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Very Non-Constrained Subspaces of Banach Spaces

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1. Introduction

We work with *real* Banach spaces. For a Banach space X, we will denote by $B(X)$, $S(X)$ and $B_X[x,r]$ respectively the closed unit ball, the unit sphere and the closed ball of radius $r > 0$ with centre at x in X. We will simply write $B[x,r]$ if there is no confusion about the ambient space. We will identify $x \in X$ with its canonical image in X^{**}. All subspaces we usually consider are *norm* closed.

We start with the notion of nicely smooth Banach spaces introduced in [12].

Definition 1.1. [12] A Banach space X is nicely smooth, if for all $x^{**} \in X^{**}$,

$$\bigcap_{x \in X} B_{X^{**}}[x, \|x^{**} - x\|] = \{x^{**}\}$$

With this as our motivating concept, we define,

Definition 1.2. A subspace Y of a Banach space X is said to be a very non-constrained (VN) subspace of X, if for all $x \in X$,

$$\bigcap_{y \in Y} B_X[y, \|x - y\|] = \{x\}.$$

Naturally, nicely smooth spaces are VN-subspaces of their biduals. Origin of the terminology will be explained soon.

Godefroy and Saphar [15] has studied nice smoothness in the context of operator spaces, and obtained the following characterization.

Definition 1.3. We say $A \subseteq B(X^*)$ is a norming set for X if $\|x\| = \sup\{x^*(x) : x^* \in A\}$. A subspace F of X^* is called a norming subspace if $B(F)$ is a norming set for X.

Theorem 1.4. [15, Lemma 2.4] *For a Banach space X, the following are equivalent :*

(a) *X is nicely smooth.*

(b) *For all $x^{**} \in X^{**} \setminus X$,*

$$\bigcap_{x \in X} B_X[x, \|x^{**} - x\|] = \emptyset$$

(c) *X^* contains no proper norming subspace.*

The proof of (b) $\Rightarrow$ (a) in [15] depends heavily on the properties of "u.s.c. hull" of $x^{**} \in X^{**}$ considered as a function on $(B(X^*), w^*)$. In a personal conversation, Godefroy asked whether one could give a proof without such topological considerations. That this can be done is a key result (Theorem 2.12) in this work.

Later, Godefroy and Kalton [14] linked this property to the Ball Generated Property (BGP) of Banach spaces. Different aspects of nicely smooth spaces were also investigated in [2, 9, 16, 18].

In course of proving Theorem 1.4 in this general set-up, we also obtain an extension of [2, Proposition 2.2]. As in [2, Theorem 2.10], we also identify some necessary and/or sufficient conditions for a subspace to be a VN-subspace (Theorem 2.20). For this, we characterize functionals with "locally unique" Hahn-Banach extensions. And here we bring back some of the topological flavour.

Definition 1.5. [6] A subspace Y of a Banach space X is said to be a U-subspace if any $y^* \in Y^*$ has a unique Hahn-Banach (i.e., norm preserving) extension in X^*.

X is said to be Hahn-Banach smooth if X is a U-subspace of X^{**}.

U-subspaces were systematically studied in [24], who referred to them as "subspaces with Property U". Godefroy proved in [12] that Hahn-Banach smooth spaces are nicely smooth. We, however, observe that, in general, a U-subspace (and even a proper M-ideal) need not be a VN-subspace and obtain characterizations of VN-subspaces among U-subspaces.

In [15], nicely smooth spaces were studied mainly as a sufficient condition for the Unique Extension Property (UEP), their main tool in studying geometry of operator spaces. Recall that a Banach space has the UEP if the only operator $T \in \mathcal{L}(X^{**})$ such that $\|T\| \leq 1$ and $T|_X = Id_X$ is $T = Id_{X^{**}}$. However, from the point of applications, a more natural generalization of the UEP is the unique ideal property introduced recently in [22].

Definition 1.6. A subspace Y of a Banach space X has the unique ideal property in X if there is at most one norm 1 projection P on X^* with $\ker(P) = Y^{\perp}$.

Recall that Y is said to be an ideal in X if such a projection exists.

Here we observe that a VN-subspace of X has the unique ideal property in X.

It is clear that a nicely smooth space, since it has the UEP, cannot be constrained, i.e., 1-complemented, in its bidual. Indeed, more is true.

Definition 1.7. A Banach space X is said to have the finite-infinite intersection property ($IP_{f,\infty}$) if every family of closed balls in X with empty intersection contains a finite subfamily with empty intersection.

In [2] it was shown that if a Banach space is both nicely smooth and has $IP_{f,\infty}$ then it is reflexive. The $IP_{f,\infty}$ was studied by Godefroy and Kalton in [14]. It is well known that dual spaces and their constrained subspaces have $IP_{f,\infty}$. By w*-compactness of the dual ball and the Principle of Local Reflexivity, it can be shown (see e.g., [14]) that X has the $IP_{f,\infty}$ if and only if any family of closed balls centred at points of X that intersects in X^{**} also intersects in X. Thus, we define

Definition 1.8. A subspace Y of a Banach space X is said to be an almost constrained (AC) subspace of X, if any family of closed balls centred at points of Y that intersects in X also intersects in Y.

It is obvious from the definitions that a proper subspace cannot simultaneously be very non-constrained and almost constrained. This explains the terminology.

Clearly, any constrained subspace is an AC-subspace. However, it has been shown recently by the first and third authors [3] that the converse is generally false, though whether they are equivalent in the case of $IP_{f,\infty}$ remains an open question.

If one considers hyperplanes, we show that there is a dichotomy between VN- and AC-subspaces and the notions of an AC-hyperplane and a constrained hyperplane coincide. We also characterize VN-hyperplanes of some classical Banach spaces.

We conclude Section 2 with some interesting applications of the existence of a separable VN-subspace.

In Section 3, we consider various stability results. In particular, we prove that for a family of Banach spaces and their subspaces, the ℓ_p $(1 \leq p \leq \infty)$ and c_0 sums of the subspaces are VN-subspaces of the sum of the superspaces if and only if the same is true of each coordinate. These are natural extensions of corresponding results in [2]. We also show that for a compact Hausdorff space K, $C(K,Y)$, the space of continuous functions from K to Y, is a VN-subspace of $C(K,X)$ if and only if Y is a VN-subspace of X. Under an assumption slightly stronger than X being nicely smooth, we show that $C(K,X)$ is a VN-subspace of $WC(K,X)$, the space of weakly continuous functions. We also show that some variants of this condition is sufficient for $\mathcal{K}(X,Y)$, the space of all compact operators from X to Y, to be a VN-subspace of $\mathcal{L}(X,Y)$, the space of all bounded operators from X to Y.

2. Main Results

Taking cue from [12, Lemma 1], we introduce the following notation.

Definition 2.1. Let Y be a subspace of a normed linear space X. For $x \in X$ and $y^* \in Y^*$, put

$$\begin{aligned} U(x,y^*) &= \inf\{y^*(y) + \|x-y\| : y \in Y\} \\ L(x,y^*) &= \sup\{y^*(y) - \|x-y\| : y \in Y\} \end{aligned}$$

For $x^* \in X^*$, we will write $U(x,x^*)$ for $U(x,x^*|_Y)$.

Remark 2.2. By [12, Lemma 1], $U(x,\cdot)$ and $L(x,\cdot)$ are analogs of the "u.s.c. hull" and "l.s.c. hull" of $x \in X$ considered as a "functional" on $B(Y^*)$. Observe that, in general, we cannot even consider $x \in X$ as a functional on Y^* as the latter may not be identifiable as a subspace of X^*.

