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A New Geometrical Framework for Time–Dependent Hamiltonian Mechanics †

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1. Introduction

In a recent paper [1, 2], a new mathematical setting for the formulation of Classical Mechanics, automatically embodying the gauge invariance of the theory under arbitrary transformations $L \to L + \frac{df}{dt}$ of the Lagrangian has been proposed. The construction relies on the introduction of a principal fiber bundle $\pi : P \to \mathcal{V}_{n+1}$ over the configuration space–time $\mathcal{V}_{n+1}$, with structural group $(\Re, +)$, referred to as the bundle of *affine scalars*. The analysis developed in [1] was mainly centered on the gauge–theoretical aspects of the subject.

In this paper, we shall examine the basic themes of Hamiltonian Mechanics in the newer context. The discussion will illustrate the simplicity and efficiency of the approach, as compared with the more traditional ones [3], ...,[11].

For convenience of the reader, the construction of the Hamiltonian bundles is briefly reviewed in §2. The resulting geometrical scheme is applied in §3 to the study of the following topics:

- symplectic structure of the Hamiltonian bundle $\mathcal{H}(\mathcal{V}_{n+1})$; associated Poisson structure of the phase space $\Pi(\mathcal{V}_{n+1})$; Hamiltonian dynamics, as the study of sections $h : \Pi(\mathcal{V}_{n+1}) \to \mathcal{H}(\mathcal{V}_{n+1})$;
- canonical transformations, as a subgroup of the group of symplectic transformations over $\mathcal{H}(\mathcal{V}_{n+1})$;
- Hamilton–Jacobi theory; in particular, geometrical interpretation of the Hamilton–Jacobi equation in the bundle $\mathcal{H}(\mathcal{V}_{n+1})$.

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2. The Hamiltonian bundles

Following [1], to any mechanical system $\mathcal{B}$ with n degrees of freedom we associate a double fibration $P \xrightarrow{\pi} \mathcal{V}_{n+1} \xrightarrow{t} \Re$, in which

- $\mathcal{V}_{n+1}$ is the configuration space-time of the dynamical system in study, with the fibration $\mathcal{V}_{n+1} \xrightarrow{t} \Re$ representing *absolute time*;
- $P \xrightarrow{\pi} \mathcal{V}_{n+1}$ is a principal fiber bundle, with structural group $(\Re, +)$, called the bundle of *affine scalars* over $\mathcal{V}_{n+1}$.

The action of $(\Re, +)$ on P results into a 1-parameter group of diffeomorphisms $\psi_\xi : P \to P$, expressed through the additive notation

$$(\nu, \xi) \in P \times \Re \;\to\; \psi_\xi(\nu) \;:=\; \nu + \xi \;\in P. \tag{2.1}$$

In what follows, we shall refer $\mathcal{V}_{n+1}$ to local coordinates t, q^i, and P to fibered coordinates $t, q^i, u \quad (i = 1, \ldots, n)$, u denoting any trivialization of $P \to \mathcal{V}_{n+1}$. The first jet space $j_1(P, \mathcal{V}_{n+1})$ associated with the fibration $P \to \mathcal{V}_{n+1}$ will be referred to jet-coordinates t, q^i, u, p_0, p_i, with transformation laws

$$\bar{t} = t + c, \qquad \bar{q}^{\,i} = \bar{q}^{\,i}(t, q^1, \ldots, q^n), \qquad \bar{u} = u + f(t, q^1, \ldots, q^n), \tag{2.2a}$$

$$\bar{p}_0 = p_0 + \frac{\partial f}{\partial t} + \left(p_k + \frac{\partial f}{\partial q^k}\right)\frac{\partial q^k}{\partial t}, \qquad \bar{p}_i = \left(p_k + \frac{\partial f}{\partial q^k}\right)\frac{\partial q^k}{\partial \bar{q}^{\,i}}. \tag{2.2b}$$

Eqs. (2.2a, b) ensure the invariance of the contact 1-form

$$\Theta = du - p_0 dt - p_i dq^i, \tag{2.3}$$

known as the *canonical 1-form* of $j_1(P, \mathcal{V}_{n+1})$.

