_i)$ we have

$$\sup \left\{ \left\| \sum_{i \in F} x_{\ell_i} \right\| \,:\, F \subseteq \mathbb{N},\ F \text{ finite} \right\} \leq k$$

and this forces (x_{ℓ_i}) to be equivalent to the unit vector basis of c_0. Indeed let $n \in \mathbb{N}$ and $(a_i)_{i=1}^n \subseteq [-1,1]$. Let $x^* \in B_{X^*}$ satisfy

$$x^* \left(\sum_{i=1}^n a_i x_{\ell_i} \right) = \left\| \sum_{i=1}^n a_i x_{\ell_i} \right\|.$$

But

$$\begin{aligned} x^* \left(\sum_{i=1}^n a_i x_{\ell_i} \right) &\leq \sum_{i=1}^n |x^*(x_{\ell_i})| = x^* \left(\sum_{i \in F} x_{\ell_i} \right) - x^* \left(\sum_{i \in G} x_{\ell_i} \right) \\ &\leq \left\| \sum_{i \in F} x_{\ell_i} \right\| + \left\| \sum_{i \in G} x_{\ell_i} \right\| \leq 2k \end{aligned}$$

where $F = \{ i \leq n : x^*(x_{\ell_i}) \geq 0 \}$ and $G = \{1, \ldots, n\} \backslash F$. ∎

There is a more advanced theorem of this kind due to J. Elton (see [47]) whose proof also uses Ramsey's theorem. We state it without proof.

THEOREM 2.11. *Let (x_n) be a normalized weakly null basic sequence. Then either*

(a) *there exists $L = (\ell_i) \in [\mathbb{N}]$ so that (x_{ℓ_i}) is equivalent to the unit vector basis of c_0 or*
(b) *there exists $L = (\ell_i) \in [\mathbb{N}]$ so that for all $(a_i) \subseteq \mathbb{R}$ with $(a_i) \notin c_0$,*

$$\sup_n \left\| \sum_{i=1}^n a_i x_{\ell_i} \right\| = \infty\,.$$

3. Looking for ℓ_p

As mentioned in the previous section it was once hoped that every X would contain an isomorph of some ℓ_p or c_0, i.e. that these spaces would form some sort of collection of of infinite dimensional atoms from which every general X must be built. The evidence for this conjecture was basically: let's check all the spaces we know. We see that it is true for them (and sometimes this was hard to show) so maybe it is always true. Also work of V.D. Milman [42],[43] provided other evidence connecting the conjecture with a widesweeping, but reasonable stability conjecture. We shall discuss these results and Tsirelson's counterexample to the ℓ_p/c_0 conjecture in this section.

Let X be our usual separable infinite dimensional Banach space and let $f : S_X \to \mathbb{R}$. We say that f is *oscillation stable* if for all $Y \subseteq X$ (recall this means that Y is a closed infinite dimensional linear subspace) and $\varepsilon > 0$, there exists $Z \subseteq Y$ with $\operatorname{osc}(f, Z) \equiv \sup\{f(y) - f(z) : y, z \in S_Z\} < \varepsilon$. Of course one must impose some conditions on f to give it a chance to be oscillation stable. A natural one is f is *Lipschitz* (i.e. there exists $K < \infty$ so that for all $x, y \in S_X$, $|f(x)-f(y)| \le K\|x-y\|$). Another one is f is uniformly continuous. It is not hard to show that every Lipschitz $f : S_X \to \mathbb{R}$ is oscillation stable if and only if every uniformly continuous such f is oscillation stable.

In the course of trying to prove some theorem, the Banach space researcher often is confronted with such a Lipschitz function f. And if f is oscillation stable, life is sweet and the proof works. A special case of a Lipschitz function f as above is when f is an equivalent norm on X. If some equivalent norm $|\cdot|$ on X is not oscillation stable we obtain $Y \subseteq X$ and $\lambda > 1$ so that for all $Z \subseteq Y$

$$\sup\left\{\frac{|y|}{|z|} : y, z \in S_{Z,\|\cdot\|}\right\} > \lambda\,.$$

We say then that $(Y, \|\cdot\|)$ is *λ-distortable* (or is λ-distorted by $|\cdot|$). Geometrically this says we can find a new norm so that restricted to any $Z \subseteq Y$ the new norm is not just (essentially) a multiple of the old norm. Milman's theorem can be stated as follows.

Theorem 3.1. *Suppose X is such that either*

(a) *For all $x \in X$ the function $f_x : S_X \to \mathbb{R}$ given by $f_x(y) = \|x+y\|$ is oscillation stable or*

(b) *Every equivalent norm on X is oscillation stable (i.e. no subspace of X is distortable).*

Then X contains an isomorph of ℓ_p for some $1 \le p < \infty$ or c_0.

What was the evidence for hoping that (a) or (b) might hold? Well, we have two theorems.

THEOREM 3.2. (SEE [44]) *Let $f : S_X \to \mathbb{R}$ be Lipschitz. Then for all $n \in \mathbb{N}$ and $\varepsilon > 0$ there exists $F \subseteq X$ with dimension $F = n$ and $\operatorname{osc}(f, F) < \varepsilon$.*

THEOREM 3.3. ([28]) *ℓ_1 and c_0 are not distortable.*

Proof. To show that ℓ_1 is not distortable it suffices to prove that if (x_n) is a normalized basic sequence (in a space X) which is equivalent to the unit vector basis of ℓ_1 then given $\varepsilon > 0$ there exists a normalized block basis (y_n) of (x_n) satisfying

$$\Big\| \sum a_i y_i \Big\| \ge (1-\varepsilon) \sum |a_i| \tag{3.1}$$

for all $(a_i) \subseteq \mathbb{R}$.

Now, there exists $c > 0$ so that

$$\Big\| \sum a_i x_i \Big\| \ge c \sum |a_i| \tag{3.2}$$

for all $(a_i) \subseteq \mathbb{R}$. The proof works by first passing to a subsequence of (x_i) which yields, essentially, a stable value of c in (3.2) (one that cannot be increased by passing to a further subsequence). And then we choose a block basis (y_n) which (essentially) gives equality in (3.2), i. e.

$$y_n = \sum_{i=p_{n-1}+1}^{p_n} a_i x_i \,, \qquad \|y_n\| = 1 \,, \qquad \sum_{i=P_{n-1}+1}^{P_n} |a_i| \approx c^{-1} \,.$$

Thus y_n's are chosen to give the worst ℓ_1 estimates and as consequence they must yield a good ℓ_1 estimate:

$$\Big\| \sum b_n y_n \Big\| = \Big\| \sum_n \sum_{i=p_{n-1}+1}^{p_n} b_n a_i x_i \Big\| \ge c \sum_n \sum_{i=p_{n-1}+1}^{p_n} |b_n||a_i| \approx \sum_m |b_n| \,.$$

The argument that c_0 is not distortable is similar. One can produce a good upper c_0 estimate much like we proved a good lower ℓ_1 estimate above by taking a block basis realizing the worst c_0 estimate in the limiting case for a sequence equivalent to the unit vector basis of c_0. Then another argument is needed (a good exercise or see [28] or [35]) to show that the good lower c_0 estimate comes along for free. ∎

We will discuss the proof of Theorem 3.1 later. First however we present Tsirelson's example [60] or actually the dual space of the original example as described by Figiel and Johnson [18]. We say that a collection $(E_i)_{i=1}^n$ of finite subsets of $\mathbb{N}$ is *admissible* if $n \le E_1 < E_2 < \cdots < E_n$. (For $F, G \subseteq \mathbb{N}$ by $F < G$ we mean $\max F < \min G$ and $n < F$ means $n < \min F$.) If $x \in c_{00}$ and $E \subseteq \mathbb{N}$ by Ex we mean the restriction of x to E. Thus $Ex(n) = x(n)$ if $n \in E$ and 0 otherwise.