The following result is immediate from the proof of the Hahn-Banach Theorem (see e.g., [29, Section 48]).

LEMMA 2.3. *Let Y be a subspace of a normed linear space X. Suppose $x_0 \notin Y$ and $y^* \in S(Y^*)$. Then $L(x_0, y^*) \leq U(x_0, y^*)$ and α lies between these two numbers if and only if there exists a Hahn-Banach extension x^* of y^* with $x^*(x_0) = \alpha$.*

Remark 2.4. It is clear that for any $x^* \in B(X^*)$ and $x \in X$, $L(x, x^*) \leq x^*(x) \leq U(x, x^*)$ and an $y^* \in S(Y^*)$ has an unique Hahn-Banach extension to X if and only if for all $x \in X$, $L(x, y^*) = U(x, y^*)$.

Here is our analogue of [12, Lemma 2]

LEMMA 2.5. *Let Y be a subspace of a Banach space X. For $x_1, x_2 \in X$, the following are equivalent :*

(a) $x_2 \in \bigcap_{y \in Y} B_X[y, \|x_1 - y\|]$.

(b) *For all $y \in Y$, $\|x_2 - y\| \leq \|x_1 - y\|$.*

(c) *For all $x^* \in B(X^*)$, $U(x_2, x^*) \leq U(x_1, x^*)$.*

Proof. Equivalence of (a) and (b) is clear.

(b) $\Rightarrow$ (c) If $\|x_2 - y\| \leq \|x_1 - y\|$, for all $y \in Y$, then for all $x^* \in B(X^*)$, $x^*(y) + \|x_2 - y\| \leq x^*(y) + \|x_1 - y\|$. And therefore, $U(x_2, x^*) \leq U(x_1, x^*)$.

(c) $\Rightarrow$ (b) Suppose $\|x_2 - y_0\| > \|x_1 - y_0\|$ for some $y_0 \in Y$. Then there exists $\varepsilon > 0$ such that $\|x_2 - y_0\| - \varepsilon \geq \|x_1 - y_0\|$. Choose $x^* \in B(X^*)$ such that $\|x_1 - y_0\| \leq \|x_2 - y_0\| - \varepsilon < x^*(x_2 - y_0) - \varepsilon/2$. Thus $U(x_1, x^*) \leq x^*(y_0) + \|x_1 - y_0\| < x^*(x_2) - \varepsilon/2 < U(x_2, x^*)$. ∎

Remark 2.6. Instead of $B(X^*)$, it suffices to consider any norming set for X.

The next lemma is a key step that allows us to do away with topological considerations.

LEMMA 2.7. *Let Y be a subspace of a Banach space X. For $x_1, x_2 \in X$, and $x^* \in B(X^*)$, $U(x_1, x^*) - U(x_2, x^*) \leq U(x_1 - x_2, x^*)$.*

Proof. For any $x^* \in B(X^*)$ and $y_1, y_2 \in Y$,

$$\begin{aligned} U(x_1, x^*) &\leq x^*(y_1 + y_2) + \|x_1 - y_1 - y_2\| \\ &= x^*(y_1) + x^*(y_2) + \|(x_2 - y_2) + (x_1 - x_2 - y_1)\| \\ &\leq x^*(y_2) + \|x_2 - y_2\| + x^*(y_1) + \|x_1 - x_2 - y_1\| \end{aligned}$$

Since $y_1, y_2 \in Y$ are arbitrary, it follows that

$$U(x_1, x^*) \leq U(x_2, x^*) + U(x_1 - x_2, x^*)$$

∎

Analogous to $\mathcal{O}(X)$ in [13], we now introduce the ortho-complement of Y in X.

DEFINITION 2.8. Let $Y \subseteq X$ be a subspace of X. We define the ortho-complement $O(Y, X)$ of Y in X as

$$O(Y, X) = \{x \in X : \|x - y\| \geq \|y\| \text{ for all } y \in Y\}.$$

Remark 2.9. Recall that (see e.g., [19]) for $x, y \in X$, one says x is orthogonal to y in the sense of Birkhoff (written $x \perp_B y$) if $\|x + \lambda y\| \geq \|x\|$, for all $\lambda \in \mathbb{R}$. Thus, $O(Y, X)$ is the collection of $x \in X$ such that $Y \perp_B x$. This justifies the terminology. We could have formulated most of the results in this paper in terms of Birkhoff orthogonality also. But we did not do it as this does not give us any better insight into the phenomenon.

Here is our analogue of [13, Lemma I.1] with some additions.

LEMMA 2.10. *Let Y be a subspace of a Banach space X. Let $x \in X$. Then, the following are equivalent :*

(a) $x \in O(Y, X)$

(b) $\ker(x)|_Y \subseteq Y^*$ *is a norming subspace for Y.*

(c) $0 \in \bigcap_{y \in Y} B_Y[y, \|x - y\|]$.

(d) *For every $x^* \in B(X^*)$, $L(x, x^*) \leq 0 \leq U(x, x^*)$.*

(e) *For every $y^* \in B(Y^*)$, $L(x, y^*) \leq 0 \leq U(x, y^*)$.*

Proof. (a) $\Rightarrow$ (b) Suppose $x \in O(Y,X)$. We need to show $\|y\| = \|y|_{\ker(x)}\|$ for all $y \in Y$. Clearly, $\|y\| \geq \|y|_{\ker(x)}\|$. Since

$$\ker(x)^* = X^{**}/\ker(x)^{\perp} = X^{**}/\mathrm{span}\{x\},$$

we have

$$\|y|_{\ker(x)}\| = d(y, \mathrm{span}\{x\}) = \inf_{\lambda\in\mathbb{R}} \|y - \lambda x\| \geq \|y\|$$

since $x \in O(Y,X)$ and from the definition, it follows that $O(Y,X)$ is closed under scalar multiplication. Hence $\|y\| = \|y|_{\ker(x)}\|$.

(b) $\Rightarrow$ (a) Since $\ker(x)|_Y$ norms Y, $\|y\| = \|y|_{\ker(x)}\| = d(y, \mathrm{span}\{x\}) = \inf_{\lambda\in\mathbb{R}} \|y - \lambda x\|$ for all $y \in Y$. Hence $\|x - y\| \geq \inf_{\lambda\in\mathbb{R}} \|y - \lambda x\| \geq \|y\|$ for all $y \in Y$. Thus, $x \in O(Y,X)$.

(a) $\Leftrightarrow$ (c) and (d)$\Rightarrow$(e) are immediate from definition, while $(c) \Rightarrow (d)$ follows from Lemma 2.5.

(e) $\Rightarrow$ (a) For every $y^* \in B(Y^*)$, $0 \leq U(x, y^*)$ implies for all $y^* \in B(Y^*)$ and $y \in Y$,

$$0 \leq y^*(y) + \|x - y\| \implies y^*(-y) \leq \|x - y\|.$$

Since this is true for all $y^* \in B(Y^*)$, $\|y\| \leq \|x - y\|$ for all $y \in Y$. That is, $x \in O(Y,X)$. ∎

Remark 2.11. In [3], it has been further shown that for a w*-closed subspace $F \subseteq X^*$, $F|_Y$ is a norming subspace for Y if and only if $F_{\perp} = \{x \in X : f(x) = 0 \text{ for all } f \in F\} \subseteq O(Y,X)$.

Now we are ready for our main characterization theorem for a VN-subspace. Compare this with Theorem 1.4 and [2, Proposition 2.2].

We will use the following notation. For $y^* \in Y^*$, the set of all Hahn-Banach extension of y^* to X is denoted by $\mathrm{HB}(y^*)$.

