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Recent Research in Hyperspace Theory

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Hyperspace theory has its beginnings in the early years of XX century with the work of Felix Hausdorff (1868-1942) and Leopold Vietoris (1891-2002). Given a topological space X, the hyperspace 2^X of all nonempty closed subsets of X is equipped with the Vietoris topology, also called the exponential topology, see [37, p. 160] or the finite topology, see [48, p. 153], introduced in 1922 by Vietoris [60]. Vietoris proved the most basic facts of the structure of 2^X, as e.g. that compactness (similarly connectedness) of 2^X is equivalent to that of X. In case when X is a metric space, the family of all bounded nonempty closed subsets of X can be metrized by the Hausdorff metric (distance), introduced by Hausdorff in 1914, [24]. Topologies on these and other families of subsets of a topological space X were studied by E. Michael in [48]. In particular, it is shown in that paper that if X is metric and compact, then the Vietoris topology coincides with the one introduced by the Hausdorff metric, [48, Proposition 3.5, p. 160]. The reader is referred to [10, Chapter 12, p. 750] for an outline of history and for a further information in this area.

Since 1942, when J. L. Kelley doctoral dissertation [36] was published, the hyperspace theory became an important way of obtaining information on the structure of a topological space X by studying properties of the hyperspace 2^X and its hyperspaces. The general task in this part of topology can be formulated as studying various properties of the hyperspaces to get more information about the structure and properties of the space itself. Since for a given space X the structure of the hyperspace 2^X and its subspaces is rather complicate and hard to be seen, in particular any geometrical models of hyperspaces are in most cases unknown, the hyperspace theory created its own methods of study to realize the above mentioned goals. One of such special methods is investigation of Whitney maps $\omega : 2^X \to [0,\infty)$ such that $\omega(\{x\}) = 0$ and $\omega(A) < \omega(B)$ for every $A, B \in 2^X$ with $A \subset B \neq A$, and of structure of the preimages $\omega^{-1}(t)$, called Whitney levels (see [33, Chapters VII-IX]). Another one is studying properties of some special mappings between hyperspaces, as induced mappings, selections and retractions, see [33, Chapter XII]. The most successful achievements in this direction have been obtained during last sixty years in study of continua (i.e., compact, connected spaces). The results are collected in two large monographs, [50] and [33], where the above mentioned methods are presented and developed.

The aim of this article is to conceive the last achievements in the theory and to bring to the reader's attention some open problems, especially ones that tie hyperspace theory and continuum theory. To be more concise, we restrict our considerations to the most important part of the hyperspace theory, namely to that of (metric) continua.

It should be underlined that up to the last years of the XX century, the study of hyperspaces of continua concentrated on investigation of two main hyperspaces: the hyperspace 2^X of all nonempty closed subsets of a continuum X and the hyperspace $C(X)$ of connected members of 2^X, i.e., of subcontinua of X. Very recently, during last several years, a more extensive study of other hyperspaces started, and the attention of investigators mostly concerns the hyperspaces $C_n(X)$ of members of 2^X that consists of at most n components, for $n > 1$. Many results related to this subject are not published yet, and they are known to the specialists from preprints only. Some of them are discussed in this article.

The paper consists of six sections. Basic concepts used in the paper are collected in Preliminaries. In the second section some models for various hyperspaces are recalled, and relations between hyperspaces are studied. A special attention is paid to results concerning the existence of a homeomorphism between the cone over a continuum X and a hyperspace $\mathcal{H}(X)$ (as e.g. the cone = hyperspace property). The third section is devoted to hyperspace determined continua and some related concepts: having unique (or almost unique) hyperspace. In Section 4 structural properties of hyperspace are considered. We give an information on recently obtained results about smoothness and the property of Kelley for hyperspaces. The last two sections deal with mappings between hyperspaces. In Section 5 necessary and sufficient conditions are proved for a mapping to be an induced one. And finally new results concerning some special induced mappings between hyperspaces are gathered. Many open problems (previously known as well as new ones) indicate a direction of a further study in the area.

1. Preliminaries

The symbols $\mathbb{N}$ and $\mathbb{R}$ stand for the sets of all positive integers and of all reals, correspondingly. The closed unit interval $[0,1]$ of reals is denoted by $\mathbb{I}$, and $\mathbb{S}^1$ stands for the unit circle. In particular, $\mathbb{I}^n$ and $\mathbb{I}^{\aleph_0}$ denote the n-cell and the Hilbert cube, respectively.

A continuum is called a *linear graph* provided that it can be written as the union of finitely many arcs any two of which are either disjoint or intersect only in one or both of their end points. For an integer $n \geq 3$ a *simple n-od* means a continuum that is homeomorphic to the cone over a discrete n-point space.

A continuum is said to be *decomposable* provided that it is the union of two proper subcontinua. Otherwise it is said *indecomposable*. A continuum is *hereditarily decomposable* (*hereditarily indecomposable*) if each of its subcontinua is decomposable (indecomposable, respectively).

If two spaces, A and B, are homeomorphic, we write $A \approx B$.

Given a continuum X with a metric d, we let 2^X to denote the hyperspace of all nonempty closed subsets of X equipped with the Hausdorff metric H defined by

$$H(A,B) = \max\{\sup\{d(a,B) : a \in A\}, \sup\{d(b,A) : b \in B\}\} \tag{1.1}$$

(see e.g. [50, (0.1), p. 1 and (0.12), p. 10]). Given a continuum X, a *hyperspace* of X means any subspace $\mathcal{H}(X)$ of 2^X equipped with the inherited topology (thus induced by the Hausdorff metric H defined by (1.1)). Recall definitions of the most important ones, which have appeared in the literature.

For each $n \in \mathbb{N}$, let

$$F_n(X) = \{A \in 2^X : \operatorname{card} A \leq n\}$$

denote the *n-fold symmetric product of X*. Thus 1-fold symmetric product of X is the hyperspace $F_1(X)$ of singletons of X, and by the definitions we have $X \approx F_1(X)$. The concept of the symmetric product has been introduced K. Borsuk and S.M. Ulam in [5]. See [50, (0.48), p. 23] and [33, p. 6 and 7] for more information. Further, define the *hyperspace $F_\infty(X)$ of finite subsets of X* by

$$F_\infty(X) = \{A \in 2^X : A \text{ is finite}\} = \bigcup\{F_n(X) : n \in \mathbb{N}\}$$

(see [33, Definition 1.8, p. 7], where $F_\infty(X)$ is denoted by $F(X)$). Note that for a continuum X all hyperspaces $F_n(X)$ are continua (see [48, 2.4.2, p. 156, and Theorem 4.10, p. 165]; compare [5, p. 877]) and $F_\infty(X)$ is a dense (and connected) subset of 2^X (see [48, 2.4.1, p. 156]).

Similarly, for each $n \in \mathbb{N}$, define

$$C_n(X) = \{A \in 2^X : A \text{ has at most } n \text{ components}\},$$
$$C_\infty(X) = \{A \in 2^X : A \text{ has finitely many components}\} = \bigcup\{C_n(X) : n \in \mathbb{N}\}.$$

If $n = 1$, the hyperspace $C_1(X)$ of all connected elements of 2^X, i.e., of all subcontinua of X, is usually denoted by $C(X)$. It is one of the most important (and extensively studied in contemporary literature) hyperspaces of X, as it can be seen from the content of the two monographs [50] and [33]. The hyperspaces $C_n(X)$ for arbitrary $n \in \mathbb{N}$ are subjects of a more intensive investigation rather recently. It is known in particular that, for each $n \in \mathbb{N}$, the hyperspaces $C_n(X)$ are (arcwise connected) continua (see [44, Theorem 3.1, p. 240]), while $C_\infty(X)$ is neither any G_δ-subset of 2^X nor locally compact (see [44, Theorems 8.1 and 8.3, p. 253 and 254, respectively]). The reader is referred to [44] and [45] for basic information about these hyperspaces.