Proposition 3.4. *There exists a norm $\|\cdot\|$ on c_{00} that satisfies the following equation: For all $x \in c_{00}$*

$$\|x\| = \max\left(\|x\|_\infty,\ \sup\left\{\frac{1}{2}\sum_{i=1}^{n}\|E_i x\| : (E_i)_{i=1}^n \text{is admissible}\right\}\right). \tag{3.3}$$

T is then defined to be the completion of $(c_{00}, \|\cdot\|)$. This gives a new way of constructing a Banach space. No longer is a norm defined on c_{00} by an explicit formula but rather the norm is described implicitly as the solution of a certain equation. Surprisingly the additional consequences of this idea were not fully exploited for another 15 years.

Proposition 3.4 can be proved by defining on c_{00}, $\|x\|_0 = \|x\|_\infty$ and inductively

$$\|x\|_{n+1} = \max\left(\|x\|_n,\ \sup\left\{\frac{1}{2}\sum_{i=1}^{m}\|E_i x\|_n : (E_i)_{i=1}^m \text{ is admissible}\right\}\right).$$

Then $\|x\| \equiv \lim\|x\|_n$ is the desired norm.

Theorem 3.5. *T has the following properties*

(1) *The unit vector basis (e_i) is a normalized unconditional basis for T with* $\mathrm{ubc}(e_i) = 1$.
(2) *T is reflexive.*
(3) *If E is any spreading model of T then E is isomorphic to ℓ_1.*
(4) *T does not contain a subspace isomorphic to ℓ_1 (or c_0 or ℓ_p $(1 < p < \infty)$).*

Proof. (1) follows by induction: one proves that $\mathrm{ubc}_{\|\cdot\|_n}(e_i) = 1$ for all n.

(4) By virtue of Proposition 1.5, if T contains an isomorph of some ℓ_p or c_0 then it must contain a normalized block basis (x_n) of (e_n) which is equivalent

to the unit vector basis of some ℓ_p or c_0. Since for all n

$$\Big\|\sum_{i=n+1}^{2n} a_i x_i\Big\| \ge \frac{1}{2}\sum_{i=n+1}^{2n}|a_i|\|x_i\| = \frac{1}{2}\sum_{i=n+1}^{2n}|a_i|$$

by (3.3), the only case left to consider is ℓ_1. So suppose (x_n) is equivalent to the unit vector basis of ℓ_1. By the proof of Theorem 3.3 we may assume (by replacing (x_n) by a block basis if necessary) that for all $(a_i)_1^n \subseteq \mathbb{R}$

$$\Big\|\sum_{i=1}^{n} a_i x_i\Big\| \ge 0.99\sum_{i=1}^{n}|a_i|\,.$$

But one can show that for n large enough

$$\Big\|\frac{1}{2}x_1 + \frac{1}{2n}\sum_{i=2}^{n+1} x_i\Big\| < 0.99$$

which is a contradiction. Indeed suppose that n is fixed and let

$$\frac{1}{2}x_1 + \frac{1}{2n}\sum_{i=2}^{n+1} x_i = x$$

and choose admissible sets $m \le E_1 < \cdots < E_m$ with $\|x\| = \frac{1}{2}\sum_{j=1}^m\|E_j x\|$. (Note: $\|x\|_\infty \le \frac{1}{2}$ so (3.3) yields that the norm must be achieved in this fashion.) Since

$$\Big\|\frac{1}{2n}\sum_{i=2}^{n+1} x_i\Big\| \le \frac{1}{2}$$

by the triangle inequality we may suppose that $E_1 x_1 \ne 0$. Thus $m \le \max(\operatorname{supp} x_1)$ (if $x = \sum a_i e_i$, where $\operatorname{supp} x \equiv \{i : a_i \ne 0\}$). Let

$$I = \big\{i \in [2, n+1] \,:\, E_j x_i \ne 0 \text{ for at least two } j\text{'s}\big\}\,.$$

Then $|I| \le m$ and so by the triangle inequality

$$\begin{aligned}
\|x\| &\le \frac{1}{2}\|x_1\| + \frac{1}{2}\sum_{j=1}^{m}\Big\|E_j\Big(\frac{1}{2n}\sum_{i=2}^{n+1} x_i\Big)\Big\| \\
&\le \frac{1}{2} + \frac{1}{2}\sum_{j=1}^{m}\Big\|E_j\Big(\frac{1}{2n}\sum_{i\in I} x_i\Big)\Big\| + \frac{1}{2}\sum_{j=1}^{m}\Big\|E_j\Big(\frac{1}{2n}\sum_{i\notin I} x_i\Big)\Big\| \\
&\le \frac{1}{2} + \frac{1}{2n}\sum_{i\in I}\|x_i\| + \frac{1}{2}\Big(\frac{1}{2n}\sum_{i\notin I}\|x_i\|\Big) \;<\; \frac{1}{2} + \frac{m}{2n} + \frac{1}{4} \;<\; 0.99
\end{aligned}$$

for large enough n (recall m is bounded above by $\max(\operatorname{supp} x_1)$).

The idea of this argument is roughly as follows. The admissible family $(E_i)_{i=1}^m$ must intersect the support of x_1 so it is bounded in length. The only way $\frac{1}{2}\sum_{j=1}^m \|E_j x_i\|$ can yield more than $\frac{1}{2}$ is if more than one E_j intersects the support of x_i. But this could only happen $m \ll n$ times and thus only contribute a negligible amount to $\|\frac{1}{2n}\sum_{i=2}^{n+1} x_i\|$.

(2) Follows from (1) and (4) and Theorem 2.5.

(3) If $(\tilde{e}_n)$ is a spreading model of a normalized block basis (x_n) of (e_n), then $(\tilde{e}_n)$ satisfies $\|\sum a_i \tilde{e}_i\| \ge \frac{1}{2}\sum |a_i|$ by an observation above (in the proof of (4)). But since T is reflexive, any normalized basic sequence (x_n) in T must be weakly null. [Otherwise, a subsequence (y_n) of (x_n) is weakly convergent to some $y \neq 0$. But a normalized basic sequence cannot converge weakly to $y \neq 0$ since by Mazur's theorem once can find given $\varepsilon > 0$ a two element block basis (z_1, z_2) of (y_n) with $\|z_i - y\| < \varepsilon$ for $i = 1, 2$. Thus $\|y\| - \varepsilon \le \|z_1\| \le \mathrm{bc}(y_i)\|z_1 - z_2\| < \mathrm{bc}(y_i) \cdot \varepsilon$ which is impossible for small ε.] Thus Proposition 1.4 gives the result. ∎

By virtue of Theorem 3.1, T must contain a distortable subspace. In fact it can be shown (and nothing more is known) that T, itself is $(2-\varepsilon)$-distortable for all $\varepsilon > 0$. The norm that witnesses this is given by (for $n = n(\varepsilon)$ sufficiently large)

$$|x|_n \equiv \sup\left\{\frac{1}{2}\sum_{i=1}^n \|E_i x\| \,:\, E_1 < \cdots < E_n\right\}.$$

X is called *arbitrarily distortable* if X is λ-distortable for all $\lambda > 1$. The following question has been open for 12 years.

(Q2) Is T arbitrarily distortable?

Remarks on the proof of Theorem 3.1. For a complete proof see [52] or [5]. We will discuss the key ingredients of the argument, with an eye towards showing how stabilization plays a major role. The argument is more complicated than the other proofs we have presented and we have only sketched certain steps.

(1) For $x \in X$ define for $y \in X$,

$$\|y\|_x = \big\|x\|y\| + y\big\| + \big\|x\|y\| - y\big\|\,.$$

It can be shown that $\|\cdot\|_x$ is an equivalent norm on X for all $x \in X$. Moreover under either hypothesis (a) or (b) one has that every $\|\cdot\|_x$ is oscillation stable on X.