THEOREM 2.12. *Let Y be a subspace of a Banach space X. Then, the following are equivalent :*

(a) *Y is a VN-subspace of X.*

(b) *For any $x \in X \setminus Y$,*

$$\bigcap_{y\in Y} B_Y[y, \|x - y\|] = \emptyset.$$

(c) $O(Y,X) = \{0\}$.

(d) *Any $A \subseteq B(X^*)$ such that $A|_Y$ is a norming set for Y, separates points of X.*

(e) *Any subspace $F \subseteq X^*$ such that $F|_Y$ is a norming subspace for Y, separates points of X.*

(f) *For all nonzero $x \in X$, there exists $y^* \in S(Y^*)$ such that every $x^* \in HB(y^*)$ takes non-zero value at x.*

Proof. Clearly, (a) $\Rightarrow$ (b)

(b) $\Rightarrow$ (c) Suppose $x \in O(Y,X)$ and $x \neq 0$. Then, $x \notin Y$ and by Lemma 2.10, it follows that $0 \in \bigcap_{y \in Y} B_Y[y, \|x-y\|]$, a contradiction.

(c) $\Rightarrow$ (a) Suppose $x_1, x_2 \in X$ such that

$$x_2 \in \bigcap_{y \in Y} B_X[y, \|x_1 - y\|].$$

By Lemma 2.5, for all $x^* \in B(X^*)$, $U(x_2, x^*) \leq U(x_1, x^*)$. By Lemma 2.7, $0 \leq U(x_1 - x_2, x^*)$. That is, $x_1 - x_2 \in O(Y,X)$, by Lemma 2.10. By (c), $x_1 = x_2$. Hence Y is a VN-subspace.

(c) $\Rightarrow$ (d) Let $A \subseteq B(X^*)$ be such that $A|_Y$ is a norming set for Y. By Lemma 2.10, $A^\perp \cap X \subseteq O(Y,X)$. By (c), therefore, $A^\perp \cap X = \{0\}$. Thus, A separates points of X.

(d) $\Leftrightarrow$ (e) Since a subspace F is norming if and only if it is the closed linear span of a norming set, this is clear.

(d) $\Rightarrow$ (f) Suppose (f) does not hold. Then there exists $x \neq 0 \in X$ such that for every $y^* \in S(Y^*)$, there exists $x^* \in \mathrm{HB}(y^*)$ such that $x^*(x) = 0$. Let

$$A = \{x^* \in S(X^*) : x^*(x) = 0\}$$

Then $A|_Y = S(Y^*)$ and hence, is a norming set for Y, but A clearly does not separate x from 0.

(f) $\Rightarrow$ (c) $x \in O(Y,X)$ implies, by Lemma 2.10, that for every $y^* \in S(Y^*)$, $L(x, y^*) \leq 0 \leq U(x, y^*)$, and by Lemma 2.3, this implies for every $y^* \in S(Y^*)$, there exists $x^* \in \mathrm{HB}(y^*)$ such that $x^*(x) = 0$. Hence the result. ∎

The following observations are quite useful in applications.

Proposition 2.13. *Let $Y \subseteq Z \subseteq X$, where Y is a VN-subspace of X. Then Z is a VN-subspace of X and Y is a VN-subspace of Z. If, moreover, Z is an AC-subspace of X, then $Z = X$.*

Proof. We observe that $O(Y,Z) \subseteq O(Y,X)$ and $O(Z,X) \subseteq O(Y,X)$. This proves the first part. In the second part, observe that Z is both a VN-subspace and an AC-subspace of X. Thus, $Z = X$. ∎

Remark 2.14. Compare this with [2, Theorems 2.16 and 2.18].

COROLLARY 2.15. *X is reflexive if and only if there is a subspace $M \subseteq X^*$ which when considered as a subspace of X^{***} is a VN-subspace.*

Proof. If X is reflexive, take $M = X^*$. Conversely, if there is an $M \subseteq X^* \subseteq X^{***}$ and M is a VN-subspace of X^{***}, then since X^* is an AC-subspace of X^{***}, we have by the above result that $X^* = X^{***}$. ∎

EXAMPLE 2.16. Even though the property under consideration here depends on the norm, it should be emphasized that if a Banach space X contains two subspaces Y and Z, which are isometrically isomorphic and one of them is a VN-subspace, the other need not be a VN-subspace.

For example, consider the usual inclusion of $c_0 \subseteq c \subseteq \ell_\infty$. The inclusion of c_0 in ℓ_∞ is the canonical embedding of c_0 in $c_0^{**} = \ell_\infty$. Since c_0 is nicely smooth, in this embedding, it is a VN-subspace of ℓ_∞. By the above result, therefore, c, in its inclusion, is a VN-subspace of ℓ_∞. However, it has been noted in [2] that c is not nicely smooth. That is, in the canonical embedding of c in $c^{**} = \ell_\infty$, c is not a VN-subspace. It follows that even c_0, as a subspace of the canonical embedding of c, is not a VN-subspace of ℓ_∞. This example illustrates the need of caution in applying the above proposition.

And here is our analogue of [2, Theorem 2.13].

PROPOSITION 2.17. *Let Y be a subspace of X. Then Y is a VN-subspace of X in every equivalent renorming of X if and only if $Y = X$.*

Proof. The converse being trivial, suppose $Y \neq X$. Let $x \in X \setminus Y$ and let $F = \{x^* \in X^* : x^*(x) = 0\}$. Define a new norm on Y by

$$\|y\|_1 = \sup\{x^*(y) : x^* \in B(F)\} \quad \text{for } y \in Y$$

It follows from arguments similar to [14, Theorem 8.2] that $\|\cdot\|_1$ is an equivalent norm on Y with $F|_Y$ as a norming subspace. Now this norm on Y extends to an equivalent norm on X by [10, Lemma II.8.1]. And clearly, with this norm, Y is not a VN-subspace of X. ∎

Inspired by [2, Theorem 2.10], we now try to identify some necessary and some sufficient conditions for a subspace to be a VN-subspace. As in [2], we start with a class of functionals with "locally unique" Hahn-Banach extensions. And here we bring back some of the topological flavour.

Let $C(x) = \{x^* \in B(X^*) : U(x, x^*) = L(x, x^*)\}$, for $x \in X$, and $C = \cap_{x \in X} C(x)$.

We now obtain characterizations of elements of $C(x)$ and C. The first is an analogue of [2, Proposition 2.7] and a refinement of [4, Proposition 3.2].

Proposition 2.18. *Let Y be a subspace of a Banach space X. Let $x^* \in B(X^*)$ and $x_0 \in X \setminus Y$. The following are equivalent :*

(a) $x^* \in C(x_0)$.

(b) $\|x^*|_Y\| = 1$ *and every* $x_1^* \in HB(x^*|_Y)$ *takes the same value at* x_0.

(c) *if* $x^*(x_0) > \alpha$ *(or,* $x^*(x_0) < \alpha$*) for some* $\alpha \in \mathbb{R}$*, then there exists a closed ball* B *in* X *with centre in* Y *such that* $x_0 \in B$ *and* $\inf x^*(B) > \alpha$ *(respectively,* $\sup x^*(B) < \alpha$*).*

(d) $\|x^*|_Y\| = 1$ *and if* $\{x_\alpha^*\} \subseteq S(X^*)$ *is a net such that* $x_\alpha^*|_Y \to x^*|_Y$ *in the* w^**-topology of* Y^**, then* $\lim_\alpha x_\alpha^*(x_0) = x^*(x_0)$.