Consider the following hyperspaces $\mathcal{H}(X)$ of a continuum X, where $n \in \mathbb{N}$.

$$\mathcal{H}(X) \in \{2^X, F_n(X), F_\infty(X), C_n(X), C_\infty(X)\}. \tag{1.2}$$

Obvious inclusions for these hyperspaces are:

$$F_n(X) \subset F_{n+1}(X) \subset F_\infty(X) \quad \text{and} \quad C_n(X) \subset C_{n+1}(X) \subset C_\infty(X) \tag{1.3}$$
$$F_n(X) \subset C_n(X) \quad \text{and} \quad F_\infty(X) \subset C_\infty(X). \tag{1.4}$$

Therefore, inclusions (1.3) and (1.4) lead to the following proposition that extends [11, Proposition 1.2, p. 6].

Proposition 1.1. *For each continuum X the hyperspace $F_1(X)$ of singletons of X is homeomorphic (even isometric) to X, and thus it is a subcontinuum of any hyperspace $\mathcal{H}(X)$ listed in* (1.2). *Consequently,*

$$X \approx F_1(X) \subset \mathcal{H}(X) \subset 2^X.$$

Let a continuum X be given. An *order arc* in a hyperspace $\mathcal{H}(X)$ is an arc $\mathcal{A}$ in $\mathcal{H}(X)$ such that for any $A, B \in \mathcal{A}$ either $A \subset B$ or $B \subset A$. An *order arc* from A_0 to A_1 means that A_0 and A_1 are end points of the order arc and that $A_0 \subset A_1$ (hence $A_0 \subset A \subset A_1$ for all points A in the order arc; we say then that $\mathcal{A}$ *begins with* A_0), see [50, Definition 1.2, p. 57] or [33, Definition 14.1, p. 110].

Let a point p in a continuum X and a hyperspace $\mathcal{H}(X)$ be given. To make our notation shorter, we put

$$\mathcal{H}(p, X) = \{A \in \mathcal{H}(X) : p \in A\}.$$

Let $f : X \to Y$ be a mapping. For a fixed hyperspace $\mathcal{H}$ in (1.2) define the *$\mathcal{H}$-induced mapping* $\mathcal{H}(f) : \mathcal{H}(X) \to \mathcal{H}(Y)$ by

$$\mathcal{H}(f)(A) = f(A) \text{ for each } A \in \mathcal{H}(X)$$

If $\mathcal{H}(X) = 2^X$, then $\mathcal{H}(f)$ is usually denoted by 2^f. It is known that for each mapping $f : X \to Y$ the induced mapping $2^f : 2^X \to 2^Y$ is continuous (see [33, Lemma 13.3, p. 106] and compare [48, 5.10.1 of Theorem 5.10, p. 170]). Since $\mathcal{H}(X) \subset 2^X$ simply by the definition, and since $\mathcal{H}(f) = 2^f|\mathcal{H}(X)$, the continuity of 2^f implies the one of each $\mathcal{H}(f)$. Similarly,

(1.5) if f is a homeomorphism, then $\mathcal{H}(f)$ is a homeomorphism

(see [50, (0.52) and (0.53), p. 29 and 30]).

The following assertions are straightforward.

Proposition 1.2. *Let a mapping $f : X \to Y$ between continua X and Y be given, and let $\mathcal{H}$ be one of the hyperspaces listed in* (1.2). *Then $\mathcal{H}(f)(F_1(X)) \subset F_1(Y)$.*

Proposition 1.3. *Let a mapping $f : X \to Y$ between continua X and Y be given, $\mathcal{H}$ be one of the hyperspaces listed in* (1.2) *and let $A, B \in \mathcal{H}(X)$ with $A \subset B$. Then $\mathcal{H}(f)(A) \subset \mathcal{H}(f)(B)$.*

2. Some models and interrelations; hyperspaces and cones

A geometric model for a hyperspace $\mathcal{H}(X)$ for a given continuum X is, roughly speaking, a picture that shows what the hyperspace looks like. For a majority of continua X it is rather hard to imagine the (geometric) structure of a continuum that is homeomorphic to the simplest hyperspaces as 2^X and $C(X)$. Let us recall that the problem whether the Hilbert cube is a model for $2^{\mathbb{I}}$ has its long and interesting story from the Wojdysławski's question of 1938 in [62] thru the affirmative 1972 answer by R. M. Schori and J. E. West in [56], [57] and [58] to the complete 1978 solution by D. W. Curtis and R. M. Schori in [17] and [18] who proved the following important result (see also [33, Curtis-Schori Theorem 11.3, p. 89]).

THEOREM 2.1. *A nondegenerate continuum X is locally connected if and only if $2^X \approx \mathbb{I}^{\aleph_0}$. If, additionally, there is no free arc in X, then also $C(X) \approx I^{\aleph_0}$.*

Thus, in particular, the Hilbert cube is a model for $2^{\mathbb{I}}$ and for $C(\mathbb{I}^2)$. The 2-cell $\mathbb{I}^2$ is a model for $C(\mathbb{I})$ and $C(\mathbb{S}^1)$, see [33, Examples 5.1 and 5.2, p. 33 and 35, respectively].

Indeed, if $X = \mathbb{I}$, then points of $C(\mathbb{I})$ are closed intervals $[a, b]$ with $0 \leq a \leq b \leq 1$. Defining a function h from $C(\mathbb{I})$ into $\mathbb{R}^2$ by $h([a, b]) = (a, b)$ we see that the image $h(C(\mathbb{I}))$ is the triangular 2-cell T in the plane $\mathbb{R}^2$ with vertices $(0, 0)$, $(0, 1)$ and $(1, 1)$. It is easy to observe that h is one-to-one and $h(C(\mathbb{I})) = T$. Since a sequence of points $\{[a_i, b_i] : i \in \mathbb{N}\}$ in $C(\mathbb{I})$ converges to $[a, b]$ if and only if the sequence of points $\{(a_i, b_i) \in \mathbb{R}^2 : i \in \mathbb{N}\}$ converges (in the plane $\mathbb{R}^2$) to (a, b), it follows that $h : C(\mathbb{I}) \to T$ is a homeomorphism.

Similarly, if $X = \mathbb{S}^1 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$, then points of $C(\mathbb{S}^1)$ are arcs $A \subset \mathbb{S}^1$, singletons $\{x\}$ with $x \in \mathbb{S}^1$, and $\mathbb{S}^1$. For any arc $A \subset \mathbb{S}^1$ let $l(A)$ denote the length of A, and $m(A)$ be the middle point of A. Define a homeomorphism h from $C(\mathbb{S}^1)$ onto the unit disc $D = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\}$ as follows. For an arc $A \in C(\mathbb{S}^1)$, let $h(A)$ be the point that lies on the straight line segment from $(0, 0)$ to $m(A)$ and that is of distance $\frac{l(A)}{2\pi}$ from $m(A)$. Now, there is only one way to define h on the rest of $C(\mathbb{S}^1)$ so that the resulting function is continuous: we put $h(\{x\}) = x$ for each $x \in \mathbb{S}^1$ and $h(\mathbb{S}^1) = (0, 0)$. It follows easily that $h : C(\mathbb{S}^1) \to D$ is a homeomorphism.