(2) In general, if E is a spreading model of X we cannot find a subspace of E which is isomorphic to a subspace of X. For example every spreading model E of T is isomorphic to ℓ_1 and by an earlier remark thus every subspace of E contains an isomorph of ℓ_1 but T does not contain ℓ_1 isomorphically. However if the spreading model of (x_n) is (e_n) which doubly generates an ℓ_p type over X (or c_0 type) then we can pull it back into the space. We explain what this means. Since X is separable if (x_n) is normalized basic we can find $(y_n) \subseteq (x_n)$ so that for all $x \in X$, $n \in \mathbb{N}$ and $(a_i)_{i=1}^n \subseteq \mathbb{R}$ the iterated limit

$$\lim_{k_1\to\infty} \cdots \lim_{k_n\to\infty} \left\| x + \sum_{i=1}^n a_i y_{k_i} \right\| \equiv \left\| x + \sum_{i=1}^n a_i e_i \right\|$$

exists. This is proved using Ramsey's theorem just like Theorem 2.2. The limit thus defines a norm on $X \oplus E$, a *spreading model over* X. (y_n) *doubly generates an* ℓ_p *type over* X if for all $x \in X$ and $(\alpha, \beta) \in S_{\ell_p^2}$

$$\lim_{i\to\infty} \lim_{j\to\infty} \|x + \alpha y_i + \beta y_j\| = \lim_{i\to\infty} \|x + y_i\| = \|x + e_1\|$$

(the c_0 case is defined similarly using $S_{\ell_\infty^2}$). One can show (it is not hard) that in this case given $\varepsilon > 0$ a subsequence of (y_n) is $(1+\varepsilon)$-equivalent to the unit vector basis of ℓ_p (or c_0).

The task is thus to use the stabilization of all $\|\cdot\|_x$ on X to produce such a spreading model. The "p" involved comes from the following famous theorem of Krivine [33] which is also a stabilization result.

(3) *Krivine's Theorem*: Given $C > 0$, $\varepsilon > 0$ and $k \in \mathbb{N}$ there exists $n = n(C, \varepsilon, k)$ so that if $(x_i)_1^n$ is basic with $\mathrm{bc}(x_i)_1^n \le C$ then there exists $p \in [1, \infty]$ and a block basis $(y_i)_1^k$ of $(x_i)_1^n$ which is $(1+\varepsilon)$-equivalent to the unit vector basis of ℓ_p^k.

(4) Every space X contains a normalized basic sequence (x_n) which generates a spreading model (e_n) over X with (e_n) monotone basic and moreover for all $x \in X$ and $e \in E$,

$$\|x + e\| = \|x - e\| .$$

Using Rosenthal's ℓ_1 theorem (Theorem 2.6), the proof of Proposition 1.3 (b) and the proof that ℓ_1 is not distortable (Theorem 3.3) it is easy to obtain a basic sequence $(b_n) \subseteq X$ so that any spreading model (e_n) of any normalized block basis (x_n) of (b_n) is monotone. The trick is to get $\|x+e\| = \|x-e\|$. This is done using the *Borsuk-Ulam antipodal mapping theorem*: If $k \ge 1$ and F is any $k+1$ dimensional Banach space and $\phi : S_F \to \mathbb{R}^k$ is a continuous *antipodal* mapping (i.e. $\phi(-x) = -\phi(x)$ for $x \in S_F$), then there exists $x \in S_F$ with $\phi(x) = 0$.

This is used as follows. Let (d_n) be dense in X. Assume $x_1, \dots, x_n$ have been chosen as a normalized block basis of (b_n) and lie in $\mathrm{span}(b_i)_1^{m-1}$. Define $\phi : S_{\langle b_i \rangle_m^{m+n+1}} \to \mathbb{R}^{n+1}$ by $\big(\phi(b)\big)_j = \|d_j + b\| - \|d_j - b\|$ for $j \le n+1$. Choose $x_{n+1} \in S_{\langle b_i \rangle_m^{m+n+1}}$ with $\phi(x_{n+1}) = 0$. A subsequence of (x_n) will generate the desired spreading model.

(5) Given a basic sequence (b_i) in X one can find a normalized block basis (x_i) of (b_i) that generates a spreading model E over X which satisfies (4) and in addition

$$\|x+e\| = \|x+e'\| \qquad \text{if } x, e, e' \in E \text{ with } \|e\| = \|e'\|\,.$$

This is accomplished by using the stability of the functions $\|\cdot\|_x$. Let (d_i) be dense in X. By a diagonal procedure given $\varepsilon_n \downarrow 0$ we can find a normalized block basis (z_n) of (b_i) so that for all n and $i \le n$,

$$\big|\|y\|_{d_i} - \|z\|_{d_i}\big| < \varepsilon_n \qquad \text{for } y, z \in S_{[(z_i)_n^\infty]}\,.$$

This property also holds for any normalized block basis (x_n) of (z_n) and thus we can assume (4) holds for some such (x_n) as well as the above estimates for (z_n) replaced by (x_n). Thus if (w_j) is any normalized block basis of (x_n) we have for $x \in X$,

$$\begin{aligned}\|x+e_1\| &= \lim_{j\to\infty} \|x+x_j\| = \lim_{j\to\infty} \frac{1}{2}\big(\|x+x_j\| + \|x-x_j\|\big)\\ &= \lim_j \frac{1}{2}\|x_j\|_x = \lim_{j\to\infty} \frac{1}{2}\|w_j\|_x = \lim_{j\to\infty}\|x+w_j\|\end{aligned}$$

and (5) follows. From this it is easy to see that

(6) $\|x+a+e\| = \|x+a+e'\|$ if $m \in \mathbb{N}$, $x \in X$, $a \in \mathrm{span}(e_i)_1^m$, $e, e' \in \mathrm{span}(e_i)_{m+1}^\infty$ and $\|e\| = \|e'\|$.

(7) Given $x \in X$ and $\varepsilon > 0$ there exists a subsequence $(x_i') \subseteq (x_i)$ so that for all $k \in \mathbb{N}$ and $\big\|\sum_1^k \alpha_j e_j\big\| \le 1$,

$$\left|\left\|x + \sum_{j=1}^k \alpha_j x_i'\right\| - \left\|x + \sum_{j=1}^k \alpha_j e_j\right\|\right| < \varepsilon\,.$$

To do this we inductively choose $(x'_i) \subseteq (x_i)$ so that if $i \in \mathbb{N}$, $(\alpha_j)_1^j \subseteq [-2,2]$, $\gamma, \beta \in [-2,2]$ then

$$\left| \left\| x + \sum_{j=1}^{i} \alpha_j x'_j + \beta x'_{i+1} + \gamma e_2 \right\| - \left\| x + \sum_{j=1}^{i} \alpha_j x'_j + \beta e_1 + \gamma e_2 \right\| \right| < \varepsilon 2^{-(i+1)}$$

and note that then the left hand side of the inequality in (7) is

$$\begin{aligned} &\le \sum_{i=1}^{k} \left| \left\| x + \sum_{j=1}^{i} \alpha_j x'_j + \sum_{j=i+1}^{k} \alpha_j e_j \right\| - \left\| x + \sum_{j=1}^{i-1} x'_j + \sum_{j=1}^{k} \alpha_j e_j \right\| \right| \\ &= \sum_{i=1}^{k} \left| \left\| x + \sum_{j=1}^{i-1} \alpha_j x'_j + \alpha_i x'_i + \left\| \sum_{j=i+1}^{k} \alpha_j e_j \right\| e_2 \right\| \right. \\ &\qquad \left. - \left\| x + \sum_{j=1}^{i-1} \alpha_j x'_j + \alpha_i e_i + \left\| \sum_{j=i+1}^{k} \alpha_j e_j \right\| e_2 \right\| \right| < \sum_{i=1}^{k} 2^{-i} \varepsilon \ < \ \varepsilon\,. \end{aligned}$$

And this holds for all further subsequences of (x'_i). Thus a diagonal argument yields

(8) Given $\varepsilon_n \downarrow 0$ there is a subsequence (w_j) of (x_i) so that for all $k, m \in \mathbb{N}$, $b \in mB_{\mathrm{span}(d_i,w_i)_1^m}$, $m < n_1 < \cdots < n_k$,

$$\left| \left\| b + \sum_{i=1}^{k} \alpha_i w_{n_i} \right\| - \left\| b + \sum_{i=1}^{k} \alpha_i e_i \right\| \right| < \varepsilon_m \qquad \text{if } \left\| \sum_{i=1}^{k} \alpha_i e_i \right\| \le 1\,.$$