(e) $\|x^*|_Y\| = 1$ *and if* $\{x_n^*\} \subseteq S(X^*)$ *is a sequence such that* $x_n^*|_Y \to x^*|_Y$ *in the* w^**-topology of* Y^**, then* $\lim x_n^*(x_0) = x^*(x_0)$.

Proof. (a) $\Leftrightarrow$ (b) Let $y^* = x^*|_Y$ and $\|y^*\| = \alpha$. Then $\alpha \leq \|x^*\| \leq 1$ and it suffices to show that $\alpha = 1$.

Working with some $x_1^* \in \mathrm{HB}(y^*)$, it follows that

$$\begin{aligned} L(x_0, x^*) &\leq \sup\{y^*(y) - \alpha\|x_0 - y\| : y \in Y\} \leq \\ x_1^*(x_0) &\leq \inf\{y^*(y) + \alpha\|x_0 - y\| : y \in Y\} \leq U(x_0, x^*) \end{aligned}$$

Thus, equality holds everywhere.

Now if $\alpha < 1$, let $0 < \delta < d(x_0, Y)$ and let $0 < \varepsilon < (1 - \alpha)\delta$. Then $(1 - \alpha)\|x_0 - y\| > \varepsilon$ for all $y \in Y$. And therefore, for all $y \in Y$,

$$y^*(y) - \|x_0 - y\| + \varepsilon < y^*(y) - \alpha\|x_0 - y\|$$

And therefore, the first inequality must be strict. Contradiction!

The result now follows from Lemma 2.3.

(a) $\Leftrightarrow$ (c) The proof is essentially same as the proof of [2, Proposition 2.7 (c) $\Leftrightarrow$ (d)]. We omit the details.

(b) $\Rightarrow$ (d) Let $\{x^*_\alpha\} \subseteq S(X^*)$ be a net such that $\lim_\alpha x^*_\alpha(y) = x^*(y)$ for all $y \in Y$. It follows that any w*-cluster point of $\{x^*_\alpha\}$ is in $\mathrm{HB}(x^*|_Y)$. By (b), therefore, $\lim x^*_\alpha(x_0) = x^*(x_0)$.

(d) $\Rightarrow$ (e) is clear.

(e) $\Rightarrow$ (b) If $x^*_1 \in \mathrm{HB}(x^*|_Y)$ with $x^*(x_0) \neq x^*_1(x_0)$, then the constant sequence $x^*_n = x^*_1$ clearly satisfies $\lim_n x^*_n(y) = x^*(y)$ for all $y \in Y$, but $\{x^*_n(x_0)\}$ cannot converge to $x^*(x_0)$. ∎

Proposition 2.19. *Let Y be a subspace of a Banach space X. Let $x^* \in B(X^*)$. The following are equivalent :*

(a) $x^* \in C$.

(b) $\|x^*|_Y\| = 1$ *and* x^* *is the unique Hahn-Banach extension of* $x^*|_Y$ *to* X.

(c) *if* $x_0 \notin Y$ *and* $x^*(x_0) > \alpha$ *(respectively,* $x^*(x_0) < \alpha$*) for some* $\alpha \in \mathbb{R}$, *then there exists a closed ball* B *in* X *with centre in* Y *such that* $x_0 \in B$ *and* $\inf x^*(B) > \alpha$ *(respectively,* $\sup x^*(B) < \alpha$*).*

(d) $\|x^*|_Y\| = 1$ *and if* $\{x^*_\alpha\} \subseteq S(X^*)$ *is a net such that* $x^*_\alpha|_Y \to x^*|_Y$ *in the w*-topology of* Y^*, *then* $x^*_\alpha \to x^*$ *in the w*-topology of* X^*.

(e) $\|x^*|_Y\| = 1$ *and if* $\{x^*_n\} \subseteq S(X^*)$ *is a sequence such that* $x^*_n|_Y \to x^*|_Y$ *in the w*-topology of* Y^*, *then* $x^*_n \to x^*$ *in the w*-topology of* X^*.

Theorem 2.20. *Let Y be a subspace of a Banach space X. Consider the following statements :*

(a) C *separates points of* X.

(b) *Any two distinct points in* X *are separated by disjoint closed balls with centres in* Y.

(b_1) *For every* $x \in X$, $C(x)$ *separates points of* X.

(b_2) *For every nonzero* $x \in X$, *there is* $x^* \in C(x)$ *such that* $x^*(x) \neq 0$.

(c) Y *is a* VN*-subspace of* X.

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c) and (a) $\Rightarrow$ (b_1) $\Rightarrow$ (b_2) $\Rightarrow$ (c).

Proof. (a) $\Rightarrow$ (b) The proof is essentially same as the proof of [2, Theorem 2.10 (a) $\Rightarrow$ (b)], except that we need to use Proposition 2.18 instead of [2, Corollary 2.8]. We omit the details.

(b) $\Rightarrow$ (c) Clear.

(a) $\Rightarrow$ (b_1) $\Rightarrow$ (b_2) follows from definitions.

(b$_2$) $\Rightarrow$ (c) By (b$_2$), for every nonzero $x \in X$, there is a $x^* \in C(x)$ such that $x^*(x) \neq 0$. By Proposition 2.18, $y^* = x^*|_Y \in S(Y^*)$ and every $x_1^* \in \mathrm{HB}(y^*)$ to X takes the same value at x. The result now follows from Theorem 2.12(f). ∎

Remark 2.21. If $C|_Y$ is a norming set for Y, then all the conditions are clearly equivalent. Notice that $C|_Y = \{y^* \in S(Y^*) : \mathrm{HB}(y^*)$ is singleton$\}$. Thus, this condition is satisfied if Y is an U-subspace of X.

It is clear from Proposition 2.19 that the class C is the analogue of w*-weak points of continuity if $Y = Z$ and $X = Z^{**}$. In this case, the above condition is satisfied if Z is an Asplund space. Thus we get back much of [2, Theorem 2.10].

As mentioned in the introduction, Hahn-Banach smooth spaces are nicely smooth. We now give an elementary example to show that, in contrast, a U-subspace need not be a VN-subspace.

Example 2.22. Let $X = \mathbb{R}^2$ with the Euclidean norm and $Y = \{(r, 0) : r \in \mathbb{R}\}$. It is easy to see that Y is a U-subspace of X. But Y is also a constrained subspace, and therefore, not a VN-subspace of X.

Example 2.23. Recall that a subspace Y of X is called an L-summand (M-summand) if there exists a projection P on X with range Y such that for all $x \in X$, $\|x\| = \|Px\| + \|x - Px\|$ (resp. $\|x\| = \max\{\|Px\|, \|x - Px\|\}$). A subspace Y of X is called an M-ideal if $Y^\perp$ is an L-summand in X^*. Y is called a proper M-ideal in X if it is an M-ideal but not an M-summand in X. The book [17] is a standard reference for M-ideals and related topics. It is known that proper M-ideals are not constrained.

A Banach space X is called M-embedded if it is a proper M-ideal in X^{**}. In [28], it is proved that if X is an M-embedded space, then it a proper M-ideal in every even dual. Also an M-ideal is an U-subspace (see [17]). However, by Corollary 2.15, such an X cannot be a VN-subspace of $X^{(4)}$. Thus, we get another example of a U-subspace which is not a VN-subspace.

In fact, this example shows that even a proper M-ideal need not be a VN-subspace. However, an M-embedded space, being Hahn-Banach smooth, is always nicely smooth.

Let us now try to understand why such examples work.