The manifold $j_1(P, \mathcal{V}_{n+1})$ is naturally embedded into the cotangent space $T^*(P)$ according to the identification

$$\eta = \left[du - p_0(\eta)dt - p_i(\eta)dq^i\right]_{\pi(\eta)} \qquad \forall\, \eta \in j_1(P, \mathcal{V}_{n+1}). \tag{2.4}$$

In view of the latter, one can easily establish two distinguished actions of the group $(\Re, +)$ on $j_1(P, \mathcal{V}_{n+1})$, respectively denoted by $\psi_{\xi *} : j_1(P, \mathcal{V}_{n+1}) \to j_1(P, \mathcal{V}_{n+1})$ and $\varphi_\xi : j_1(P, \mathcal{V}_{n+1}) \to j_1(P, \mathcal{V}_{n+1})$, and described by the equations

$$\psi_{\xi *}(\eta) := (\psi_{-\xi})^*_*(\eta) = \left[du - p_0(\eta)dt - p_i(\eta)dq^i\right]_{\pi(\eta)+\xi} \tag{2.5a}$$

$$\varphi_\xi(\eta) := \eta - \xi(dt)_{\pi(\eta)} = \left[du - \left(p_0(\eta) + \xi\right)dt - p_i(\eta)dq^i\right]_{\pi(\eta)} \tag{2.5b}$$

for all $\eta \in j_1(P, \mathcal{V}_{n+1})$, $\xi \in \Re$. The first of these, written symbolically as

$$\psi_{\xi *}: \ (t, q^i, u, p_0, p_i) \longrightarrow (t, q^i, u + \xi, p_0, p_i)$$

is the pull-back of the (inverse of) the action (2.1). Let $\mathcal{H}(\mathcal{V}_{n+1})$ denote the quotient of $j_1(P, \mathcal{V}_{n+1})$ by this action. The following properties are entirely straightforward [1]:

- $\mathcal{H}(\mathcal{V}_{n+1})$ is an affine bundle over $\mathcal{V}_{n+1}$, with coordinates t, q^i, p_0, p_i, modelled on the cotangent bundle $T^*(\mathcal{V}_{n+1})$;
- the quotient map makes $j_1(P, \mathcal{V}_{n+1}) \to \mathcal{H}(\mathcal{V}_{n+1})$ into a principal fibre bundle, with structural group $(\Re, +)$ and fundamental vector field $\frac{\partial}{\partial u}$;
- the canonical 1-form (2.3) endows $j_1(P, \mathcal{V}_{n+1}) \to \mathcal{H}(\mathcal{V}_{n+1})$ with a distinguished connection, henceforth referred to as the *canonical connection*.

The second action of $(\Re, +)$ on $j_1(P, \mathcal{V}_{n+1})$, described by eq. (2.5b), and summarized into the symbolic relation

$$\varphi_\xi: \ (t, q^i, u, p_0, p_i) \longrightarrow (t, q^i, u, p_0 + \xi, p_i)$$

comes from the invariant character of the 1-form dt. The quotient of $j_1(P, \mathcal{V}_{n+1})$ by this action will be denoted by $\mathcal{H}^c(\mathcal{V}_{n+1})$. Once again, one has the properties:

- $\mathcal{H}^c(\mathcal{V}_{n+1})$ is a fibre bundle over $\mathcal{V}_{n+1}$, with coordinates t, q^i, u, p_i;
- the action (2.5b) makes $j_1(P, \mathcal{V}_{n+1}) \to \mathcal{H}^c(\mathcal{V}_{n+1})$ into a principal fibre bundle, with structural group $(\Re, +)$ and fundamental vector field $\frac{\partial}{\partial p_0}$.