(9) (Another stabilization result [48]) Given $C > 0$, $\varepsilon > 0$ and $k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ so that if $\dim F = n$ and F has a basis $(x_i)_1^n$ with $\mathrm{bc}(x_i)_1^n \le C$ and $f : S_E \to \mathbb{R}$ is C-Lipschitz then there exists a normalized block basis $(y_i)_1^k$ of $(x_i)_1^n$, so that

$$|f(x) - f(y)| < \varepsilon \qquad \text{for all } x, y \in S_{\mathrm{span}(y_i)_1^k}\,.$$

From this and Krivine's theorem we can find a normalized block basis $(a_i, b_1, a_2, b_2, \ldots)$ of (e_i) and $p \in [1, \infty]$ so that if $m \in \mathbb{N}$, $x \in mB_{\mathrm{span}((d_i)_1^{m-1} \cup (a_i)_1^{m-1} \cup (b_i)_1^{m-1})}$ and $|\alpha|, |\beta| \le 1$ then

(10) $\big| \|x + \alpha a_m + \beta b_m\| - \|x + (|\alpha|^p + |\beta|^p)^{1/p} b_m\| \big| < 2^{-m}$. Now we also have in this case for $m < n_1 < n_2$

$$\|x + \alpha a_{n_1} + \beta a_{n_2}\| = \|x + \alpha a_{n_1} + \beta b_{n_2}\|$$

if $(\alpha, \beta) \in S_{\ell_p^2}$.

From this we can obtain a sequence (from (a_i)) that doubly generates an ℓ_p type over X.

The key thing to note in this proof is how stability arguments played a crucial role.

4. Gowers' Dichotomy Theorem

Following the solution of the problem "Does every X contain an isomorph of c_0 or ℓ_p for some $1 \leq p < \infty$?" more attention became focused on the problem "Does every X contain an unconditional basic sequence?" If so in every X one could find a subspace Y with very nice structure. In particular one could construct many nontrivial projections on Y.

In 1977 Maurey and Rosenthal [40] constructed a normalized weakly null basis (x_n) such that no subsequence of (x_n) was unconditional. Thus to find an unconditional basic sequence inside a given space with a basis, one would have to search amongst the block bases and not just the subsequences. So the known Ramsey theory was inadequate. One would need a "block Ramsey" type theorem. Such block theorems did exist but they were not the right kind or at least nobody could see how to apply them. Here is an example of such a theorem. It generalized Ramsey's theorem (Theorem 2.1). We state it in a Banach space sort of way.

Theorem 4.1. (Hindman–Milliken [27], [41]) *Let $m \in \mathbb{N}$, $n \in \mathbb{N}$ and let β denote the class of all block bases of a given basis (e_i). Let $f : \beta \to \{1, \ldots, n\}$ be a function such that if $(x_i), (y_i) \in \beta$ and $\operatorname{supp}(x_i) = \operatorname{supp}(y_i)$ for $1 \leq i \leq m$ then $f\big((x_i)\big) = g\big((y_i)\big)$. Then there exists $(x_i) \in \beta$ so that $f\big((x_i)\big) = g\big((y_i)\big)$ for all block bases (y_i) of (x_i).*

So it remained to either find a combinatorial–analytical theorem that could be successfully used to study the block bases of a given basis or to construct a space X without any unconditional basic sequence inside it. As it turned out both things happened. First Gowers and Maurey constructed the space X without any unconditional basic sequence [23]. Then Gowers proved a block Ramsey theorem and from this deduced his famous Dichotomy Theorem [20].

Gowers and Maurey were inspired by the construction of Schlumprecht's space S in 1991 [57]. A number of variants of Tsirelson's space T were constructed in the 15 year interval before S appeared but the focal point of these examples had nothing to do with distortion. Schlumprecht set out to construct the first arbitrarily distortable Banach space. The definition of T involved the parameter $\frac{1}{2}$ and T is $(2-\epsilon)$-distortable. If the parameter is changed to $\frac{1}{n}$ the new space becomes $(n-\epsilon)$-distortable. Motivated by this, Schlumprecht defined S as the completion of c_{00} under the implicit norm

$$\|x\| = \max\left(\|x\|_\infty,\ \sup\left\{\frac{1}{f(n)}\sum_{i=1}^{n}\|E_i x\| : n \geq 2, E_1 < \cdots < E_n\right\}\right)$$

where $f(n) = \log_2(n+1)$ for $n \in \mathbb{N}$. He proved that S was arbitrarily distorted by the collection of norms

$$\|x\|_n = \max\left(\|x\|_\infty,\ \sup\left\{\frac{1}{f(n)}\sum_{i=1}^{n}\|E_i x\| : E_1 < \cdots < E_n\right\}\right).$$

As observed by Gowers and Maurey his arguments actually showed that S is *biorthogonally distortable*. This means that there exists $\epsilon_n \downarrow 0$ and sets $A_n \subseteq B_S$, $B_n \subseteq B_{S^*}$ so that:

(1) For all $X \subseteq S$ and $n \in \mathbb{N}$, $A_n \cap X \neq \emptyset$.
(2) For all $n \in \mathbb{N}$ and $x \in A_n$ there exists $x^* \in B_n$ with $x^*(x) > \frac{1}{2}$.
(3) If $x \in A_n$ and $x^* \in B_m$ with $n \neq m$ then $|x^*(x)| < \epsilon_{\min(n,m)}$.

And they observed that for every K a biorthogonally distortable space could (using ideas from [40]) be renormed so as to not contain an unconditional basic sequence (x_i) with $\mathrm{ubc}(x_i) \leq K$. Of course, it is easy to see that S has a 1-subsymmetric unconditional basis so renorming S won't cause unconditionality to disappear. Also the implicit equation used to describe the norm in S (or T) is inherently unconditional. Gowers and Maurey defined a new Banach space GM by describing (see section (1) a norming set of functionals $\mathcal{F}$ in the dual in such a manner as to produce the useful properties of S but so as to lose the unconditionality as in [40]. Thus the first space not containing an unconditional basic sequence was born.

Gowers and Maurey went on to show that their space GM was HI (*hereditarily indecomposable*) which means that it has no nontrivial complemented subspaces and every subspace has this property. The Hahn Banach theorem easily yields that if $F \subseteq X$, F finite dimensional, then F is complemented in X (i.e. F is the range of a projection on X). Let us write $X = Y \oplus Z$ if Y and Z are subspaces of X so that for some projection P on X, $Y = P(X)$ and $Z = (I-P)(X)$. Equivalently, $X = Y \oplus Z$ if $Y + Z = X$

and $0 < d(S_Y, S_Z) \equiv \inf\{\|y-x\| : y \in S_Y, z \in S_Z\}$. X is HI if for all $Y \subseteq X$, $Y = W \oplus Z$ implies that W or Z must be finite dimensional. Hence X is HI means that for all infinite dimensional $Y, Z \subseteq X$ and $\varepsilon > 0$ there exists $y \in S_Y$, $z \in S_Z$ with $\|y - z\| < \varepsilon$.

As we noted above, having an unconditional basis guarantees the existence of many nontrivial projections on a space. Being HI says that no nontrivial projections exist. Remarkably, by passing to a subspace one must encounter one or the other of these alternatives.

Theorem 4.2. (Gowers' Dichotomy Theorem [20]) *For every Banach space X there exists $Y \subseteq X$ so that either Y has an unconditional basis or Y is HI.*

Gowers deduced this beautiful theorem from another beautiful theorem which he created, a block Ramsey type theorem. Let X have a basis (e_i). We let Σ denote the set of all finite normalized block bases (including the empty sequence) of (e_i). Let $\sigma \subseteq \Sigma$ and let $Y \prec X$ (by this we mean Y is a subspace of X generated by a block basis of (e_i)). We say σ *covers* Y if there exists $n \in \mathbb{N}$ and $(x_i)_{i=1}^n \in \sigma \cap Y^n$. σ is called *large* if for all $Y \prec X$, σ covers Y.