DEFINITION 2.24. A subspace Y of X is said to be a $(*)$-subspace of X if the set

$$A = \{x^* \in S(X^*) : \|x^*|_Y\| = 1\} = \mathrm{HB}(S(Y^*))$$

separates points of X.

Here are some natural examples of $(*)$-subspaces.

(a) X is a $(*)$-subspace of X^{**}.

(b) If $Y \subseteq Z \subseteq X$ and Y is a $(*)$-subspace of X, then Z is a $(*)$-subspace of X and Y is a $(*)$-subspace of Z.

(c) For any two Banach spaces X and Y, $\mathcal{K}(X,Y)$ is a $(*)$-subspace of $\mathcal{L}(X,Y)$.

(d) If Y is a $(*)$-subspace of Z, then for any Banach space X, $X \otimes_\pi Y$ is a $(*)$-subspace of $X \otimes_\pi Z$. In particular, $X \otimes_\pi Y$ is a $(*)$-subspace of $X \otimes_\pi Y^{**}$.

(e) $C(K,X)$ is a $(*)$-subspace of $WC(K,X)$.

(f) More generally, if Y is an ideal in X and satisfies the conditions of [27, Lemma 1(i)], then Y is a $(*)$-subspace of X. See [27] for details.

PROPOSITION 2.25. *Let Y be a subspace of a Banach space X.*

(a) *If Y is a VN-subspace, Y is a $(*)$-subspace.*

(b) *If Y is a $(*)$-subspace as well as a U-subspace of X, then Y is a VN-subspace.*

Proof. By Theorem 2.12(d), if Y is a VN-subspace, A separates points of X.

And if Y is a U-subspace, $A \subseteq C$ and therefore, if Y is $(*)$-subspace, C separates points of X. ∎

As mentioned in the introduction, it was shown in [15, Proposition 2.5] that nicely smooth spaces have the UEP. Here we show that

PROPOSITION 2.26. *Let Y be a VN-subspace of X. Then*

(a) *the only operator $T \in \mathcal{L}(X)$ such that $\|T\| \leq 1$ and $T|_Y = Id_Y$ is $T = Id_X$.*

(b) *Y has the unique ideal property in X.*

Proof. (a) This is essentially the same proof as [15, Proposition 2.5].

(b) Let P_i, $i = 1, 2$ be norm 1 projections on X^* with $\ker(P_i) = Y^\perp$. It is enough to show that for all $x \in B(X)$, $P_1^*(x) = P_2^*(x)$.

We make the following observations :

(i) Let $\sigma_i = \sigma(X, P_i X^*)$, $i = 1, 2$ be the topologies induced on X by $P_i X^*$. Since Y is a VN-subspace of X and $P_i X^*$ are norming for Y, we have $(B(X), \sigma_i)$ are Hausdorff spaces. Also $B(Y)$ is σ_i-dense in $B(X)$ (see [17], Remark 1.13).

(ii) $P_i^*|_Y = Id|_Y$.

(iii) $P_1^* = P_2^* P_1^*$.

(iv) On $B(P_1^* X^{**})$, we can consider the two topologies τ_1 and τ_2 induced by $P_1 X^*$ and $P_2 X^*$ respectively. It is easy to note that these two are compact Hausdorff topologies on $B(P_1^* X^{**})$ and from (c), the identity map is τ_1-τ_2 continuous. Thus these two topologies are identical.

(v) $\tau_i|_Y = \sigma_i|_Y$, $i = 1, 2$.

Now, given $x \in B(X)$, take a net $\{y_\alpha\} \subseteq B(Y)$ such that $y_\alpha \xrightarrow{\sigma_1} x$. Since σ_1 is Hausdorff, x is the unique σ_1-cluster point of $\{y_\alpha\}$. Therefore, $y_\alpha \xrightarrow{\sigma_2} x$ also. Thus for all $x^* \in X^*$, $(P_1^* y_\alpha)(x^*) = x^*(y_\alpha) \longrightarrow (P_1^* x)(x^*)$ and $(P_2^* y_\alpha)(x^*) = x^*(y_\alpha) \longrightarrow (P_2^* x)(x^*)$. Thus $P_1^* x = P_2^* x$ as desired. ∎

Remark 2.27. In case of X in X^{**}, as noted in [22], (a) $\Leftrightarrow$ (b). We do not know if (a) $\Rightarrow$ (b). However, (b), in general, does not imply (a). See Remark 2.29 below.

Recall that a hyperplane H in a Banach space X is a subspace such that $H = \ker(x^*)$ for some $x^* \in S(X^*)$. By Proposition 2.13, any hyperplane containing a VN-subspace is itself a VN-subspace. Indeed, a VN-subspace is the intersection of all VN-hyperplanes containing it. So, when is a hyperplane a VN-subspace?

PROPOSITION 2.28. *For a hyperplane H in a Banach space X, the following are equivalent :*

(a) *H is a VN-subspace of X.*

(b) *H is not an AC-subspace of X.*

(c) *the only operator $T \in \mathcal{L}(X)$ such that $\|T\| \leq 1$ and $T|_H = Id_H$ is $T = Id_X$.*

(d) *H is not constrained in X.*

Proof. Clearly, (a) $\Rightarrow$ (b) $\Rightarrow$ (d) and (a) $\Rightarrow$ (c) $\Rightarrow$ (d).

(d) $\Rightarrow$ (a) Let $x^* \in S(X^*)$ be such that $H = \ker(x^*)$. Suppose H is not a VN-subspace in X. Then there is a $x_0 \in O(H, X)$, $x_0 \neq 0$. Since $O(H, X)$ is closed under scalar multiplication, we may assume $x^*(x_0) = 1$.

Clearly, $P : X \to X$ defined by $P(x) = x - x^*(x)x_0$ is a bounded linear projection onto H. It suffices to show that $\|P(x)\| \leq \|x\|$ for all $x \in X$.

Let $x \in X$. Since $O(H, X)$ is closed under scalar multiplication, $x^*(x)x_0 \in O(H, X)$ and therefore, $\|x\| = \|x^*(x)x_0 + P(x)\| \geq \|P(x)\|$. ∎

Remark 2.29. Observe that even in this case, we cannot replace (c) above by "H has the unique ideal property in X".

For example, let K be a compact Hausdorff space and $X = C(K)$. Let $k_0 \in K$ be an isolated point and let $H = \{f \in C(K) : f(k_0) = 0\}$. Then H is an M-summand in X and therefore, is not a VN-subspace. However, H is an M-ideal and hence a U-subspace of X. It follows that H has the unique ideal property in X.

However, such a situation cannot occur for X in X^{**} and we obtain

Corollary 2.30. *Let X be a Banach space such that* $\dim(X^{**}/X) = 1$. *Then the following are equivalent :*

(a) *X is not nicely smooth.*

(b) *X has the $IP_{f,\infty}$.*

(c) *X is constrained in X^{**}.*

(d) *X fails the UEP.*

We now characterize VN-hyperplanes in some classical Banach spaces.

Proposition 2.31. *For a Banach space X, the following are equivalent :*

(a) *X is a Hilbert space.*

(b) *No proper subspace of X is a VN-subspace.*

(c) *No hyperplane in X is a VN-subspace.*

Proof. In a Hilbert space, every subspace is constrained, hence no proper subspace is a VN-subspace. Thus (a) $\Rightarrow$ (b) $\Rightarrow$ (c).

(c) $\Rightarrow$ (a) If no hyperplane is a VN-subspace, by Proposition 2.28, every hyperplane is constrained. It is well known (see for example [1, Corollary 2.2]) that this implies X is a Hilbert space. ∎

At the other end of the spectrum are spaces in which all hyperplanes are VN-subspaces. Examples of such spaces are available even among reflexive spaces. Let us recall the following result.