The concluding step in the definition of the Hamiltonian bundles relies on the observation that the group actions (2.5a, b) *commute*. Either of them may therefore be used to induce an action on the space of orbits associated with the other. As proved in [1], this makes both $\mathcal{H}(\mathcal{V}_{n+1})$ and $\mathcal{H}^c(\mathcal{V}_{n+1})$ into principal fiber bundles over a common "double quotient" space, henceforth denoted by $\Pi(\mathcal{V}_{n+1})$ and identified with the *phase space* of the system. The situation is illustrated by the commutative diagram

$$\begin{array}{ccc} j_1(P, \mathcal{V}_{n+1}) & \longrightarrow & \mathcal{H}^c(\mathcal{V}_{n+1}) \\ \big\downarrow & & \big\downarrow \\ \mathcal{H}(\mathcal{V}_{n+1}) & \longrightarrow & \Pi(\mathcal{V}_{n+1}) \end{array} \tag{2.6}$$

in which all arrows denote principal fibrations, with structural groups isomorphic to $(\Re, +)$, and group actions arising in a straightforward way from eqs. (2.5a, b). The principal bundles $\mathcal{H}(\mathcal{V}_{n+1}) \to \Pi(\mathcal{V}_{n+1})$ and $\mathcal{H}^c(\mathcal{V}_{n+1}) \to \mathcal{H}(\mathcal{V}_{n+1})$ are called respectively the *Hamiltonian* and the *co-Hamiltonian* bundle over $\Pi(\mathcal{V}_{n+1})$.

The geometrical environment based on the diagram (2.6) provides the necessary tool for a gauge–invariant formulation of Hamiltonian Mechanics. To this end, referring to [1] for the necessary clarifications, we recall that, through the Legendre transformation, every regular Lagrangian system determines a section $h : \Pi(\mathcal{V}_{n+1}) \to \mathcal{H}(\mathcal{V}_{n+1})$ of the Hamiltonian bundle, expressed in coordinates as

$$p_0 + H(t, q^1, \dots, q^n, p_1, \dots, p_n) = 0. \tag{2.7}$$

The function $H(t, q^i, p_i)$ involved in the representation (2.7) is known as the *Hamiltonian* of the system.

Taking the diagram (2.6) into account, it is easily seen that every section $h : \Pi(\mathcal{V}_{n+1}) \to \mathcal{H}(\mathcal{V}_{n+1})$ may be lifted to a section $\hat{h} : \mathcal{H}^c(\mathcal{V}_{n+1}) \to j_1(P, \mathcal{V}_{n+1})$, described locally by the same equation (2.7). This gives rise to a principal bundle homomorphism, summarized into the commutative diagram

$$\begin{array}{ccc} \mathcal{H}^c(\mathcal{V}_{n+1}) & \xrightarrow{\hat{h}} & j_1(P, \mathcal{V}_{n+1}) \\ \big\downarrow & & \big\downarrow \\ \Pi(\mathcal{V}_{n+1}) & \xrightarrow{h} & \mathcal{H}(\mathcal{V}_{n+1}) \end{array} \tag{2.8}$$

By means of eq. (2.8), the canonical connection of $j_1(P, \mathcal{V}_{n+1}) \to \mathcal{H}(\mathcal{V}_{n+1})$ may be pulled back to a connection over $\mathcal{H}^c(\mathcal{V}_{n+1}) \to \Pi(\mathcal{V}_{n+1})$, with connection 1-form

$$\hat{h}^*(\Theta) = du + H(t, q^i, p_i)dt - p_i dq^i.$$

The difference $du - \hat{h}^*(\Theta)$ is then (the pull-back of) a 1-form ϑ_h over $\Pi(\mathcal{V}_{n+1})$, expressed in coordinates as

$$\vartheta_h = -H dt + p_i dq^i \tag{2.9}$$

and called the *Poincaré-Cartan* 1*-form* associated with the section h.