Now suppose that $\sigma \subseteq \Sigma$ is large. We consider a game played by two players, S and V. S chooses $Y_1 \prec X$ and V chooses a vector x_1 in the unit sphere of Y_1. Then S chooses $Y_2 \prec X$ and V chooses a vector x_2 in the unit sphere of Y_2 and so on. We say V *wins* if for some $k \in \mathbb{N}$, $(x_1, \ldots, x_k) \in \sigma$. S *wins* if $(x_1, x_2, \ldots, x_k) \notin \sigma$ for all k. Let $\Delta = (\delta_1, \delta_2, \ldots)$ be a sequence of positive numbers. Set

$$\sigma_\Delta = \left\{ (y_1, \ldots, y_k) \in \Sigma : \begin{array}{c} \text{there exists } (x_1, \ldots, x_k) \in \sigma \\ \text{with } \|x_i - y_i\| < \delta_i \text{ if } i \le k \end{array} \right\}.$$

Theorem 4.3. *Let $\sigma \subseteq \Sigma$ be large for X. Let Δ be any sequence of positive numbers. Then there exists $Z \prec X$ so that V has a winning strategy inside Z for σ_Δ.*

The conclusion means that if S is restricted to choosing block subspaces of Z, then V has a winning strategy (to find $(x_1, \ldots, x_k) \in \sigma_\Delta$). To deduce Theorem 4.2 from this, one argues as follows. Let us say X is HI(ε) if for all $Y, Z \prec X$ there exists $y \in Y$, $z \in Z$ with

$$\|y - z\| < \varepsilon \|y + z\|\,.$$

It is an easy exercise to show that X is HI if and only if X is HI(ε) for all $\varepsilon > 0$. Gowers used Theorem 4.3 to prove that if X contains no sequence

(x_n) with $\mathrm{ubc}(x_n) \le C$ then X contains a $\mathrm{HI}(\frac{1}{2C})$ block subspace. A diagonal argument (see below) then yields the theorem. The above is proved by using

$$\sigma \equiv \left\{ (x_i)_{i=1}^k \in \Sigma \,:\, \left\| \sum_{i=1}^k (-1)^i \lambda_i x_i \right\| < \frac{1}{C} \left\| \sum_{i=1}^k \lambda_i x_i \right\| \text{ for some } (\lambda_i)_{i=1}^k \subseteq \mathbb{R} \right\}.$$

We will not prove Theorem 4.3 but we will prove Gowers' Dichotomy Theorem. In fact we give two proofs. The first proof we give is due to Maurey [36], [37] and the flavor of the proof is the same as the proof of Theorem 2.7.

Maurey's proof of Theorem 4.2. We may assume X has a basis (e_i). We need only show that given $\varepsilon > 0$, X contains either a block basis (x_n) with $\mathrm{ubc}(x_n) \le \frac{2}{\varepsilon}$ or else X contains a block subspace which is $\mathrm{HI}(2\varepsilon)$.

Indeed suppose we can do this. We write $Y \prec X$ if Y is a block subspace (a subspace spanned by a block basis) of (e_i). Letting $\varepsilon = \frac{2}{n}$ for $n \in \mathbb{N}$ we use the above claim (yet to be proved) to inductively choose $X \succ X_1 \succ X_2 \succ \cdots$ so that for all n either X_n contains an unconditional basic sequence (with unconditional basis constant not exceeding n) or else X_n is $\mathrm{HI}(4/n)$. We write $Y \overset{a}{\prec} Z$ if Y is *almost* a block subspace of Z, i.e., Y has a basis which is a block basis of Z except for finitely many terms. By a diagonal procedure we can choose $Y \prec X$ with $Y \overset{a}{\prec} X_n$ for $n \in \mathbb{N}$. If for all n, X_n is $\mathrm{HI}(4/n)$ then so is Y and thus Y is HI. Otherwise X contains an unconditional block basis.

To prove the claim we work from now on in

$$X_0 \equiv \left\{ x = \sum_{i=1}^n a_i e_i \,:\, n \in \mathbb{N}, (a_i)_{i=1}^n \subseteq \mathbb{Q} \right\}$$

where $\mathbb{Q}$ is the set of rationals. Thus by $Y \prec X_0$ we now mean that Y is a $\mathbb{Q}$-linear subspace of X_0 generated by a block basis (in X_0) of (e_i). Set $A = \big\{ (x,y) \in X_0 \times X_0 : \|x - y\| < \varepsilon \|x + y\| \big\}$. For $(x,y) \in X_0 \times X_0$ and $Z \prec X_0$ we say (x,y) *accepts* Z if for all $U, V \prec Z$ there exists $u \in U$,$v \in V$ with $(x+u, y+v) \in A$. (x,y) *rejects* Z if for all $W \prec Z$, (x,y) does not accept W.

Note that if (x,y) accepts Z then for all $W \overset{a}{\prec} Z$, (x,y) accepts W and the same holds if we replace "accepts" by "rejects." Also for all $(x,y) \in X_0 \times X_0$ and $Z \prec X_0$ there exists $W \prec Z$ so that (x,y) either accepts or rejects W.

List the elements of $X_0 \times X_0$ as $(x_n, y_n)_{n=1}^\infty$. Inductively we can choose $X_1 \succ X_2 \succ \cdots$ so that for all n, (x_n, y_n) rejects or accepts X_n. Let $Z_0 \prec X$

with $Z_0 \overset{a}{\prec} X_n$ for all n. If $(0,0)$ accepts Z_0 then $\overline{Z}_0$ is HI(2ε) (ε increases to 2ε to allow approximation by elements of Z_0). Otherwise we shall construct a block basis (z_k) of Z_0 with $1 \le \|z_k\| \le 2$ for all k so that for all $m \in \mathbb{N}$ and $\varepsilon_i = \pm 1$

$$\Big\| \sum_{k=1}^{m} a_k z_k \Big\| \le \frac{1}{\varepsilon} \Big\| \sum_{k=1}^{m} \varepsilon_k a_k z_k \Big\| \tag{4.1}$$

provided that each $a_k = \frac{j_k}{N2^k}$ for some integer $|j_k| \le N2^k$ for $k \le m$ where $N > \frac{16}{\varepsilon}$ is a fixed integer. This is enough by standard approximation arguments to show (4.1) holds for all $(a_k) \subseteq \mathbb{R}$ (we may assume $\mathrm{bc}(e_i) < 2$) with $\frac{1}{\varepsilon}$ replaced by $\frac{2}{\varepsilon}$.

It suffices to construct the z_k's so that $\|x - y\| \ge \varepsilon \|x + y\|$ whenever $(a_k)_{k=1}^m$ is as above, I and J partition $\{1, 2, \dots, m\}$ and $x = \sum_{k \in I} a_k z_k$, $y = \sum_{k \in J} a_k z_k$. In other words we wish for all such x and y that $(x, y) \notin A$.

Thus we wish to construct $(z_k) \prec Z_0$ so that every pair (x, y) of the form given above with respect to (z_k), we call such (x, y) a *reasonable pair*, rejects Z_0 (which will imply $(x, y) \notin A$). Our starting point is $(0, 0)$ rejects Z_0.

Note that if (x, y) rejects Z_0 then for all $W \prec Z_0$ there exists $W' \prec W$ so that for all $w' \in W'$, $(x + w', y)$ rejects Z_0. If not there exists $W \prec Z_0$ so that for all $W' \prec W$ there exists $w' \in W'$ so that $(x + w', y)$ accepts Z_0. Thus for all $U \prec W$ there exists $(w'', v) \in W' \times U$ so that $(x + w' + w'', y + v) \in A$. Hence (x, y) accepts W which contradicts (x, y) rejects Z_0.