THEOREM 2.32. [7, Proposition VI.3.1] *Let $1 < p < \infty$, $p \neq 2$ and $1/p + 1/q = 1$. Let $f \in L_q(\Omega, \Sigma, \mu)$, $f \neq 0$. Then the hyperplane* $\ker(f)$ *is constrained in $L_p(\Omega, \Sigma, \mu)$ if and only if f is of the form $f = \alpha\chi_A + \beta\chi_B$, where A and B are atoms of μ and $\alpha, \beta \in \mathbb{R}$.*

Thus for μ nonatomic, the spaces $L_p(\mu)$, $1 < p < \infty$, $p \neq 2$, provide examples of reflexive spaces in which all hyperplanes are VN-subspaces. Since there are constrained subspaces in these spaces, this also shows that intersection of VN-subspaces need not be a VN-subspace.

Even for $L_1(\mu)$ with μ nonatomic, it is known that, there is no constrained hyperplane. Thus again, all hyperplanes in $L_1(\mu)$ are VN-subspaces. Actually, in this case, no subspace of finite co-dimension is constrained (see [17], Corollary IV.1.15). Are these all VN-subspaces?

Coming to the sequence spaces, Theorem 2.32 also shows that for $1 < p < \infty$, $p \neq 2$ and $1/p + 1/q = 1$, for $\phi \in \ell_q$, the hyperplane $\ker(\phi)$ is constrained in ℓ_p if and only if at most 2 coordinates of ϕ are nonzero.

The same statement is also true for ℓ_1. This was proved by [8, Theorem 3]. But their argument is quite involved. Here is a simple proof.

Proof. Suppose $\phi = (s_1, s_2, 0, 0, \ldots) \in \ell_\infty$ and $H = \ker(\phi)$. Let

$$\boldsymbol{z} = \frac{1}{|s_1| + |s_2|}(sgn(s_1), sgn(s_2), 0, 0, \ldots) \in \ell_1.$$

Then $\phi(\boldsymbol{z}) = 1$ and it is not difficult to verify that the projection defined by $P(\boldsymbol{x}) = \boldsymbol{x} - \phi(\boldsymbol{x})\boldsymbol{z}$ is of norm 1.

Conversely, suppose $\phi = (s_1, s_2, s_3, \ldots) \in \ell_\infty$ has at least three nonzero coordinates and $H = \ker(\phi)$. Without loss of generality, assume s_1, s_2, s_3 are nonzero. We will show that H cannot be an AC-subspace. Since $\boldsymbol{x_0} = (1/s_1, 1/s_2, 1/s_3, 0, 0, \ldots) \notin H$, if H were an AC-subspace, we would have an $\boldsymbol{y_0} \in \cap_{y \in H} B_H[\boldsymbol{y}, \|\boldsymbol{x_0} - \boldsymbol{y}\|]$. Let $\boldsymbol{y_0} = (y_1, y_2, y_3, \ldots)$. Note that $\boldsymbol{z_0} =$

$\boldsymbol{x_0} - \boldsymbol{y_0} \in O(H, X)$. Now if we put $\boldsymbol{y} = (1/s_1, 1/s_2, -2/s_3, 0, 0, \ldots) - \boldsymbol{y_0}$, then $\boldsymbol{y} \in H$. And therefore, $\|\boldsymbol{z_0} - \boldsymbol{y}\| \geq \|\boldsymbol{y}\|$. That is,

$$\left|\frac{3}{s_3}\right| \geq \left|\frac{1}{s_1} - y_1\right| + \left|\frac{1}{s_2} - y_2\right| + \left|\frac{2}{s_3} + y_3\right| + \sum_{i=4}^{\infty} |y_i|.$$

And hence,

$$\left|\frac{1}{s_3} - y_3\right| \geq \left|\frac{1}{s_1} - y_1\right| + \left|\frac{1}{s_2} - y_2\right| + \sum_{i=4}^{\infty} |y_i|.$$

Similarly taking $\boldsymbol{y} = (1/s_1, -2/s_2, 1/s_3, 0, 0, \ldots) - \boldsymbol{y_0}$, we get

$$\left|\frac{1}{s_2} - y_2\right| \geq \left|\frac{1}{s_1} - y_1\right| + \left|\frac{1}{s_3} - y_3\right| + \sum_{i=4}^{\infty} |y_i|,$$

and taking $\boldsymbol{y} = (-2/s_1, 1/s_2, 1/s_3, 0, 0, \ldots) - \boldsymbol{y_0}$, we get

$$\left|\frac{1}{s_1} - y_1\right| \geq \left|\frac{1}{s_2} - y_2\right| + \left|\frac{1}{s_3} - y_3\right| + \sum_{i=4}^{\infty} |y_i|.$$

This is surely not possible. ∎

Coming to c_0, it is shown in [8, Theorem 1] that for $\phi = (s_1, s_2, s_3, \ldots) \in \ell_1$ with $\|\phi\| = 1$, the hyperplane $\ker(\phi)$ is constrained in c_0 if and only if $|s_n| \geq 1/2$ for some n. Thus, whenever $|s_n| < 1/2$ for all n, the hyperplane $\ker(\phi)$ is a VN-subspace.

It follows from the results of [5] that for $\phi \in \ell_\infty^*$ with $\|\phi\| = 1$, if we write $\phi = \phi_1 + \phi_2$, where $\phi_1 = (s_1, s_2, s_3, \ldots) \in \ell_1$ and $\phi_2 \in c_0^\perp$, then the hyperplane $\ker(\phi)$ is a VN-subspace of ℓ_∞ if and only if $|s_n| < 1/2$ for all n.

For $X = C(K)$, the hyperplane $H = \{f \in C(K) : f(k) = 0\}$ is an M-ideal for any $k \in K$ and is constrained only when k is an isolated point of K. Thus for every other k, we get a VN-subspace.

Remark 2.33. It would be interesting to characterize all $x^* \in S(X^*)$ such that $\ker(x^*)$ is a VN-subspace of X. This clearly is same as characterizing all $x^* \in S(X^*)$ such that $\ker(x^*)$ is constrained.

If X is an M-embedded space, then any constrained subspace of X^* is w*-closed. Therefore, for any $x^{**} \in X^{**} \setminus X$, $\ker(x^{**})$ is a VN-subspace of X^*. But as the ℓ_1 example above shows, this does not exhaust all the possibilities.

Example 2.34. Observe that a 1-dimensional subspace is always constrained, and therefore, cannot be a VN-subspace. Can a space have a finite dimensional VN-subspace? It is easy to see that in a polyhedral Banach space, for example c_0, finite dimensional subspaces, since they have only finitely many extreme points, cannot be VN-subspaces. But in c we can exhibit a two-dimensional VN-subspace.

Consider the subspace $Y \subseteq c$ spanned by $\boldsymbol{x} = (\sin 1/n)$ and $\boldsymbol{y} = (\cos 1/n)$. Taking vectors of the form $\sin 1/k \cdot \boldsymbol{x} + \cos 1/k \cdot \boldsymbol{y}$, one can see that any norming subspace for Y in ℓ_1 contains all the unit vectors e_n. Hence Y is a VN-subspace.

We now discuss some consequences of the existence of a separable VN-subspace.