Only for very simple continua X as some specific linear graphs, locally connected fans (i.e., n-ods and the infinite locally connected fan F_ω = the hairy point) and a few other ones the geometric models for $C(X)$ and 2^X are known (see e.g. [33, Chapters II and III, p. 31-96]).

Recently it was shown by R. Schori that $C_2(\mathbb{I}) \approx \mathbb{I}^4$. In the proof (given in [30, Lemma 2.2, p. 349]) one considers two auxiliary subsets of $C_2(\mathbb{I})$, namely $D(1) = \{A \in C_2(\mathbb{I}) : 1 \in A\}$ and $D(0, 1) = \{A \in C_2(\mathbb{I}) : 0, 1 \in A\}$. By constructing the corresponding homeomorphisms it is shown that $D(0, 1) \approx$

$\mathbb{I}^2$, $D(1) \approx \text{Cone}(D(0,1))$ and $C_2(\mathbb{I}) \approx \text{Cone}(D(1))$, whence the conclusion follows.

Further, $C_2(\mathbb{I})$ and $C_2(\mathbb{S}^1)$ are not homeomorphic (see [30, Lemma 2.3, p. 349]). In fact, since $C_2(\mathbb{I}) \approx \mathbb{I}^4$, it follows that $C_2(\mathbb{I}) \setminus \{A\}$ is unicoherent for each $A \in C_2(\mathbb{I})$. By defining a suitable essential mapping from $C_2(\mathbb{S}^1) \setminus \{\mathbb{S}^1\}$ onto $\mathbb{S}^1$ we conclude that $C_2(\mathbb{S}^1) \setminus \{\mathbb{S}^1\}$ is not unicoherent, which completes the proof.

It is not known if $C_n(\mathbb{I}) \approx C_n(\mathbb{S}^1)$ for $n \geq 3$ (see [30, Question 5.1, p. 362]; see also [45, Question, p. 273]). Also, models for $C_3(\mathbb{I})$ and $C_2(\mathbb{S}^1)$ are unknown (see [30, Problem 5.5, p. 362]). Thus the following set of problems is natural.

Problem 2.2. Given a continuum X, describe geometric models for hyperspaces $\mathcal{H}(X)$ listed in (1.2). Specifically, give such models if X is a simple linear graph as an arc, a circle, a noose, an n-od, etc.

As it can easily be observed, since $C(\mathbb{I}) \approx \mathbb{I}^2 \approx C(\mathbb{S}^1)$, the hyperspaces $C(X)$ for $X \in \{\mathbb{I}, \mathbb{S}^1\}$ are homeomorphic to the (geometric) cones over X. There are also other continua X such that the hyperspace $C(X)$ is homeomorphic to $\text{Cone}(X)$, the *cone over* X, defined as the quotient space obtained from the product $X \times [0,1]$ by shrinking $X \times \{1\}$ to a point. Continua for which there is the above mentioned homeomorphism are called *C-H continua*. A continuum X is said to have the *cone = hyperspace property* if there exists a homeomorphism $h : \text{Cone}(X) \to C(X)$ such that h maps the vertex of the cone to the point X in $C(X)$ and maps the base of the cone onto the set of singletons of X in $C(X)$. The simplest continua that enjoy this property are an arc and a circle, as well as any solenoid and the Brouwer-Janiszewski-Knaster continuum (called also the buckedhandle continuum), see [53, Theorem 2, p. 167] and compare [50, p. 303]. For an extension of this result to all *Knaster continua* (i.e. the inverse limits of inverse sequences of arcs with open bonding mappings) see [19, Corollary 12, p. 639].

The following question (see [50, Questions 8.35, p. 332]) is related to the above recalled concepts.

Question 2.3. For what continua X does there exist a continuum Z such that $C(X) \approx \text{Cone}(Z)$?

A characterization of continua having the cone = hyperspace property in terms of some selections is given in [28] (see also [33, Theorem 80.5, p. 428] and compare [59, Theorem 3.1, p. 1032]).

A very complete discussion about the cone = hyperspace property is contained in [50, Chapter VIII] and [33, Sections 40 and 80]. In particular, the following results are known.

THEOREM 2.4. *Let X be a continuum.*

(2.4.1) *If* $\dim X$ *is finite and X has the cone = hyperspace property, then either $X \approx \mathbb{I}$, or $X \approx \mathbb{S}^1$, or X is indecomposable and each one of its proper nondegenerate subcontinua is an arc (see* [54, Theorem 1, p. 279]*).*

(2.4.2) *If X is hereditarily decomposable and C-H, then X is homeomorphic to one of the eight continua listed in* [49, Theorem 1.1, p. 322]*; (these continua are pictured in* [33, Fig. 20, p. 63]*). Furthermore, each of these eight continua is a C-H continuum (see also* [50, Theorem 8.3, p. 322]*).*

(2.4.3) *If* $\dim X$ *is finite and X is a C-H continuum but not hereditarily decomposable, then* $\dim X = 1$*, X is atriodic, it contains a unique nondegenerate indecomposable subcontinuum Y (see* [55, Theorem 8, p. 286]*). Furthermore, Y has the cone = hyperspace property and $X \setminus Y$ is arcwise connected (see* [25, Theorem, p. 286]*).*

(2.4.4) *If X is locally connected, then $C(X) \approx \mathrm{Cone}(Z)$ for some finite dimensional continuum Z if and only if X is an arc, a circle or a simple n-od (see* [40, Theorem 4, p. 3071] *and compare* [33, 80.13 and 80.14, p. 430-431]*).*

A further progress in the area is obtained in [38] (see also [39] and [32]. The results are summarized in the following theorem ([38, Theorem, p. ix]; compare also [32, Theorem 0, p. 378]). Since [38] is rather hard to access, and [32] does not contain any argument for this theorem, an outline of the proof of the result is enclosed.

THEOREM 2.5. *Let X be a continuum for which there are a finite dimensional continuum Z and a homeomorphism $h : C(X) \to \mathrm{Cone}(Z)$, and let Y be a subcontinuum of X such that $h(Y)$ is the vertex $v(Z)$ of* $\mathrm{Cone}(Z)$*. Then the following assertions are satisfied.*

(2.5.1) $\dim X = 1$.

(2.5.2) *Each subcontinuum of X that does not contain Y is an arc or a point.*

(2.5.3) *If X is hereditarily decomposable, then Y is a point, an arc or a circle.*

(2.5.4) *If X is not hereditarily decomposable, then Y is the only nondegenerate indecomposable subcontinuum of X.*

(2.5.5) *If Y is nondegenerate, it has the cone = hyperspace property.*

(2.5.6) *If X is indecomposable, then $X \approx Z$. Consequently, X has the cone = hyperspace property.*

(2.5.7) *$X \setminus Y$ is locally connected.*

(2.5.8) *Each component of $X \setminus Y$ is homeomorphic to either the real line $\mathbb{R} = (-\infty, \infty)$ or the real half-line $[0, \infty)$, and it coincides with an arc component of $X \setminus Y$.*

(2.5.9) *$X \setminus Y$ has a finite number of arc components.*

Proof. (Outline of proof) Since $\dim Z < \infty$, it follows that $\dim \mathrm{Cone}(Z) = \dim Z + 1$, [50, Lemma 8.0, p. 301], whence $\dim C(X) < \infty$. Since $\dim X \geq 2$ implies $\dim C(X) = \infty$, [33, Theorems 72.5 and 73.9, p. 348 and 354, respectively], (2.5.1) follows.