Assume $(z_i)_1^n$ have been chosen so that all reasonable pairs formed from $(z_i)_1^n$ reject Z_0. Since there are only finitely many such reasonable pairs by our observation above there exists $W \prec Z_0$ so that for all $w \in W$ and all reasonable pairs (x, y), $(x + w, y)$ rejects Z_0. Choose $1 \le \|z_{n+1}\| < 2$ with $z_{n+1} \in W$. Consider any reasonable pair $(x', y') = (x + a z_{n+1}, y)$ or $(x, y + a z_{n+1})$ for some a and some reasonable pair (x, y) formed from $(z_i)_1^n$. In both cases (x', y') rejects Z_0 (the second case uses the symmetry of A and reasonable pairs). ∎

The second proof we give of Gower's dichotomy theorem is due to Figiel, Frankeiwicz, Komorowski and Ryll-Nardzewski, We thank the authors for allowing us to present this pretty argument. Additional and more general results are obtained in [17].

Second Proof [17] As in Maurey's proof we work in X_0 and employ the same notation $Y \prec Z$ and $Y \overset{\alpha}{\prec} Z$. We begin with some additional terminology.

For $U,V \prec X_0$, $\phi(U,V) \equiv \sup\{\|u-v\| : u \in U, v \in V, \|u+v\| = 1\}$. Note that $\phi(U,V) < \infty$ iff $d(S_U,S_V) > 0$ iff $[U+V] = [U] \oplus [V]$. For $Y \prec X_0$, $\tilde\phi(Y) \equiv \inf\{\phi(U,V) : U,V \prec Y\}$. Thus $\tilde\phi(Y) = \infty$ iff Y is HI. $\vec E = (E_1,E_2)$ will denote that $E_1,E_2 \prec X_0$ and both spaces are finite dimensional, and possibly $\{0\}$. We let $\tilde\phi(\vec E,Y) \equiv \inf\{\phi(E_1+U,E_2+U) : U,V \prec Y\}$.

(i) If $W \overset{\alpha}{\prec} Y$ then $\tilde\phi(\vec E,Y) \le \tilde\phi(\vec E,U)$. Indeed let $U,V \prec W$. Let $U_0 \prec U,Y$ and $V_0 \prec V,Y$. Then

$$\phi(E_1+U_0,E_2+V_0) \le \phi(E_1+U,E_2+V)$$

which yields (i). From this we obtain

(ii) There exists $Y \prec X_0$ so that for all $\vec E = (E_1,E_2)$ and $Y \prec W$,

$$\tilde\phi(\vec E,Y) = \tilde\phi(\vec E,W)\,.$$

Indeed this follows from (i) and these three things. There are only countably many $\vec E$'s. There does not exist a collection of reals $(r_\alpha)_{\alpha<\omega_1}$ with $r_\alpha < r_\beta$ if $\alpha < \beta$. Given $\beta < \omega_1$ and $(Y_\alpha)_{\alpha<\beta}$ with $Y_\alpha \overset{\alpha}{\succ} Y_\gamma$ if $\alpha < \gamma < \beta$ there exists Y_β with $Y_\beta \overset{\alpha}{\prec} Y_\alpha$ for all $\alpha < \beta$. From (ii) and (i) we obtain

(iii) If $W \overset{\alpha}{\prec} Y$ then $\tilde\phi(\vec E,Y) = \tilde(\vec E,W)$.

(iv) Let $\mathcal{E}$ be a collection of finitely many $\vec E's$. Assume $\tilde\phi(\vec E,Y) < d$ for all $\vec E \in \mathcal{E}$. Then there exists $Z \prec Y$ so that for all $\vec E = (E_1,E_2) \in \mathcal{E}$ and all $z \in Z$,

(a) $\tilde\phi\big((\langle E_1,z\rangle,E_2),Y\big) < d$,
(b) $\tilde\phi\big(E_1,\langle E_2,z\rangle),Y\big) < d$.

It suffices to prove by (ii) that for a fixed $\vec E = (E_1,E_2)$ there exists $Z \prec Y$ satisfying (a) and (b) with Y replaced by Z. There exists $U,V \prec Y$ with $\phi(E_1+U,E_2+V) < d$. Thus for all $u \in U$, $\tilde\phi\big((\langle E_1,u\rangle,E_2),Y,\big) = \tilde\phi\big((\langle E_1,u\rangle,E_2),U\big) < d$. We repeat this argument in the second coordinate to obtain $Z \prec U$ with the desired property.

If $\tilde\phi(Y) = \tilde\phi\big((0,0),Y\big) = \infty$ then Y is HI. Otherwise let $\tilde\phi(Y) < d$. We will prove by induction that

(v) There exists a block basis (x_i) of Y so that for all n and all partitions G_1,G_2 of $\{1,2,\dots,n\}$ if $E_i = \langle x_j\rangle_{j\in G_i}$ and $\vec E = (E_1,E_2)$ then $\tilde\phi(\vec E,Y) < d$.

To do this use $\tilde{\phi}\big((0,0),Y\big) < d$ to choose $Z_1 \prec Y$ by (iv). Thus if $x_1 \in Z_1$, $\tilde{\phi}\big((\langle x_1\rangle,0),Z_1\big) < d$ and $\tilde{\phi}\big((0,\langle x_1\rangle),Z_1\big) < d$. We apply (iv) again to these two choices of $\vec{E}$ to obtain $Z_2 \prec Z_1$ and let $x_2 \in Z_2$ with $x_2 > x_1$ and so on. It follows immediately that (x_i) is unconditional:

$$\Big\|\sum_1^n \varepsilon_i a_i x_i\Big\| \le d \Big\|\sum_{i=1}^n a_i x_i\Big\|$$

for $\varepsilon_i = \pm 1$, $(a_i)_1^n \subseteq \mathbb{Q}$ and hence for all $(a_i)_1^n \subseteq \mathbb{R}$. ∎

5. Odds and ends

In this section we continue the discussion of some of the topics raised above and state some more open problems.

First let us return to distortion or more generally the oscillation stability of a Lipschitz function $f : S_X \to \mathbb{R}$. c_0 and ℓ_1 are not distortable. Are there other nondistortable spaces X? What about ℓ_2 or ℓ_p $(1 < p < \infty)$? Are there spaces X on which every Lipschitz function f is oscillation stable?

Theorem 5.1. ([49]) *For $1 < p < \infty$, ℓ_p is arbitrarily distortable and in fact biorthogonally distortable.*

In the case of ℓ_2 we obtain sets $A_n \subseteq S_{\ell_2}$ for $n \in \mathbb{N}$ and $\varepsilon_n \downarrow 0$ so that:
(i) for $n \in \mathbb{N}$ and $X \subseteq \ell_2$, $A_n \cap X \neq \emptyset$;
(ii) for $x \in A_n$, $y \in A_m$ with $n \neq m$, $|\langle x,y\rangle| < \varepsilon_{\min(n,m)}$.
Thus the sets (A_n) are each very large but A_n and A_m are nearly orthogonal for $n \neq m$.

The proof of this theorem is indirect. One does not construct the sets A_n by working directly in ℓ_2 but rather infers the existence of such sets using the properties of S. Thus S is not just some pathological Banach space but rather its existence has implications in the "real world" of ℓ_2.

The companionship of c_0 and ℓ_1 in both being nondistortable separates when we consider the oscillation stability of arbitrary Lipschitz functions.

Theorem 5.2. ([22]) *Every Lipschitz $f : S_{c_0} \to \mathbb{R}$ is oscillation stable*

The proof is a delicate argument involving untrafilters.

Theorem 5.3. ([49]) *If every Lipschitz $f : S_X \to \mathbb{R}$ is oscillation stable then for all $Y \subseteq X$ some subspace of Y is isomorphic to c_0.*

The relationship between the properties of being distortable, arbitrarily distortable, or biorthogonally distortable are still unclear.

(Q3) If X is distortable does X contain an arbitrarily distortable subspace?

(Q4) If X is arbitrarily distortable does X contain a biorthogonally distortable subspace?

Theorem 5.4. ([49]) *If X is not distortable then for all $Y \subseteq X$, Y contains an isomorph of either c_0 or ℓ_1.*

Regarding (Q3) some partial results are known about the structure of a space which is distortable but does not contain an arbitrarily distortable subspace if it does exist.