Theorem 2.35. *Let X be a Banach space with a separable VN-subspace Y. Then,*

(a) *there is a countable set $\{\xi_n^*\} \subseteq S(X^*)$ which separates points of X.*

(b) *Weakly compact subsets of X are metrizable.*

(c) *Let K be a compact Hausdorff space. Then every $f \in WC(K, X)$ is Baire class-1, i.e., there exists a sequence $\{f_n\} \subseteq C(K, X)$ such that $f_n \to f$ pointwise.*

Proof. (a) Consider the duality map $D : S(Y) \to S(Y^*)$ given by $D(y) = \{y^* \in S(Y^*) : y^*(y) = 1\}$. Let $\{y_n\}$ be a dense subset of $S(Y)$. Let y_n^* be a selection of $D(y_n)$. Then $\{y_n^*\}$ is norming for Y. For each y_n^*, choose $x_n^* \in \mathrm{HB}(y_n^*)$. Then $\overline{\mathrm{sp}}\{x_n^*\}$ is a subspace of X^* which is norming for Y.

Since Y is a VN-subspace of X, $\overline{\mathrm{sp}}\{x_n^*\}$ separates points of X. Now $\overline{\mathrm{sp}}\{x_n^*\}$ being norm separable, there is a norm dense set $\{\xi_n^*\} \subseteq \overline{\mathrm{sp}}\{x_n^*\}$ which separates points of X.

(b) Let K be a weakly compact subset of X. Since $\{\xi_n^*\}$ separates points of K and are weakly continuous, we have the result.

(c) Let $W = f(K)$. W is a weakly compact subset of X. Hence by (a), it is weakly metrizable. It follows that W is weakly separable and so, it is norm separable. Now follow the arguments of [26]. ∎

The following result is also immediate.

Proposition 2.36. *Suppose Y is a separable subspace of a Banach space X such that Y is a VN-subspace of X^{**}. Then $B(X^{**})$ is w*-metrizable.*

Remark 2.37. If a Banach space X satisfies the hypothesis of Proposition 2.36, then (X, w) is σ-fragmentable (see [21] for details).

THEOREM 2.38. *Let X be a WCG Banach space with a separable VN-subspace. Then X itself is separable.*

Proof. It is well known that any separable subspace of a WCG space is actually contained in a separable constrained subspace (see, e.g., [10, page 238]). Hence the result follows from Proposition 2.13. ∎

3. Stability Results

THEOREM 3.1. *Let Γ be an index set. For all $\alpha \in \Gamma$, let Y_α be a subspace of X_α. Then the following are equivalent :*

(a) *For all $\alpha \in \Gamma$, Y_α is a VN-subspace of X_α.*

(b) *For some $1 \leq p \leq \infty$, $\oplus_{\ell_p} Y_\alpha$ is a VN-subspace of $\oplus_{\ell_p} X_\alpha$.*

(c) *For all $1 \leq p \leq \infty$, $\oplus_{\ell_p} Y_\alpha$ is a VN-subspace of $\oplus_{\ell_p} X_\alpha$.*

(d) *$\oplus_{c_0} Y_\alpha$ is a VN-subspace of $\oplus_{\ell_\infty} X_\alpha$.*

(e) *$\oplus_{c_0} Y_\alpha$ is a VN-subspace of $\oplus_{c_0} X_\alpha$.*

Proof. (c) $\Rightarrow$ (b) is trivial.

(b) or (e) $\Rightarrow$ (a) Let $X = \oplus X_\alpha$ and $Y = \oplus Y_\alpha$, where the sum is any of c_0- or ℓ_p- $(1 \leq p \leq \infty)$ sum. Similar to [2, Theorem 3.1], it is immediate that if for every $\alpha \in \Gamma$, $x_\alpha \in O(Y_\alpha, X_\alpha)$, then $x \in O(Y, X)$. Hence $O(Y, X) = \{0\}$ implies $O(Y_\alpha, X_\alpha) = \{0\}$ for all $\alpha \in \Gamma$.

(a) $\Rightarrow$ (c) for $1 \leq p < \infty$. This also is similar to [2, Theorem 3.1]. We omit the details.

(d) $\Rightarrow$ (e) and (c) for $p = \infty$. This follows from Proposition 2.13.

(a) $\Rightarrow$ (d) This follows from an argument similar to [2, Theorem 3.3]. We again omit the details. ∎

Remark 3.2. Since $(\oplus_{\ell_p} X_\alpha)^{**} = \oplus_{\ell_p} X_\alpha^{**}$ for $1 < p < \infty$ and $(\oplus_{c_0} X_\alpha)^{**} = \oplus_{\ell_\infty} X_\alpha^{**}$, [2, Theorem 3.1 and 3.3] are immediate corollaries.

It also follows that for any family $\{X_\alpha\}$ of Banach spaces, $\oplus_{c_0} X_\alpha$ is a VN-subspace of $\oplus_{\ell_\infty} X_\alpha$.

We now consider $C(K, Y)$ in $C(K, X)$ where Y is a subspace of X.

Lemma 3.3. *Let K be compact Hausdorff space. Let Y be a subspace of X. Consider $y^* \otimes \delta(k) \in S(C(K,Y)^*)$ where $y^* \in S(Y^*)$ and for $k \in K$, $\delta(k)$ denotes the Dirac measure at k. Then $HB(y^* \otimes \delta(k)) = HB(y^*) \otimes \delta(k)$.*

Proof. Let $G = y^* \otimes \delta(k)$. Then $G \in S(C(K,Y)^*)$. Let $F \in \mathrm{HB}(G)$, then $\|F\| = \|G\| = 1$ and considering the total variation of F, it is not difficult to see that F is also a point mass at k. That is,

$$F = x^* \otimes \delta(k),$$

where $x^* \in \mathrm{HB}(y^*)$. ∎

Theorem 3.4. *Let K be a compact Hausdorff space. Let Y be a subspace of X. $C(K,Y)$ is a VN-subspace of $C(K,X)$ if and only if Y is a VN-subspace of X.*

Proof. Observe that if $x \in O(Y,X)$, then the constant function $x \in O(C(K,Y), C(K,X))$. Hence if $C(K,Y)$ is a VN-subspace of $C(K,X)$, then Y is a VN-subspace of X.

Conversely, let Y be a VN-subspace of X. By Theorem 2.12, it suffices to show that for all nonzero $f \in C(K,X)$, there exists $G \in S(C(K,Y)^*)$ such that every $F \in \mathrm{HB}(G)$ takes non-zero value at f.

Let $f \neq 0 \in C(K,X)$. Choose $k_0 \in K$ such that $f(k_0) \neq 0$. Since Y is a VN-subspace of X, by Theorem 2.12, there exists $y^* \in S(Y^*)$ such that every $x^* \in \mathrm{HB}(y^*)$ takes nonzero value at $f(k_0)$. Define G by

$$G = y^* \otimes \delta(k_0)$$

Let $F \in \mathrm{HB}(G)$. Then by the above lemma,

$$F = x^* \otimes \delta(k_0),$$

where $x^* \in \mathrm{HB}(y^*)$. Hence $F(f) = x^*(f(k_0)) \neq 0$. ∎

Remark 3.5. Compare this result with the result of [2] that $C(K,X)$ is nicely smooth if and only if X is nicely smooth and K is finite.

We now state a lemma whose first part is [22, Lemma 3.1] and the second part can be obtained essentially along the same line.