Referring once again to [1] for clarifications and comments, we finally recall that, through the Legendre transformation, the dynamical flow of the system is expressed in the form of a vector field Z over $\Pi(\mathcal{V}_{n+1})$, known as the *Hamiltonian flow*, completely determined by the conditions

$$\langle Z, dt \rangle = 1, \quad Z \,\lrcorner\, d\vartheta_h = 0. \tag{2.10}$$

In local coordinates, a straightforward comparison with eq. (2.9) yields the representation

$$Z = \frac{\partial}{\partial t} + \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i} \tag{2.11}$$

mathematically equivalent to Hamilton's equations of motion

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}\,, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}\,.$$

3. Hamiltonian mechanics

3.1. Poisson brackets Pursuing the plan indicated in the Introduction, we shall now illustrate a few basic themes of Classical Hamiltonian Mechanics within the geometrical environment developed in §2. A major role in the discussion will be played by the canonical connection of the bundle $j_1(P, \mathcal{V}_{n+1}) \to \mathcal{H}(\mathcal{V}_{n+1})$, described by the contact 1-form

$$\Theta = du - p_0 dt - p_i dq^i. \tag{3.1}$$

The curvature 2-form of Θ, defined, up to a sign, by

$$\Omega := -d\Theta = dp_0 \wedge dt + dp_i \wedge dq^i\,, \tag{3.2}$$

makes the base manifold $\mathcal{H}(\mathcal{V}_{n+1})$ into a $(2n+2)$-dimensional *symplectic manifold*, thereby endowing it with a canonical *Poisson structure*. In fiber coordinates, the latter results in the Poisson brackets

$$\{f, g\} = \frac{\partial f}{\partial t}\frac{\partial g}{\partial p_0} + \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_0}\frac{\partial g}{\partial t} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}. \tag{3.3}$$

The link between eq. (3.3) and the ordinary language of Hamiltonian Mechanics rests on two basic observations:
(i) through an obvious pull–back procedure, the ring of differentiable functions over the phase space $\Pi(\mathcal{V}_{n+1})$ may be identified with the sub-ring

$$\mathcal{F}_0 = \left\{ f\,:\; f \in \mathcal{F}\left(\mathcal{H}(\mathcal{V}_{n+1})\right),\; \frac{\partial f}{\partial p_0} = 0 \right\} \tag{3.4}$$

of the ring of differentiable functions over $\mathcal{H}(\mathcal{V}_{n+1})$. A comparison with eq. (3.3) provides the relation

$$f \in \mathcal{F}_0 \quad \Longleftrightarrow \quad \{t, f\} = 0 \tag{3.5}$$

whence, taking the Jacobi identity into account

$$f, g \in \mathcal{F}_0 \;\Rightarrow\; \{t, \{f, g\}\} = \{f, \{t, g\}\} - \{g, \{t, f\}\} \equiv 0 \;\Rightarrow\; \{f, g\} \in \mathcal{F}_0. \quad (3.6)$$

In other words, the sub-ring $\mathcal{F}_0$ *inherits* the Poisson bracket operation present in $\mathcal{F}(\mathcal{H}(\mathcal{V}_{n+1}))$, thus inducing a Poisson structure over $\Pi(\mathcal{V}_{n+1})$. Of course, due to the identification (3.4), for each pair $f, g \in \mathcal{F}(\Pi(\mathcal{V}_{n+1}))$, eq. (3.3) simplifies into

$$\{f, g\} = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}\,. \quad (3.7)$$

(ii) a similar argument shows that a function $\sigma \in \mathcal{F}(\mathcal{H}(\mathcal{V}_{n+1}))$ is a (local) trivialization of the bundle $\mathcal{H}(\mathcal{V}_{n+1}) \to \Pi(\mathcal{V}_{n+1})$ if and only if it satisfies the Poisson-bracket relation

$$\{t, \sigma\} = 1. \quad (3.8)$$

From this, and from the Jacobi identity, it is easily seen that, for any such function, the condition $f \in \mathcal{F}_0$ implies also $\{f, \sigma\} \in \mathcal{F}_0$. In particular, if we let $\sigma_h = p_0 + H$ denote the trivialization associated with a given Hamiltonian section $h : \Pi(\mathcal{V}_{n+1}) \to \mathcal{H}(\mathcal{V}_{n+1})$, by comparison with eq. (3.3) we get the relation

$$\{f, \sigma_h\} = \frac{\partial f}{\partial t} + \{f, H\} = \frac{\partial f}{\partial t} + \frac{\partial H}{\partial p_i}\frac{\partial f}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial f}{\partial p_i}. \quad (3.9)$$