Theorem 5.5. ([59], [45], [38]) *Let X be a space that does not contain an arbitrarily distortable subspace. Then X contains an unconditional basic sequence (x_i). Moreover there exists $p \in [1, \infty]$ and $C < \infty$ so that for all n if $(y_i)_1^n$ is a normalized block basis of $(x_i)_{i=1}^\infty$ then $(y_i)_1^n$ is C-equivalent to the unit vector basis of ℓ_p^n. (We call (x_i) an asymptotic ℓ_p basis.) If ℓ_1 is not finitely representable in $\left[(x_i)_{i=1}^\infty\right]$ then $\left[(x_i)_{i=1}^\infty\right]$ contains an arbitrarily distortable subspace.*

While ℓ_2 is arbitrarily distortable it is not known if one can also distort all of its spreading models.

(Q5) For $K < \infty$ does there exist an equivalent norm $|\cdot|$ on ℓ_p $(1 < p < \infty)$ so that if (e_i) is a spreading model of $(\ell_p, |\cdot|)$ then (e_i) is not K-equivalent to the unit vector basis of ℓ_p?

We now turn to some additional results and problems involving spreading models. First Theorem 5.5 raises the question as to whether a space with an asymptotic ℓ_1 basis can be arbitrarily distortable (or if a space with all spreading models equivalent to the unit vector basis of ℓ_1 can be arbitrarily distortable).

THEOREM 5.6. ([2]) *There exists a space X with a asymptotic ℓ_1 basis which is arbitrarily distortable. In fact one can find such a space X which is HI. And one can find such a space with an unconditional basis.*

Every space X has a nice spreading model E: E has an unconditional basis and is either reflexive or contains an isomorph of c_0 or ℓ_1.

Can one do more? Does every X have a spreading model isomorphic to c_0 or some ℓ_p or at least contain such? Must every X have a spreading model E which is reflexive or isomorphic to c_0 or ℓ_1? The answer to these questions is no.

THEOREM 5.7. ([50],[1]) (a) *There exists a space X so that if E is any spreading model of X then E does not contain an isomorph of c_0 or ℓ_p $(1 \leq p < \infty)$.*

(b) *There exists a space X so that every spreading model E of X must contain an isomorph of ℓ_1 but E is never isomorphic to ℓ_1.*

A spreading model of a space X is finitely representable in X but generally is not isomorphic to a subspace of X. Are there hypotheses on the spreading models that do allow one to make deductions about the subspace structure of X? Yes in some instances.

THEOREM 5.8. ([51]) *Let X have a basis (e_i) so that every spreading model of a normalized block basis of (e_i) is 1-equivalent to the unit vector basis of ℓ_1 (respectively, c_0). Then X contains an isomorph of ℓ_1 (respectively, c_0).*

THEOREM 5.9. ([53]) *Every X can be given an equivalent norm so that the following holds.*

(a) *If some spreading model of X is 1-equivalent to the unit vector basis of ℓ_1 then X contains an isomorph of ℓ_1.*
(b) *If some spreading model of X is 1-equivalent to the unit vector basis of c_0 then X contains an isomorph of c_0.*
(c) *If some spreading model (e_i) of X satisfies $\|e_1 + e_2\| = 1$ then X is not reflexive.*

Renorming is essential in this theorem. For example every subspace of T admits a spreading model 1-equivalent to the unit vector basis of ℓ_1 but T contains no isomorph of ℓ_1. [51]

There remain some open problems.

(Q6) Let X have a basis (x_i) and let $1 < p < \infty$. Suppose that every spreading model of any normalized block basis of (e_i) is 1-equivalent to the unit vector basis of ℓ_p. Must X contain an isomorph of ℓ_p?

(Q7) Suppose that for some $1 < p < \infty$ every spreading model of X is equivalent to the unit vector basis of ℓ_p. Does X contain an asymptotic ℓ_p basic sequence?

And here are two similar questions raised separately by S. Argyros and H. Rosenthal.

(Q8) (a) Suppose that all spreading models of a space X are equivalent. Must they be equivalent to the unit vector basis of c_0 or ℓ_p for some $1 \leq p < \infty$? (The answer is yes if they are uniformly equivalent from the proof of Krivine's Theorem.)

(b) Let X have a basis (e_i). Assume that for all normalized block bases of (e_i) some subsequence is equivalent to (e_i). Is (e_i) equivalent to the unit vector basis of c_0 or ℓ_p for some $1 \leq p < \infty$? (Again the answer is yes if they are all K-equivalent to (e_i) for some $K < \infty$.)

We mentioned briefly the Banach space $C[0,1]$ and more generally the spaces $C(K)$ where K is compact metric. If K is a convergent infinite sequence then it is not hard to show that $C(K)$ is isomorphic to c_0. More generally if $\alpha < \omega_1$ is a countable ordinal we can regard α as having the order topology (a base for the topology is all open intervals of ordinals (γ, δ) with $\gamma < \delta$) and then $C(\alpha)$ denotes the Banach space thus obtained (if α is a limit ordinal $(C(\alpha) \equiv C(\alpha^+)$ where α^+ is the successor ordinal of α to insure compactness). Thus $C(\omega) \sim c_0$ as noted above. The Banach spaces $\{C(\alpha) : \omega \leq \alpha < \omega_1\}$ increase in complexity and it can be shown that every $C(K)$ space is isomorphic to either $C[0,1]$ or some $C(\alpha)$ ([46], [6]). While $C[0,1]$ contains isometric copies of every X, the subspace structure of the $C(\alpha)$'s is much simpler. They are *c_0-saturated* (every $X \subseteq C(\alpha)$ contains an isomorph of c_0). But their spreading model structure is more complex. Every spreading model of c_0 is equivalent to either the unit vector basis of c_0 or the summing basis. But we have:

PROPOSITION 5.10. *Let (e_i) be a normalized basic sequence which is 1-subsymmetric. Then (e_i) is 1-equivalent to some spreading model of $C(\omega^\omega)$.*

Proof. If $(T,\le)$ is a countable *tree* (by which we shall mean that it is partially ordered set so that the predecessors of any element form a finite linearly ordered subset) we can define a Banach space X_T as follows. We first define a norm on $c_{00}(T)$, the finitely supported real valued functions on T, as follows. If $f\in c_{00}(T)$,

$$\|f\|=\sup\left\{\left|\sum_{\alpha\in\beta}f(\alpha)\right| : \beta \text{ is a branch of } T \text{ or an initial segment of such}\right\}.$$

A *branch* of T is a maximal linearly ordered subset of T. X_T is the completion of $\big(c_{00}(T),\|\cdot\|\big)$. Now the Banach spaces $C(\alpha)$ for $\alpha<\omega_1$ can all be realized in this manner. For example $C(\omega)$ is X_{T_1} where T_1 is the tree with a unique initial (smallest) node and a countably infinite number of successors. A *basis* for X_T is the *node basis* $\{e_t : t\in T\}$ where $e_t(t')=\delta_{t,t'}$ for $t'\in T$. This is a monotone basis when listed in any order that is compatible with the order on T: e. g. $(e_{t_i})_{i=1}^\infty$ where $i<j$ implies $t_i<t_j$ or t_i and t_j are incomparable. Each e_t corresponds to the indicator of some clopen subset F_t of α with $t<t'$ implying $F_t\supset F_{t'}$.

In any event $C(\omega^\omega)$ can be realized as X_{T_ω} where T_ω is a tree with unique initial node followed by a disjoint sequence of subtrees T_n. We have defined T_1. T_{n+1} is obtained from T_n by adding a new initial node and then attaching a disjoint sequence of copies of T_n.

The second ingredient in the proof of the proposition is to realize that the proof we gave earlier that every separable X embeds isometrically into ℓ_∞ can be localized to show that given $n\in\mathbb{N}$ and $\varepsilon_n>0$ there exists m_n so that we can embed $\mathrm{span}(e_1,\dots,e_n)$ into $\ell_\infty^{m_n}$ via an into isomorphism T_n satisfying

$$(1-\varepsilon_n)\left\|\sum_1^n a_ie_i\right\|\le\left\|\sum_1^n a_iT_n(e_i)\right\|\le\left\|\sum_1^n a_ie_i\right\|$$

for all $(a_i)_1^n\subseteq\mathbb{R}$ and $\|T_ne_i\|=1$ for $i\le n$.