LEMMA 3.6. *Suppose Y is a subspace of Z.*

(a) *Consider $X \otimes Y$ as a subspace of $\mathcal{L}(X^*, Z)$. Let x^* be a w*-denting point of $B(X^*)$. Then, for any $y^* \in S(Y^*)$,*

$$HB(x^* \otimes y^*) = x^* \otimes HB(y^*).$$

(b) *Consider $\mathcal{K}(X, Y)$ as a subspace of $\mathcal{K}(X, Z)$. Let x be a denting point of $B(X)$. Then,*

$$HB(x \otimes y^*) = x \otimes HB(y^*).$$

Remark 3.7. Note that $\delta(k)$ is a w*-denting point of $B(C(K)^*)$ if and only if k is an isolated point of K. As we have seen in Lemma 3.3 above, even without any such assumption, we have $\mathrm{HB}(y^* \otimes \delta(k)) = \mathrm{HB}(y^*) \otimes \delta(k)$.

THEOREM 3.8. *Let X and Z be Banach spaces and Y is a subspace of Z. If $X \otimes_\varepsilon Y$ is a VN-subspace of $\mathcal{L}(X^*, Z)$, then Y is a VN-subspace of Z. And if w*-denting points of $B(X^*)$ separate points of X^{**}, then the converse also holds.*

In particular, if X satisfies this condition, then $C(K, X)$ is a VN-subspace of $\mathcal{L}(X^*, C(K))$, and hence, also of $WC(K, X)$.

Proof. Suppose $X \otimes_\varepsilon Y$ is a VN-subspace of $\mathcal{L}(X^*, Z)$. Then by Proposition 2.13, $X \otimes_\varepsilon Y$ is a VN-subspace of $X \otimes_\varepsilon Z$. We show that in that case, Y is a VN-subspace of Z.

Let $F \subseteq Z^*$ be a subspace such that $F|_Y$ norms Y. By definition of the injective norm, $B(X^*) \otimes B(Y^*)$ is a norming set for $X \otimes_\varepsilon Y$. It follows that $B(X^*) \otimes B(F)$ is a norming set for $X \otimes_\varepsilon Y$. If F does not separate points of Z, there is a $z \in Z$ such that $z^*(z) = 0$, for all $z^* \in F$. Take any $x \in X$, $x \neq 0$. Observe that for any $x^* \in X^*$, $x^* \otimes z^*(x \otimes z) = 0$. Hence $B(X^*) \otimes B(F)$ does not separate points of $X \otimes Z$. This contradicts the assumption that $X \otimes_\varepsilon Y$ is a VN-subspace of $X \otimes_\varepsilon Z$.

Now, suppose w*-denting points of $B(X^*)$ separates points of X^{**} and Y is a VN-subspace of Z.

As before, by Theorem 2.12, it suffices to show that for all nonzero $T \in \mathcal{L}(X^*, Z)$, there exists $\phi \in S((X \otimes Y)^*)$ such that every $\Phi \in \mathrm{HB}(\phi)$ takes nonzero value at T.

Let $T \in \mathcal{L}(X^*, Z)$, $T \neq 0$. Passing to T^*, get a w*-denting point x^* of $B(X^*)$ such that $Tx^* \neq 0$. Then, since Y is a VN-subspace of Z, there is

$y^* \in S(Y^*)$ such that if $z^* \in \mathrm{HB}(y^*)$, we have $z^*(Tx^*) \neq 0$. By Lemma 3.6, $\mathrm{HB}(x^* \otimes y^*) = x^* \otimes \mathrm{HB}(y^*)$ and therefore, $\phi = x^* \otimes y^*$ works.

Since $C(K, X) = C(K) \otimes_\varepsilon X$, it follows from the first part that $C(K, X)$ is a VN-subspace of $\mathcal{L}(X^*, C(K))$.

For the other assertion, we embed $WC(K, X)$ in $\mathcal{L}(X^*, C(K))$. For $f \in WC(K, X)$ define $T_f \in \mathcal{L}(X^*, C(K))$ by $(T_f x^*)(k) = x^*(f(k))$. Then we have $C(K, X) \subseteq WC(K, X) \subseteq \mathcal{L}(X^*, C(K))$. Hence by Proposition 2.13, we have the result. ∎

Remark 3.9. If $C(K, X)$ is a VN-subspace of $\mathcal{L}(X^*, C(K))$, then $C(K, X)$ is a VN-subspace of $\mathcal{K}(X^*, C(K)) = C(K, X^{**})$. It follows, from Theorem 3.8, that X is nicely smooth. Thus, we get a sufficient condition for nice smoothness. On the other hand, if X is Asplund (or, separable) as well as nicely smooth, then w*-denting points of $B(X^*)$ separate points of X^{**}, and therefore, $C(K, X)$ is a VN-subspace of $\mathcal{L}(X^*, C(K))$.

It is known that $C(K, X) = WC(K, X)$ for any K if and only if X has the Schur property [25]. And that when K is infinite and X fails the Schur property, $C(K, X)$ is not constrained in $WC(K, X)$ [11]. Will $C(K, X)$ be a VN-subspace of $WC(K, X)$ in such case?

Conditions under which $C(K, X)$ is an M-ideal in $WC(K, X)$ are discussed in [25]. Since $C(K, X)$ is a $(*)$-subspace of $WC(K, X)$, and M-ideals are U-subspaces, by Proposition 2.25, it follows that such $C(K, X)$ will be a VN-subspace of $WC(K, X)$. We do not know the relations between these conditions and ours.

A Banach space X with the Mazur Intersection Property (MIP) satisfies the hypothesis of Theorem 3.8. By [20, Corollary 2.8], any Banach space embeds isometrically into a Banach space with the MIP. Now, since the Schur property is hereditary, starting with any Banach space Z failing the Schur property, we can produce a Banach space X with the MIP and failing the Schur property. This will produce examples when $C(K, X)$ is a proper VN-subspace of $WC(K, X)$.

Proposition 3.10. *Suppose denting points of $B(X)$ separate points of X^*. Let Y be a VN-subspace of Z. Then $\mathcal{K}(X, Y)$ is a VN-subspace of $\mathcal{K}(X, Z)$.*

Proof. This follows from Lemma 3.6 (b) along the same line as in Theorem 3.8. ∎

Finally, we consider $\mathcal{K}(X,Y)$ in $\mathcal{L}(X,Y)$. There are discussions in the literature (see e.g. [17]) on the situation when $\mathcal{K}(X,Y)$ is an M-ideal in $\mathcal{L}(X,Y)$. Since $\mathcal{K}(X,Y)$ is a $(*)$-subspace, by Proposition 2.25 again, it follows that each such $\mathcal{K}(X,Y)$ is a VN-subspaces of $\mathcal{L}(X,Y)$. Here are some more situations when $\mathcal{K}(X,Y)$ is a VN-subspace of $\mathcal{L}(X,Y)$.

Theorem 3.11. *Suppose*

(a) *X is any Banach space and Y is such that w*-denting points of $B(Y^*)$ separate points of Y^{**}, or*

(b) *Y is any Banach space and X is such that denting points of $B(X)$ separate points of X^*,*

then $\mathcal{K}(X,Y)$ is a VN-subspace of $\mathcal{L}(X,Y)$.

Proof. Let $S = A \otimes B$, where in (a), A denotes the extreme points of $B(X^{**})$ and B denotes the set of w*-denting points of $B(Y^*)$; and in (b), A denotes the set of denting points of $B(X)$ and B denotes the set of extreme points of $B(Y^*)$. By [23, Theorem 3.7] (this also follows from Lemma 3.6), in both cases, $S \subseteq C$ as in Theorem 2.20. And by the assumptions on X and Y, in both cases, S separates points of $\mathcal{L}(X,Y)$. ∎

Remark 3.12. The assumption "(w*-)denting points separating points" in the above discussion allows us to make use the special form of Hahn-Banach extensions on tensor product spaces as in Lemma 3.6. It would be interesting to know if the results are true even without this assumption.

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