If X contains an m-od, then $C(X)$ contains an m-cell, [50, Theorem 1.100, p. 140], whence $\dim C(X) \geq m$. Since $\dim C(X)$ is finite, there exists an $m \in \mathbb{N}$ such that X does not contain m-ods. Since it can be shown that each subcontinuum A of X that does not contain Y is locally connected, A is a linear graph. (2.5.2) is attained by showing that A does not contain any ramification points.

Assume that X is hereditarily decomposable. Thus Y is decomposable, and by (2.5.2) each nondegenerate proper subcontinuum of Y is an arc, whence (2.5.3) follows.

To show (2.5.4) let $Y' \neq Y$ be a nondegenerate indecomposable subcontinuum of X. Thus $h(Y') \neq v(Z)$. It follows that $C(X) \setminus \{Y'\}$ has infinitely many arc components. On the other hand, if $p \in \mathrm{Cone}(Z) \setminus \{v(Z)\}$, then $\mathrm{Cone}(Z) \setminus \{p\}$ has at most two arc components. Hence $Y' = Y$, and (2.5.4) is shown.

To prove (2.5.5) consider two cases. If Y is decomposable, the result is a consequence of (2.5.3). If Y is indecomposable, we first prove that $h(F_1(Y))$ is a subset of the base of $\mathrm{Cone}(Z)$. Then it follows that the cone over $h(F_1(Y))$

is a subset of Cone(Z) such that the two cones have the same vertex $v(Z) = h(Y)$. Finally, the equality Cone$(h(F_1(Y))) = h(C(Y))$ is shown to verify that h has all the needed properties.

If X is indecomposable, then $Y = X$ by (2.5.4), so (2.5.6) follows from [50, Theorem 8.7, p. 308].

Finally, to show properties (2.5.7), (2.5.8) and (2.5.9) we utilize the concept and various attributes of the *semi-boundary* of $C(Y)$ in $C(X)$ defined as

$$\{A \in C(Y) : \text{there exists an order arc } \mathcal{A} \text{ in } C(X) \text{ such that } \cap\mathcal{A} = A \quad \text{and} \quad B \not\subseteq Y \text{ for every } B \in \mathcal{A} \setminus \{A\}\}$$

(see [33, Section 69, p. 333 ff.]). In particular, (2.5.7) implies that components and arc components of $X \setminus Y$ coincide. ∎

A characterization of hereditarily decomposable continua X whose hyperspace $C(X)$ is homeomorphic to Cone(Z) for some finite dimensional continuum Z is given in [32].

Interesting results on mutual relations between hyperspaces in (1.2) have been proved in [42]. Namely we have the following.

Theorem 2.6. *Let a continuum X be finite-dimensional. Then*

(2.6.1) $C(X) \approx F_2(X) \implies X \approx \mathbb{I}$ *(see* [42, Theorem 9, p. 178]*);*

(2.6.2) *if $n \in \mathbb{N}$ and $n \geq 3$, then the hyperspaces $C(X)$ and $F_n(X)$ are not homeomorphic (see* [42, Theorem 12, p. 180]*).*

The following problems are related to Theorem 2.6.

Problem 2.7. Let a continuum X be given.

(2.7.1) Which ones of the hyperspaces $\mathcal{H}(X)$ in (1.2) are homeomorphic? Note that 2^X, $F_n(X)$ and $C_n(X)$ are compact, while $F_\infty(X)$ and $C_\infty(X)$ are not.

(2.7.2) Are there some characterizations of continua X distinct from $\mathbb{I}$ in terms of the existence of a homeomorphism between some hyperspaces $\mathcal{H}_1(X)$ and $\mathcal{H}_2(X)$ (i.e., an analog of (2.2) for $X \neq \mathbb{I}$)?

In the light of the above mentioned results one can ask if the concepts of C-H continua X and continua X having the cone = hyperspace property can be extended from the hyperspace $C(X)$ to other hyperspaces $\mathcal{H}(X)$. Obviously an answer is negative for $C_\infty(X)$ and $F_\infty(X)$ (since these spaces are not compact). For the hyperspaces C_n and $F_n(X)$ with $n > 1$ the following results are known.

THEOREM 2.8. *Let a continuum X be finite-dimensional. If $n > 1$ then:*

(2.8.1) *$C_n(X)$ is not homeomorphic to* $\mathrm{Cone}(X)$ *(see* [46, Theorem 3.2, p. 257]*);*

(2.8.2) *if X is the cone over a totally disconnected set, then the hyperspaces $C_n(X)$, $F_n(X)$ and 2^X are cones over some continua (see* [46]*).*

3. Hyperspace determined continua

The following statement is an immediate consequence of implication (1.5).

STATEMENT 3.1. If X and Y are continua, then

$$X \approx Y \implies 2^X \approx 2^Y \text{ and } C(X) \approx C(Y). \tag{3.1.1}$$

Indeed, if $h : X \to Y$ is a homeomorphism, then the induced mapping $2^h : 2^X \to 2^Y$ is a homeomorphism, and thus $2^h|C(X) = C(h) : C(X) \to C(Y)$ also is a homeomorphism (see [50, (0.52) and (0.53), p. 29 and 30]).

The converse to (3.1.1) is not true in general because of the following examples (already mentioned in the previous section; see [50, (0.58), p. 32] and compare [33, Chapters II and III, p. 31-96]).

EXAMPLE 3.2. $2^{\mathbb{I}} \approx \mathbb{I}^{\aleph_0} \approx 2^{\mathbb{S}^1}$ and $C(\mathbb{I}) \approx \mathbb{I}^2 \approx C(\mathbb{S}^1)$. Moreover, if X is a nondegenerate locally connected continuum, then $2^X \approx \mathbb{I}^{\aleph_0}$, and if additionally there is no free arc in X, then also $C(X) \approx \mathbb{I}^{\aleph_0}$.

However, the implication

$$C(X) \approx C(Y) \implies X \approx Y \tag{3.1}$$

holds if continua X and Y satisfy some additional conditions. The above discussed results led to creating the following concept (see [50, Definition (0.61), p. 33]). The members of a class Λ of continua are said to be C-*determined* provided that implication (3.3) is true for any two members X and Y of Λ. The known results about C-determined continua can be gathered as follows.

THEOREM 3.3. (a) *The members of the following classes of continua are known to be C-determined:*

(3.3.1) *linear graphs, at least one of which is different from an arc and a circle (see* [20, 9.1, p. 283]) and [21]*);*

(3.3.2) *hereditarily indecomposable continua (see* [50, (0.60), p. 33] and compare also [22, Section 4, paragraph 3 on p. 1032]*);*

(3.3.3) *smooth fans (see* [23, Corollary 3.3, p. 285]*);*

(3.3.4) *indecomposable continua such that all of their nondegenerate proper subcontinua are arcs (see* [41, Theorem 3, p. 261]*);*

(3.3.5) *compactifications of the ray* $[0, \infty)$ *with a nondegenerate remainder (see* [1, Corollary 5, p. 44]*);*

(3.3.6) *continua of the form* $S_1 \cup R \cup S_2$*, where* $S_1 \cup R$ *and* $S_2 \cup R$ *are compactifications of the disjoint rays* S_1 *and* S_2 *with the common remainder* R *(see* [1, Theorem 10, p. 48]*);*

(3.3.7) *compactifications of the real line* $\mathbb{R} = (-\infty, \infty)$ *different from an arc (see* [3, Theorem 4.5, p. 24]*);*

(3.3.8) *circle-like continua being compactifications of the real line* $\mathbb{R} = (-\infty, \infty)$ *with a connected remainder (see* [2, Theorem 3.6, p. 184]*);*