On account of eq. (2.11) the right-hand side of eq. (3.9) expresses the *time derivative* of f in the course of the evolution determined by the Hamiltonian flow Z. Accordingly, the relation

$$\frac{df}{dt} = \{f, \sigma_h\} \quad (3.10)$$

embodies the full content of Hamilton's equations in the newer context. Equivalently, we may say that the Hamiltonian flow Z on $\Pi(\mathcal{V}_{n+1})$ is h-related to the unique vector field $\hat{Z}$ on $\mathcal{H}(\mathcal{V}_{n+1})$ defined by the condition

$$\hat{Z} \,\lrcorner\, \Omega = -d\sigma_h\,.$$

3.2. Canonical transformations In the analysis developed so far, all spaces have been referred to a distinguished class of local coordinates, generically called *jet coordinates*, entirely determined by the assignment of a local chart on $\mathcal{V}_{n+1}$ and of a trivialization u of P. We shall now remove this restriction, by introducing a suitable analogue of the concept of *canonical transformation*.

Once again the analysis is centered around the 2-form (3.2). A coordinate system $\xi^0, \ldots, \xi^n$, $\zeta_0, \ldots, \zeta_n$ in $\mathcal{H}(\mathcal{V}_{n+1})$ is called *symplectic* if and only if it gives rise to a representation of the form

$$\Omega = d\zeta_0 \wedge d\xi^0 + d\zeta_i \wedge d\xi^i \tag{3.11}$$

A symplectic coordinate system of the special type $t, \xi^1, \ldots, \xi^n, \zeta_0, \ldots, \zeta_n$ embodying the time variable t as one of the coordinate functions –precisely, as the coordinate function conjugate to ζ_0– will be called *special symplectic*.

Recalling eqs. (3.5), (3.8), it is an easy matter to verify that, under the latter assumption, the functions t, ξ^i, ζ_i belong to the ring $\mathcal{F}_0$, while ζ_0 is a (local) trivialization of the bundle $\mathcal{H}(\mathcal{V}_{n+1}) \to \Pi(\mathcal{V}_{n+1})$. Therefore:
a) every special symplectic coordinate system in $\mathcal{H}(\mathcal{V}_{n+1})$ induces a corresponding coordinate system in $\Pi(\mathcal{V}_{n+1})$;
b) in special symplectic coordinates, every trivialization σ of $\mathcal{H}(\mathcal{V}_{n+1})$ admits a local representation of the form

$$\sigma = \zeta_0 + f \tag{3.12}$$

with $f = f(t, \xi^1, \ldots, \xi^n, \zeta_1, \ldots, \zeta_n) \in \mathcal{F}_0$.

A coordinate system $t, \xi^1, \ldots, \xi^n, \zeta_1, \ldots, \zeta_n$ in $\Pi(\mathcal{V}_{n+1})$ will be called *canonical* if and only if it arises from a special symplectic coordinate system in $\mathcal{H}(\mathcal{V}_{n+1})$ in the way indicated above. A straightforward argument shows that a necessary and sufficient condition for this to happen is the validity of the Poisson bracket relations

$$\{\xi^i, \xi^j\} = \{\zeta_i, \zeta_j\} = 0; \quad \{\xi^i, \zeta_j\} = \delta^i_j, \qquad i, j = 1, \ldots, n. \tag{3.13}$$