We describe first how to use this to find a normalized basic sequence $(x_n)_{n=2}^\infty\subseteq X_{T_\omega}$ so that for all a_1, a_2, $\lim_{i\to\infty}\lim_{j\to\infty}\|a_1x_{n_i}+a_2x_{n_j}\|=\|a_1e_1+a_2e_2\|$. Let us label the nodes of T_2 as $\emptyset$, (1), (2), (3), …, (1,2), $(1,3,\dots)$, (2,3), (2,4), …, (3,4), (3,5), … (ordered by inclusion). Let $T_n(e_j)$ $=\big(a_1^{n,j},\dots,a_{m_n}^{n,j}\big)\in\ell_\infty^{m_n}$ for $n\in\mathbb{N}$ and $j\le n$. We let x_2 take the values $\big(a_1^{2,1},\dots,a_{m_2}^{2,1}\big)$ on the nodes $(1),(2),\dots,(m_2)$. x_3 will take the values $\big(a_1^{3,1},\dots,a_{m_3}^{3,1}\big)$ on the nodes $(m_2+1),\dots,(m_2+m_3)$. In addition x_3 will take on the values $\big(a_1^{2,2},\dots,a_{m_2}^{2,2}\big)$ on nodes $(1,2),(2,3),\dots,(m_2,m_2+1)$. x_4

take the values $\left(a_1^{2,2},\dots,a_{m_2}^{2,2}\right)$ on $(1,3),(2,4),\dots,(m_2,m_2+2)$ the values $\left(a_1^{3,2},\dots,a_{m_3}^{3,2}\right)$ on $(m_2+1,m_2+2),\dots,(m_2+m_3,m_2+m_3+1)$ and the values $\left(a_1^{4,1},\dots,a_{m_4}^{4,1}\right)$ on $(m_2+m_3+1),\dots,(m_2+m_3+m_4)$. In general x_n takes the values $\left(a_1^{n,1},\dots,a_{m_n}^{n,1}\right)$ on the first level nodes directly following those used by $x_1,\dots,x_{n-1}$ and takes the values $\left(a_1^{i,2},\dots,a_{m_i}^{i,2}\right)$ for $i<n$ on immediate successors of the first level nodes where x_i took the values $\left(a_1^{i,1},\dots,a_{m_i}^{i,1}\right)$.

To get $\|a_1x_{i_1}+a_2x_{i_2}+a_3x_{i_3}\|$ looking like $\|a_1e_1+a_2e_2+a_3e_3\|$ for $3\le i_1<i_2<i_3$ we repeat the above argument in T_3 and so on. x_2 has been completely defined already. x_3 will be complete after this step, x_4 after repeating this for 4-tuples in T_4 and so on. While it is cumbersome to write the construction, once understood (pictures help here) it is easy to check that it works (the necessary 1-subsymmetry of (e_n) is used as well). ∎

(Q9) Let X be a space such that every subsymmetric normalized basis is equivalent to a spreading model of X. What can be said about X? Must X contain an isomorph of c_0?

We raise one last stability question. Earlier we mentioned that ℓ_p is minimal (for all $X\subseteq\ell_p$, X contains an isomorph of ℓ_p). Not every Banach space contains a minimal subspace (T is such a space [9]). Also there are minimal spaces not containing any isomorph of c_0 or ℓ_p $(1\le p<\infty)$. Indeed S and T^* are such spaces ([58], [8]). The situation is unclear for spreading models.

(Q10) For all X does there exist $Y\subseteq X$ so that for all $Z\subseteq Y$, if (e_i) is a spreading model of Z then (e_i) is equivalent to a spreading model of Y? Or even for all $Z\subseteq Y$ does there exist a spreading model of Z which is equivalent to a spreading model of Y.

There are other ways of studying the asymptotic nature of a space X besides the theory of spreading models. One is the notion of asymptotic structure [39] which we describe in the simplest case. Let X have a basis (x_n), Let $(e_i)_{i=1}^\infty$ be a normalized basic sequence of length n. Then we say $(e_i)_{i=1}^n\in\{X\}_n$ (w.r.t. (x_i)) if

$$\begin{array}{lll}\forall\,\varepsilon>0 & \forall\,k_1 & \exists\,y_1\in S_{\langle x_i\rangle_{i\ge k_1}}\\ \forall\,\varepsilon>0 & \forall\,k_2 & \exists\,y_2\in S_{\langle x_i\rangle_{i\ge k_2}}\\ & \vdots & \\ \forall\,\varepsilon>0 & \forall\,k_n & \exists\,y_n\in S_{\langle x_i\rangle_{i\ge k_n}}\end{array}$$

with $(y_i)_{i=1}^n$ being $(1+\varepsilon)$-equivalent to $(e_i)_{i=1}^n$ ($\langle\cdot\rangle$ denotes linear span).

$\{X\}_n$ is called the n^{th} *asymptotic structure* of X (w.r.t. (x_i)). The theory of asymptotic structure is in one sense more complete that that of spreading models. For example a normalized block basis of a spreading model of X need not be a spreading model of X while if $(y_i)_{i=1}^m$ is a normalized block basis of $(e_i)_{i=1}^n \in \{X\}_n$ then $(y_i)_{i=1}^m \in \{X\}_m$ Also from Krivine's theorem, there exists $p \in [1,\infty]$ so that the unit vector basis of $\ell_p^n \in \{X\}_n$ for all $n \in \mathbb{N}$. On the other hand, asymptotic structures lose some of the infinite ties exhibited by spreading models.

Spreading models arise from every normalized basic sequence admits a subsequence . . . $\{X\}_n$ can be described in terms of trees. $(e_i)_{i=1}^n \in \{X\}_n$ if and only if there exists a block basic tree $(y_\alpha)_{\alpha\in T_n}$ which "converges" to $(e_i)_{i=1}^n$. By this we mean that the successors to any node form a normalized block basis of (x_i), the basis for X, each branch is a block basis of (x_i) and for all $\varepsilon > 0$ there exists $m \in \mathbb{N}$ so that if $k_1 \geq m$ then $(x_{(k_1,\dots,k_r)})_{r=1}^n$ is $(1+\varepsilon)$-equivalent to $(e_i)_{i=1}^n$. So one big difference is that the combinatorics of trees T_n is different from that of choosing subsequences via Ramsey's theorem.

Another type of asymptotic behavior is that of an *asymptotic model* [26]. These are generated by basic arrays using Ramsey's theorem as follows. $(x_i^n)_{n,i\in\mathbb{N}}$ is a *basic array* if for some $K < \infty$ each row $(x_i^n)_{i\in\mathbb{N}}$ is a K-basic normalized sequence and for all $n \leq i_i < i_2 < \cdots < i_n$, $(x_{i_j}^j)_{j=1}^n$ is also K-basic. An array (y_i^n) is a *subarray* of (x_i^n) if for some $k_0 < k_1 < \cdots$, $y_i^n = x_i^{k(n)}$ for all i. In other words once selects infinitely many columns of the array (x_i^n). A basic sequence (e_i) is an asymptotic model of X if there exists a basic array $(x_i^n) \subseteq X$ so that for some $\varepsilon_n \downarrow 0$ if $n \leq i_1 < \cdots < i_n$ then $(x_{i_j}^j)_{j=1}^n$ is $(1+\varepsilon_n)$-equivalent to $(e_i)_{i=1}^n$. Much like the proof of Theorem 3.2 one can show that every basic array admits a subarray which generates an asymptotic model. Asymptotic models include all spreading models (if (x_i) generates the spreading model (e_i) then the array (x_i^n) where $x_i^n = x_i$ for all i and n generates the asymptotic model (e_i)) as well as all normalized block bases of spreading models. More results and problems about asymptotic models can be found in [26].

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