(3.3.9) *continua of the form* $S \cup ab \cup bc$*, where* ab *and* bc *are arcs,* $S \cup ab$ *is a compactification of a ray* S *with the arc* ab *as the remainder, and* $bc \cap (S \cup ab) = \{b\}$ *(see* [2, Theorem 4.6, p. 187]*);*

(3.3.10) *continua of the form* $(S \cup ab)/\{a, e\}$*, where* $S \cup ab$ *is a compactification of a ray* S *with the arc* ab *as the remainder, and* e *is the end point of* S *(see* [2, Theorem 4.10, p. 187]*);*

(3.3.11) *arcwise connected circle-like continua (see* [2, Theorem 4.12, p. 188]*);*

(3.3.12) *circle-like continua which are either arcwise connected or compactifications of the real line* $\mathbb{R} = (-\infty, \infty)$ *with a connected remainder (see* [2, Theorem 4.14, p. 188]*).*

(b) *The members of the following classes of continua are known not to be C-determined:*

(3.3.13) *chainable continua (see* [26]*);*

(3.3.14) *fans (see* [27]*).*

In connection with the concept of C-determined continua one can introduce the following. Let $\mathcal{H}(X)$ denote one of the hyperspaces of a continuum X listed in (1.2). We say that the members of a class Λ of continua are said to be $\mathcal{H}$-*determined* provided that implication

$$\mathcal{H}(X) \approx \mathcal{H}(Y) \implies X \approx Y \tag{3.2}$$

is true for any two members X and Y of Λ.

If $\mathcal{H}$ denotes the hyperspace C of subcontinua, then the known results about the subject are collected in Theorem 3.3. Only a few results are known if $\mathcal{H}$ is not C. Extending Nadler's result (3.3.2) of Theorem 3.3, S. Macías has shown that implication (3.2) holds for hereditarily indecomposable continua if $\mathcal{H}(X)$ means either the hyperspace 2^X (see [43, Corollary, p. 417]) or $C_n(X)$, for any $n \in \mathbb{N}$ (see [45, Theorem 6.1, p. 273]). In connection with these results the following questions seem to be natural.

Question 3.4. For what hyperspaces $\mathcal{H} \in \{C_\infty, F_n, F_\infty\}$, where $n \in \mathbb{N}$ and $n > 1$, hereditarily indecomposable continua are $\mathcal{H}$-determined?

Question 3.5. Are members of the classes in (3.3.3)-(3.3.14) $\mathcal{H}$-determined if $\mathcal{H}$ is as in (1.2)?

Let $\mathcal{S}$ be a family of topological spaces and $\mathbb{M}$ be a class of mappings between members of $\mathcal{S}$. Then $\mathcal{S}$ can be *quasi-ordered with respect to* $\mathbb{M}$ writing for any $X, Y \in \mathcal{S}$

$$(Y \leq_{\mathbb{M}} X) \iff (\text{ there exists a surjection } f \in \mathbb{M} \text{ of } X \text{ onto } Y\ ),$$
$$(X =_{\mathbb{M}} Y) \iff (Y \leq_{\mathbb{M}} X \text{ and } X \leq_{\mathbb{M}} Y).$$

Replacing in the definition of $\mathcal{H}$-determined continua the relation $\approx$ by $\approx_{\mathbb{M}}$ for a given class $\mathbb{M}$ of mappings between continua one gets the following concept. Let $\mathcal{H}(X)$ denote one of the hyperspaces of a continuum X listed in (1.2), and let $\mathbb{M}$ be a class of mappings between continua. The members of a class Λ of continua are said to be $(\mathcal{H}, \mathbb{M})$-*determined* provided that implication

$$\mathcal{H}(X) \approx_{\mathbb{M}} \mathcal{H}(Y) \implies X \approx_{\mathbb{M}} Y$$

is true for any two members X and Y of Λ.

The above defined notion of a class Λ of $(\mathcal{H}, \mathbb{M})$-determined continua can be a subject of a further study.

The concept of C-determined continua has been modified by S. Macías in [43, p. 416] and by G. Acosta in [1] in the following way (see also [4, p. 745]).

For a given continuum X, consider a family $\mathcal{F}(X)$ of continua Y such that:

(a) no two distinct members of $\mathcal{F}(X)$ are homeomorphic;

(b) $C(Y)$ is homeomorphic to $C(X)$ for each member Y of $\mathcal{F}(X)$;

(c) $\mathcal{F}(X)$ is the maximal family satisfying conditions (a) and (b), i.e., if Z is a continuum such that $C(Z) \approx C(X)$, then $Z \approx Y$ for some $Y \in \mathcal{F}(X)$.

A continuum X is said to have *unique hyperspace* $C(X)$ provided that the family $\mathcal{F}(X)$ consists of one element only, viz. of X, [1, Definition 1, p. 34]; *almost unique hyperspace* provided that the family $\mathcal{F}(X)$ is finite and consists of more than one element (see [2, Definition 1.1, p. 176]). The known results about continua X with the unique hyperspace $C(X)$ are collected below. Some special irreducible continua of type λ that have almost unique hyperspace are studied in [4].

THEOREM 3.6. (a) *The following continua X have unique hyperspace $C(X)$:*

(3.6.1) *linear graphs different from an arc and a circle (see* [1, Theorem 1, p. 38]*);*

(3.6.2) *hereditarily indecomposable continua (see* [1, Theorem 2, p. 38]*);*

(3.6.3) *compactifications of a ray with a nondegenerate remainder (see* [1, Theorem 4, p. 42]*);*

(3.6.4) *indecomposable continua such that all of their nondegenerate proper subcontinua are arcs (see* [2, Theorem 2.3, p. 177]*).*

(b) *The following continua X do not have unique hyperspace $C(X)$:*

(3.6.5) *continua of the form $S_1 \cup R \cup S_2$, where $S_1 \cup R$ and $S_2 \cup R$ are compactifications of the disjoint rays S_1 and S_2 with the common remainder R (see* [2, Theorem 3.3, p. 183]*);*

(3.6.6) *compactifications of the real line* $\mathbb{R} = (-\infty, \infty)$ *with a connected remainder (see* [2, Theorem 3.4, p. 184]*);*

(3.6.7) *continua of the form* $S \cup ab \cup bc$*, where* ab *and* bc *are arcs,* $S \cup ab$ *is a compactification of a ray* S *with the arc* ab *as the remainder, and* $bc \cap (S \cup ab) = \{b\}$ *(see* [2, Theorems 4.3 and 4.4, p. 185 and 186, respectively]*);*

(3.6.8) *continua of the form* $(S \cup ab)/\{a, e\}$*, where* $S \cup ab$ *is a compactification of a ray* S *with the arc* ab *as the remainder, and* e *is the end point of* S *(see* [2, Theorem 4.7, p. 187]*);*

(3.6.9) *arcwise connected circle-like continua (see* [2, Theorem 4.11, p. 187]*).*

To illustrate the methods used in the proofs, we present outlines of arguments for (3.6.2) and (3.6.5).