Indeed, under the assumption (3.13), by pulling everything back to $\mathcal{H}(\mathcal{V}_{n+1})$, and denoting by $\omega \to X_\omega$ the "process of raising the indices" defined by the requirement $\omega = X_\omega \lrcorner\, \Omega$, it is easily seen that the (exact) 2-form $\Omega - d\zeta_k \wedge d\xi^k$ satisfies the identities

$$X_{d\xi_i} \lrcorner \left(\Omega - d\zeta_k \wedge d\xi^k\right) = d\xi^i - \{d\xi^i, d\zeta_k\} d\xi^k + \{d\xi^i, d\xi^k\} d\zeta_k = 0,$$

$$X_{d\zeta_i} \lrcorner \left(\Omega - d\zeta_k \wedge d\xi^k\right) = d\xi^i - \{d\zeta^i, d\zeta_k\} d\xi^k + \{d\zeta^i, d\xi^k\} d\zeta_k = 0,$$

$$X_{dt} \lrcorner \left(\Omega - d\zeta_k \wedge d\xi^k\right) = dt,$$

i.e. it has algebraic rank 2, and belongs to the ideal generated by dt. By Darboux theorem, this implies the existence of a local representation of the form

$$\Omega - d\zeta_k \wedge d\xi^k = d\zeta_0 \wedge dt\,,$$

proving the required result.

An important consequence of the previous characterization is that, in canonical coordinates, every Hamiltonian flow Z admits the representation

$$Z = \frac{\partial}{\partial t} + \frac{\partial K}{\partial \zeta_i}\frac{\partial}{\partial \xi^i} - \frac{\partial K}{\partial \xi^i}\frac{\partial}{\partial \zeta_i} \tag{3.14}$$

in terms of a suitable "Hamiltonian" $K(t,\xi^1,\dots,\xi^n,\zeta_1,\dots,\zeta_n)$. The conclusion follows at once by completing t,ξ^i,ζ_i to a special symplectic coordinate system t,ξ^i,ζ_0,ζ_i in $\mathcal{H}(\mathcal{V}_{n+1})$, and observing that, in view of eq. (3.12), given any Hamiltonian section $h:\Pi(\mathcal{V}_{n+1}) \to \mathcal{H}(\mathcal{V}_{n+1})$, the associated trivialization σ_h may be expressed locally as

$$\sigma_h = \zeta_0 + K(t,\xi^i,\zeta_i). \tag{3.15}$$

On account of eq. (3.10), the equations of evolution have therefore the form

$$\frac{df}{dt} = \{f,\sigma_h\} = \{f,\zeta_0 + K\} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \xi^i}\frac{\partial K}{\partial \zeta_i} - \frac{\partial f}{\partial \zeta^i}\frac{\partial K}{\partial \xi_i}$$

mathematically equivalent to the representation (3.14).

The interplay between special symplectic and canonical coordinates is particularly worthwhile in the study of *canonical transformations*, since it allows to "lift" the algorithm from $\Pi(\mathcal{V}_{n+1})$ to $\mathcal{H}(\mathcal{V}_{n+1})$, relating it to the (simpler) theory of symplectic transformations. In this connection, the central result – to be found in any textbook– is that, up to elementary "transpositions"

$$\bar\xi^\lambda \to \bar\zeta_\lambda\,, \quad \bar\zeta_\lambda \to -\bar\xi^\lambda\,, \qquad \lambda = i_1,\dots,i_r \subset \{0,\dots,n\}, \tag{3.16}$$

the most general transformation between symplectic coordinates may be expressed in terms of a "generating function" $\hat F(\xi^0,\dots,\xi^n,\bar\zeta_0,\dots,\bar\zeta_n)$, subject to the regularity requirement $\det|\partial^2\hat F/\partial\xi^\alpha\partial\bar\zeta_\beta| \neq 0$, in the implicit form

$$\zeta_\alpha = \frac{\partial \hat F}{\partial \xi^\alpha}\,, \qquad \bar\xi^\alpha = \frac{\partial \hat F}{\partial \bar\zeta_\alpha}\,, \qquad \alpha = 0,\dots,n. \tag{3.17}$$