Proof. Proof of (3.6.2) Let continua X and Y be given such that X is hereditarily indecomposable, and that $C(X) \approx C(Y)$. Since a continuum Z is hereditarily indecomposable if and only if the hyperspace $C(Z)$ is uniquely arcwise connected (see [50, Theorem 1.61, p. 111]), we infer that $C(X)$ is uniquely arcwise connected, whence it follows that $C(Y)$ is and, consequently, Y is hereditarily indecomposable. Since hereditarily indecomposable continua are C-determined (see (3.3.2) above; in fact, if $h : C(X) \to C(Y)$ is a homeomorphism, then $h(F_1(X)) = F_1(Y)$; since $F_1(X) \approx X$ and $F_1(Y) \approx Y$, we conclude $X \approx Y$), the result follows. ∎

In [23, Example 4.5, p. 286] two continua X and Y are constructed such that X is decomposable, Y is indecomposable, and $C(X) \approx C(Y)$. Therefore, there are indecomposable continua which do not have unique hyperspace, so hereditary indecomposability is essential in the result.

Proof. (Outline of proof of (3.6.5)) Let a continuum X be such that $X = S_1 \cup R \cup S_2$, where $S_1 \cup R$ and $S_2 \cup R$ are compactifications of the disjoint rays S_1 and S_2 with the common remainder R. Let a and b denote the end points of the rays S_1 and S_2, respectively, and let M be an arc with its end points a and b such that $M \cap X = \{a, b\}$. Put $Y = M \cup X$.

The following general result is shown in [2, Theorem 3.1, p. 179]. Let X be a continuum such that

(1) X is irreducible between points a and b;

(2) $C(a, X)$ and $C(b, X)$ are arcs in the hyperspace $C(X)$;

(3) if $X' = A \cup X \cup B$ is a continuum obtained from X by attaching two disjoint arcs A and B ending at the points a and b, respectively, so that $A \cap X = \{a\}$ and $B \cap X = \{b\}$, then $X' \approx X$.

If M is an arc such that $M \cap X = \{a, b\}$ and $Y = X \cup M$, then $C(Y) \approx C(X)$.

Observe that the continuum $X = S_1 \cup R \cup S_2$ satisfies conditions (1)-(3). Thereby we get $C(Y) \approx C(X)$, so (3.6.5) follows. ∎

Extending the above concept, one can say that a continuum X has *(almost) unique hyperspace* $\mathcal{H}(X)$ provided that in the conditions (b) and (c) of the definition of the family $\mathcal{F}(X)$ the hyperspace $C(X)$ is replaced by $\mathcal{H}(X)$. There are only a few results related to this concept, and they concern (similarly to preliminary results about C-determined continua, compare [50, p. 33]) the two "diametrically opposite" classes of continua, namely the simplest continua, linear graphs, and the most complicated continua, viz. hereditarily indecomposable ones. The results are reported below. To illustrate the methods used in the proof, we present outline of argument for (3.7.4).

Theorem 3.7. (a) *Let X be a linear graph. Then*

(3.7.1) *X has unique hyperspaces $F_2(X)$ and $F_3(X)$ (see* [6]*);*

(3.7.2) *if X is different from an arc and from a simple closed curve, then it has unique hyperspace $C(X)$ (see* [20, 9.1, p. 283]) and [21]*);*

(3.7.3) *X has unique hyperspace $C_n(X)$ for each integer $n \geq 2$* (see [30, Theorem 4.1, p. 356] for $n = 2$ and [31] for $n > 2$; the methods of proofs are distinct for the two cases).

(b) *Let a continuum X be hereditarily indecomposable. Then*

(3.7.4) *X has unique hyperspace 2^X (see* [43, Theorem, p. 416]*);*

(3.7.5) *X has unique hyperspace $C_n(X)$ for each $n \in \mathbb{N}$ (see* [45, Theorem 6.1, p. 273]*).*

Proof. (Outline of proof of (3.7.4)) Let X and Y be continua such that X is hereditarily indecomposable and there is a homeomorphism $h : 2^X \to 2^Y$. Then by [50, Theorem 1.136, p. 154] 2^X and 2^Y are locally connected at X and Y, respectively. Since X is indecomposable, X is the only point at which

2^X is locally connected, see [50, Theorem 1.139, p. 155], whence Y is the only point of local connectedness of 2^Y. Then $h(X) = Y$.

Take $A \in C(X)\setminus F_1(X)$ and observe that $2^X\setminus\{A\}$ is not arcwise connected, [50, Theorem 11.15, p. 368], whence $2^Y \setminus \{h(A)\}$ is not arcwise connected, and thereby $h(A) \in C(Y)$, [50, Theorem 11.3, p. 358]. Since $C(Y)$ is closed in 2^Y, [50, Theorem 0.8, p. 7], and $F_1(X) \subset \mathrm{cl}\,(C(X) \setminus F_1(X))$, we conclude that $h(F_1(X)) \subset C(Y)$.

Suppose that there is $\{x\} \in F_1(X)$ such that $h(\{x\}) \in C(Y) \setminus F_1(Y)$. By [50, Corollary 12.4, p. 376] the point $h(\{x\})$ is arcwise accessible from $2^Y \setminus C(Y)$. Since $h(C(X)) \subset C(Y)$, it follows that $\{x\}$ is arcwise accessible from $2^X\setminus C(X)$, but this contradicts the fact that X is hereditarily indecomposable, [50, Corollary 12.9, p. 378]. Therefore $h(F_1(X)) \subset F_1(Y)$.

Let $Y' \in C(Y)$ be such that $F_1(Y') = h(F_1(X))$. Then $C(Y') \subset C(Y)$. Since $h^{-1}(C(Y'))$ is an arcwise connected subcontinuum of 2^X and X is hereditarily indecomposable, it follows from [50, Theorem 12.29, p. 390] that $C(X)\cap h^{-1}(C(Y'))$ is arcwise connected. Note that $F_1(X) \subset C(X) \cap h^{-1}(C(Y'))$.

Considering the cases $h^{-1}(Y') \in C(X)$ and $h^{-1}(Y') \in 2^X\setminus C(X)$ separately and using [50, Theorems 1.50, 12.9 and 12.30, p. 102, 378 and 391, respectively], one can show that $Y' = Y$, whence it follows that $F_1(Y) = h(F_1(X))$. Since $F_1(X)$ and $F_1(Y)$ are homeomorphic to X and Y, correspondingly, we have $X \approx Y$, as needed. ∎

It is immediate to ask about similar results for other continua, not necessarily assuming that they are linear graphs or hereditarily indecomposable. The following result is related to this (see [45, Theorem 5.7, p. 272]).

THEOREM 3.8. *Let $n > 1$ be an integer. If a continuum X is such that $C_n(X)$ is homeomorphic to either $C_n(\mathbb{I})$ or $C_n(\mathbb{S}^1)$, then X homeomorphic to either $\mathbb{I}$ or $\mathbb{S}^1$.*

We close this section with a result about dendrites. Recall that a *dendrite* means a locally connected continuum containing no simple closed curve. A point of order 2 in a dendrite X (i.e., that is neither an end point of X nor a ramification point of X) is called an *ordinary point* of X.

The following result is shown in [29, Theorems 1 and 8, p. 77 and 90, respectively].

THEOREM 3.9. *Let X be a dendrite and Y be a continuum such that $F_2(X) \approx F_2(Y)$. Then Y is a dendrite. Further, if the sets of ordinary points of X and of Y are open, then $X \approx Y$.*

The following questions are related to Theorem 3.9.

QUESTION 3.10. (a) Let X be a dendrite and Y be a continuum such that $F_n(X) \approx F_n(Y)$ for some integer $n \geq 3$. Is then Y a dendrite?

(b) Let X and Y be dendrites such that $F_n(X) \approx F_n(Y)$ for some integer $n \geq 3$, and that the sets of ordinary points of X and of Y are open. Does it follow that $X \approx Y$?

(c) Is the assumption of openness of the sets of ordinary points in X and Y essential in Theorem 3.9?

4. Structural properties

A continuum X is said to have the *property of Kelley* provided that for each point $x \in X$, for each subcontinuum K of X containing x and for each sequence of points x_n converging to x there exists a sequence of subcontinua K_n of X containing x_n and converging to the continuum K (see e.g. [50, Definition 16.10, p. 538]).

The property, introduced by J. L. Kelley as property 3.2 in [36, p. 26], has been used there to study hyperspaces, in particular their contractibility (see e.g. Chapter 16 of [50], where references for further results in this area are given). Now the property, which has been recognized as an important tool in investigation of various properties of continua, is interesting by its own right, and has numerous applications to continuum theory. Many of them are not related to hyperspaces. A pointed version of this property has been introduced by Wardle in [61, p. 291], where it is shown that homogeneous continua have the property of Kelley. This result has been extended to openly homogeneous continua in [8, Statement, p. 380], and Kato has proved in [34] that it cannot be enlarged to continua that are homogeneous with respect to confluent mappings (introduced in [7]).

In [35] Kato defined a stronger version of the property of Kelley and showed that if a continuum X has this stronger property, then the hyperspace $C(X)$ of all nonempty subcontinua of X, as well as all Whitney levels in $C(X)$ have the property of Kelley.

A continuum X is said to be *smooth at a point* $p \in X$ provided that for each $\varepsilon > 0$ there is $\delta > 0$ such that if $a, b \in X$, $d(a, b) < \delta$ and $A \in C(a, X) \cap C(p, X)$, then there exists $B \in C(b, X) \cap C(p, X)$ such that $H(A, B) < \varepsilon$. A continuum X is said to be *smooth* if it is smooth at some point.

The following problem, being a research program concerning hyperspaces of continua rather than any particular question, is discussed in [13].

Problem 4.1. Let $\mathcal{H}(X)$ denote a hyperspace of a continuum X. Find necessary and/or sufficient conditions under which implications are true between any two of the following four assertions:

(4.1.1) X is smooth;

(4.1.2) X has the property of Kelley;

(4.1.3) $\mathcal{H}(X)$ is smooth;

(4.1.4) $\mathcal{H}(X)$ has the property of Kelley.

The next result, [13, Theorems 2 and 9] shows that smoothness or the property of Kelley of some hyperspaces of continua implies the property of Kelley for the continua.

Theorem 4.2. *Let a hyperspace $\mathcal{H}(X)$ of a continuum X satisfy the following conditions:*

(4.2.a) $F_1(X) \subset \mathcal{H}(X)$*;*

(4.2.b) *if an order arc $\mathcal{A}$ in 2^X begins with $A_0 \in \mathcal{H}(X)$, then $\mathcal{A} \subset \mathcal{H}(X)$.*

Then the following implications hold.

(4.2.1) *If the hyperspace $\mathcal{H}(X)$ is smooth, then the continuum X has the property of Kelley.*

(4.2.2) *If $\mathcal{H}(X)$ has the property of Kelley, then X has the property of Kelley, too.*

As a consequence we get the following corollaries, [13, Corollaries 3-5 and 10], the first of which generalizes [16, Theorem 1, p. 88].

Corollary 4.3. *Let X be a continuum.*

(4.3.1) *Smoothness of either 2^X or $C_n(X)$ for some $n \in \mathbb{N}$ implies that X has the property of Kelley.*

(4.3.2) *If there exists a smooth hyperspace $\mathcal{H}(X)$ satisfying conditions (4.2.a) and (4.2.b) of Theorem 4.2, then for each $n \in \mathbb{N}$ the hyperspace $C_n(X)$ is contractible.*

(4.3.3) *If some of the hyperspaces* 2^X *or* $C_n(X)$ *is smooth, then for each* $n \in \mathbb{N}$ *the hyperspace* $C_n(X)$ *is contractible.*

(4.3.4) *If either* 2^X *or* $C_n(X)$ *for some* $n \in \mathbb{N}$ *has the property of Kelley, then* X *also has the property of Kelley.*

The next questions are asked also in the above quoted paper [13].

Question 4.4. Let X be a continuum.

(4.4.1) For what (not locally connected) continua X does the property of Kelley for X imply that a) 2^X, b) $C_n(X)$ for some $n \in \mathbb{N}$ is smooth?

(4.4.2) Let a hyperspace $\mathcal{H}(X)$ be given. Consider the following two conditions:
(a) $\mathcal{H}(X)$ has the property of Kelley; (b) X has the property of Kelley. What are necessary and/or sufficient conditions under which: (a) implies (b)? (b) implies (a)?

Note that (4.3.4) gives a partial answer to (4.4.2).
The next results are shown as Theorems 11, 18 and 20-23 of [13].

Theorem 4.5. *Let* X *be a continuum.*

(4.5.1) *If the hyperspace* $C(X)$ *is smooth, then each Whitney level for* $C(X)$ *has the property of Kelley.*

(4.5.2) *If the hyperspace* $F_n(X)$ *is smooth for some integer* $n > 1$*, then* X *has the property of Kelley.*

(4.5.3) *If* Cone(X) *is smooth, then* X *has the property of Kelley.*

(4.5.4) Cone(X) *has the property of Kelley if and only if the product* $X \times [0,1]$ *has the property of Kelley.*

(4.5.5) *If* Cone(X) *has the property of Kelley, then it is smooth at its vertex.*

(4.5.6) *If the hyperspace* $C(X)$ *(the hyperspace* 2^X*) has the property of Kelley, then* $C(X)$ *(*2^X*, respectively) is smooth at* X*.*

We finish this section recalling an important problem (formulated in [13, Problem 24].

PROBLEM 4.6. Consider the following conditions that a continuum X may or may not satisfy.

(4.6.1) X has the property of Kelley;

(4.6.2) $X \times [0,1]$ has the property of Kelley;

(4.6.3) $C(X)$ has the property of Kelley;

(4.6.4) Cone(X) has the property of Kelley;

(4.6.5) the Whitney levels for $C(X)$ have the property of Kelley;

(4.6.6) Cone(X) is smooth at $v(X)$;

(4.6.7) $C(X)$ is smooth at X.

The main open problem in this area is if (4.6.1) implies any of the properties (4.6.2)-(4.6.6). As it is shown in [13], to this aim it is enough to prove that (4.6.1) implies (4.6.2). On the other hand, we do not know if (4.6.6) implies (4.6.4) or if (4.6.7) implies (4.6.3).

5. Inducible mappings

In connection with the concept of the induced mappings one can ask under what conditions an arbitrary mapping between the hyperspaces $\mathcal{H}(X)$ and $\mathcal{H}(Y)$ is an induced one. An answer to this question was known for the mappings between hyperspaces of all nonempty closed subsets of the continuum (i.e., 2^X and 2^Y) or between hyperspaces of all nonempty subcontinua (i.e., $C(X)$ and $C(Y)$), see [11, Theorem 2.2, p. 7], and next it has been extended to mappings between the hyperspaces C_n for any $n \in \mathbb{N}$ in [14, Theorem 49, p. 802]. The result can further be proved for all hyperspaces listed in (1.2). To do this we will use similar auxiliary concepts and notation as introduced in [11, p. 6].

Let $\mathcal{H}$ be one of the hyperspaces in (1.2). Given two mappings between hyperspaces $g_1, g_2 : \mathcal{H}(X) \to \mathcal{H}(Y)$ we will write $g_1 \prec g_2$ provided that $g_1(A) \subset g_2(A)$ for each $A \in \mathcal{H}(X)$. The following properties of the relation $\prec$ on the set of all mappings between hyperspaces are consequences of the above definition.