In particular, if the transformation (3.17) is meant to relate two *special symplectic* coordinate systems, the identification $\xi^0 = t$, $\bar{\xi}^0 = \bar{t} = t + c$ implies the restriction

$$\hat{F} = (t+c)\bar{\zeta}_0 + F(t, \xi^1, \ldots, \xi^n, \bar{\zeta}_1, \ldots, \bar{\zeta}_n), \tag{3.18}$$

while the regularity condition goes over into $\det |\partial^2 F/\partial \xi^i \partial \bar{\zeta}_j| \neq 0$. The resulting transformation then splits into the system

$$\zeta_i = \frac{\partial F}{\partial \xi^i}, \qquad \bar{\xi}^i = \frac{\partial F}{\partial \bar{\zeta}_i}, \tag{3.19a}$$

$$\zeta_0 = \frac{\partial \hat{F}}{\partial t} = \bar{\zeta}_0 + \frac{\partial F}{\partial t}. \tag{3.19b}$$

Returning to $\Pi(\mathcal{V}_{n+1})$, eqs. (3.19a) express – up to elementary transpositions– the most general transformation between canonical coordinates, while eq. (3.19b), together with eq. (3.15), provides the transformation law for the Hamiltonian

$$\overline{H} = \sigma_h - \bar{\zeta}_0 = \sigma_h - \zeta_0 + \frac{\partial F}{\partial t} = H + \frac{\partial F}{\partial t}. \tag{3.20}$$

3.3. Hamilton–Jacobi theory As a concluding remark, we discuss the interplay between Hamiltonian bundles and Hamilton–Jacobi theory. The argument is entirely classical: in what follows, we shall illustrate how it fits within the geometrical scheme developed so far. To this end, given any section $h : \Pi(\mathcal{V}_{n+1}) \to \mathcal{H}(\mathcal{V}_{n+1})$, we consider once again the associated trivialization $\sigma_h : \mathcal{H}(\mathcal{V}_{n+1}) \to \Re$, expressed locally as $\sigma_h = p_0 + H$. A straightforward argument then shows that the closed 2-form

$$\hat{\Omega} := \Omega - d\sigma_h \wedge dt = dp_i \wedge dq^i - dH \wedge dt \tag{3.21}$$

has class $2n$ [1]. By Darboux' theorem we may therefore express $\hat{\Omega}$ in terms of $2n$ independent "first integrals" $\xi^i, \zeta_i,\ i = 1, \ldots, n$ in the canonical form

$$\hat{\Omega} = d\zeta_i \wedge d\xi^i.$$

[1] The characteristic distribution

$$\mathcal{D} := \left\{ X : X \in T\left(\mathcal{H}(\mathcal{V}_{n+1})\right),\ X \lrcorner\, \hat{\Omega} = 0 \right\}$$

is in fact locally generated by the pair of vector fields $\frac{\partial}{\partial p_0}$, $\frac{\partial}{\partial t} + \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i}$.

A comparison with eq. (3.21) provides the representation

$$\Omega = d\sigma_h \wedge dt + d\zeta_i \wedge d\xi^i, \tag{3.22}$$

indicating that the $2n+2$ variables $t, \xi^i, \sigma_h, \zeta_i$ form a special symplectic coordinate system on $\mathcal{H}(\mathcal{V}_{n+1})$, or, equivalently, that the $2n+1$ variables t, ξ^i, ζ_i form a canonical coordinate system on $\Pi(\mathcal{V}_{n+1})$.

Given any local chart in $\mathcal{V}_{n+1}$, we now make use of the fact that, up to elementary transpositions (3.16), it is always possible to arrange that (the pull-back of) the coordinate functions $t, q^1, \ldots, q^n$, completed with $\sigma_h, \zeta_1, \ldots, \zeta_n$, form a local coordinate system in $\mathcal{H}(\mathcal{V}_{n+1})$. The transformation between the coordinates t, q^i, p_0, p_i and $t, \xi^i, \sigma_h, \zeta_i$ is then expressed in terms of a generating function $S(t, q^i, \zeta_i)$ in the implicit form (see eqs. (3.19))

$$p_i = \frac{\partial S}{\partial q^i}\,, \qquad \xi^i = \frac{\partial S}{\partial \zeta_i}\,, \qquad p_0 = \sigma_h + \frac{\partial S}{\partial t}. \tag{3.23}$$