Proposition 5.1. *The relation $\prec$ is an order on the set of all mappings between hyperspaces $\mathcal{H}(X)$ and $\mathcal{H}(Y)$ (where $\mathcal{H}$ is in* (1.2)*), that is, the following properties are true for every mappings g_1, g_2, g_3 between corresponding hyperspaces:*

(5.1.1) $g_1 \prec g_2$ *and* $g_2 \prec g_3$ *implies* $g_1 \prec g_3$*;*

(5.1.2) $g_1 \prec g_2$ *and* $g_2 \prec g_1$ *implies* $g_1 = g_2$*;*

(5.1.3) $g_1 \prec g_1$.

Let X and Y be continua. A mapping between hyperspaces, $g : \mathcal{H}(X) \to \mathcal{H}(Y)$ is said to be *inducible* provided that there exists a mapping $f : X \to Y$ such that $g = \mathcal{H}(f)$. We have the following characterization of inducible mappings.

Theorem 5.2. *Let continua X and Y be given and let $\mathcal{H}$ denote one of the hyperspaces listed in* (1.2)*. A mapping between hyperspaces, $g : \mathcal{H}(X) \to \mathcal{H}(Y)$, is inducible if and only if each of the following three conditions is satisfied:*

(5.2.1) $g(F_1(X)) \subset F_1(Y)$*;*

(5.2.2) $A \subset B$ *implies* $g(A) \subset g(B)$ *for every* $A, B \in \mathcal{H}(X)$*;*

(5.2.3) *g is minimal with respect to the order $\prec$, i.e., if a mapping $g_0 : \mathcal{H}(X) \to \mathcal{H}(Y)$ satisfies* (5.2.2)*, and $g_0 \prec g$, then $g = g_0$.*

Proof. In fact, the proof is almost the same as for the hyperspace 2^X in [11, proof of Theorem 2.2, p. 7]. We repeat the arguments here for sake of completeness only.

Assume g is inducible, i.e., $g = \mathcal{H}(f)$ for some $f : X \to Y$. Then (5.2.1) and (5.2.2) follow from Propositions 1.2 and 1.3 correspondingly. Let g_0 satisfy (5.2.2), and let $g_0 \prec \mathcal{H}(f)$. Then for each $x \in X$ we have $g_0(\{x\}) \subset \mathcal{H}(f)(\{x\}) = \{f(x)\}$, and thus

(5.2.4) $g_0(\{x\}) = \{f(x)\}$.

For each $A \in \mathcal{H}(X)$ and for each $x \in A$ we have $\{x\} \subset A$, whence $g_0(\{x\}) \subset g_0(A)$ by (5.2.2). Taking the union over all points $x \in A$ and using (5.2.4) we get

$$\bigcup\{g_0(\{x\}) : x \in A\} = \bigcup\{\{f(x)\} : x \in A\} = f(A) \subset g_0(A).$$

Since the last inclusion holds for each $A \in \mathcal{H}(X)$, we conclude that $\mathcal{H}(f) \prec g_0$. This implies $\mathcal{H}(f) = g_0$ by (5.1.2), thus (5.2.3) follows. If a mapping g satisfies conditions (5.2.1)-(5.2.3), then one can define $f : X \to Y$ putting $f(x)$ to be the only point in the set $g(\{x\})$. Thus for each $A \in \mathcal{H}(X)$ we have $\mathcal{H}(f)(A) = f(A) = \bigcup\{\{f(x)\} : x \in A\} = \bigcup\{g(\{x\}) : x \in A\} \subset g(A)$ by (5.2.2). Thus $\mathcal{H}(f) \prec g$, and by (5.2.3) we get $\mathcal{H}(f) = g$. The proof is finished. ∎

Recall that in [11, Section 3, p. 8-9] examples are constructed which shown that conditions (5.2.1), (5.2.2) and (5.2.3) of Theorem 5.2 are independent in the sense that no one of them is implied by the two others.

6. Special induced mappings

Let $\mathbb{M}$ denote a class of mappings between topological spaces (as e.g. open, monotone, light, confluent, weakly confluent and others — see [33] and [52] for the definitions, and compare [47, Table II, p. 28] for interrelations).

Problem 6.1. Let $f : X \to Y$ be a mapping between continua X and Y and let $\mathcal{H}$ be as in (1.2).

(6.1.1) Under what conditions $f \in \mathbb{M}$ implies $\mathcal{H}(f) \in \mathbb{M}$?

(6.1.2) Under what conditions $\mathcal{H}(f) \in \mathbb{M}$ implies $f \in \mathbb{M}$?

Several results related to the above problems, and concerning mostly the induced mappings 2^f and $C(f)$ are already known. The reader is referred to [9] and [33, Chapter XII, Section 77, p. 381], where the recent work in the area is gathered. In connection with these results the following particular question can be asked.

Question 6.2. (a) What of the above mentioned theorems for the induced mappings $C(f) = C_1(f)$ can be extended to the induced mappings $C_n(f)$ for $n > 1$?

(b) What about similar results for $F_n(f)$ with $n > 1$?

The following results were obtained in [14] (see [33], [50] and [52] for the definitions of the needed concepts).

Theorem 6.3. *Let a mapping $f : X \to Y$ between continua be given. Then for each $n \in \mathbb{N}$ and $n > 1$ the following statements hold.*

(6.3.1) *$C_n(f)$ is a surjection if and only if f is weakly confluent* [14, Proposition 1, p. 784].

(6.3.2) *$C_n(f)$ is monotone if and only if f is monotone* [14, Theorem 4, p. 784].

(6.3.3) *If $C_n(f)$ is open, then f is open; the converse is not true* [14, Theorem 8 and Remark 9, p. 786].

(6.3.4) *If f is an open monotone surjection, then $C_n(f)$ is an r-mapping* [14, Theorem 20, p. 792].

(6.3.5) *If Y is in Class (W) and f is refinable, then $C_n(f)$ is refinable* [14, Theorem 38, p. 799].

Further, the following holds for all $n \in \mathbb{N}$.

(6.3.6) *If f is confluent and $A \in C_n(A)$, then $\cup(C_n(f))^{-1}(A) = f^{-1}(A)$ (here $\cup$ means the union mapping)* [14, Lemma 2, p. 784].

S. B. Nadler, Jr. proved in [51, Lemma 2.1, p. 750] that, for a surjective mapping f between continua,

(6.1) f is a monotone surjection if and only if $C(f)$ is a surjective CE-mapping.

This result has been extended in [14, Corollary 7, p. 786] from $C(f)$ to $C_n(f)$ for $n > 1$. Thus the following result holds, which is a stronger form of (6.3.2) and (6.1).

THEOREM 6.4. *Let a mapping $f : X \to Y$ between continua be given. Then the conditions below are equivalent.*

(6.4.1) *There exists an $n \in \mathbb{N}$ such that the induced mapping $C_n(f)$ is monotone.*

(6.4.2) *There exists an $n \in \mathbb{N}$ such that the induced mapping $C_n(f)$ is a surjective CE-mapping.*

(6.4.3) *For each $n \in \mathbb{N}$ the induced mapping $C_n(f)$ is monotone.*

(6.4.4) *For each $n \in \mathbb{N}$ the induced mapping $C_n(f)$ is a surjective CE-mapping.*

(6.4.5) *f is monotone.*

Remark 6.5. Besides induced mappings considered above some special mappings between hyperspaces as e.g. retractions or selections were considered in the literature. Concerning recent results on this topic see [15]. For open problems see [12].

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