A comparison with the representation $\sigma_h = p_0 + H$ shows that S satisfies the partial differential equation

$$0 = \frac{\partial S}{\partial t} + H\left(t, q^1, \ldots, q^n, \frac{\partial S}{\partial q^1}, \ldots, \frac{\partial S}{\partial q^n}\right), \tag{3.24}$$

known as the *Hamilton-Jacobi equation*. More precisely, due to the regularity requirement $\det|\partial^2 S/\partial q^i \partial \zeta_j| \neq 0$ –embodied into the notion of generating function– the function S is easily recognized to provide a *complete integral* for eq. (3.24). Conversely, given any complete integral $S(t, q^i, \zeta_i)$ of eq. (3.24), expressed in terms of the variables t, q^i and of n "essential" parameters ζ_i, eqs. (3.23) define implicitly $2n+1$ functions $\zeta_i(t,q,p)$, $\sigma_h(t,q,p)$, $\xi^i(t,q,p)$ satisfying the relation

$$\begin{aligned}\Omega &= d\left(\sigma_h + \frac{\partial S}{\partial t}\right) \wedge dt + d\left(\frac{\partial S}{\partial q^i}\right) \wedge dq^i \\ &= d\sigma_h \wedge dt + d\left(dS - \frac{\partial S}{\partial \zeta_i} d\zeta_i\right) = d\sigma_h \wedge dt + d\zeta_i \wedge d\xi^i\end{aligned}$$

with

$$\sigma_h(t, q^i, p_i) = p_0 - \frac{\partial S}{\partial t} = p_0 + H(t, q^i, p_i)$$

thus yielding back the representation (3.22).

From a dynamical viewpoint, the relevance of the previous construction stems from the fact that eq. (3.22) implies the Poisson-bracket relations

$$\{\xi^i, \sigma_h\} = \{\zeta_i, \sigma_h\} = 0.$$

From these, recalling the representation (3.10) of the evolution equations, we conclude that the $2n$ functions $\xi^i(t,q,p)$, $\zeta_i(t,q,p)$ are *constant* along the integral curves of the Hamiltonian flow Z determined by the section h. In this respect, the first pair of eqs. (3.23), solved for p_i, q^i as functions of t, ξ^i, ζ_i provide the *general integral* of the Hamilton equations.

Still another way of looking at the Hamilton-Jacobi equation is obtained by focusing on the image space $h(\Pi(\mathcal{V}_{n+1}))$, viewed as a submanifold of $\mathcal{H}(\mathcal{V}_{n+1})$. Of course, no matter how we choose to represent it, the knowledge of $h(\Pi(\mathcal{V}_{n+1}))$ is mathematically equivalent to the assignment of the section h, and therefore carries a complete information on the dynamical equations. In this respect, the content of eqs. (3.23), (3.24) may be summarized into the following

Proposition 3.1. *Finding a complete integral of the Hamilton-Jacobi equation is mathematically equivalent to determining a (local) parametric representation of the submanifold $h(\Pi(\mathcal{V}_{n+1})) \subset \mathcal{H}(\mathcal{V}_{n+1})$ of the special form*

$$p_i = \frac{\partial S}{\partial q^i}(t, q^1, \dots, q^n, \zeta_1, \dots, \zeta_n), \quad p_0 = \frac{\partial S}{\partial t}(t, q^1, \dots, q^n, \zeta_1, \dots, \zeta_n) \tag{3.25}$$

Proof. The proof is entirely straightforward. By consistency with the "cartesian" representation $\sigma_h = p_0 + H(t, q^i, p_i) = 0$, the validity of eqs. (3.25) is in fact equivalent to the simultaneous validity of the Hamilton-Jacobi equation (3.24) and of the regularity condition $\det|\partial^2 S/\partial q^i \partial \zeta_j| \neq 0$, expressing the functional independence of the variables t, q^i, p_i on $h(\Pi(\mathcal{V}_{n+1}))$. ∎

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