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Upper and Lower Solutions for First Order Problems with Nonlinear Boundary Conditions

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1. Introduction

We are interested in solutions of the nonlinear equation

$$u'(t) = f(t, u(t))\,,\; t \in I = [0, T]\,,\; T > 0 \tag{1}$$

satisfying the condition

$$g(u(0), u(T)) = 0\,, \tag{2}$$

where $f : I \times \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ are continuous functions.

If $g(x, y) = x - c$ with $c \in \mathbb{R}$, then (2) is the initial condition

$$u(0) = c\,. \tag{3}$$

Similarly, if $g(x, y) = x - y$, then (2) is the periodic condition

$$u(0) = u(T)\,. \tag{4}$$

Finally, the antiperiodic boundary condition

$$u(0) = -u(T)\,. \tag{5}$$

corresponds to the case $g(x, y) = x + y$.

Nonlinear boundary conditions are discussed in several papers and the usual hypothesis is that g must be monotone nonincreasing in the second variable (see [1, 2, 3, 9] and the references therein) or nondecreasing (we considered this type of condition in [4, 5]).

Here we discuss a more general situation since we only require monotonicity in the second variable and no monotonicity assumptions are imposed on the nonlinearity f. To achieve this we introduce a new definition of upper and lower solution.

The results that we present are new and improve and complement those in [1, 2, 3, 4, 5, 6, 9, 11].

It is possible to define the concept of subsolution and supersolution for equation (1) as follows.

Definition 1. We say that a function $\alpha \in C^1(I)$ is a subsolution of equation (1) if

$$\alpha'(t) \leq f(t, \alpha(t)) \, , \; t \in I \, . \tag{6}$$

Analogously, we say that $\beta \in C^1(I)$ is a supersolution of (1) if

$$\beta'(t) \geq f(t, \beta(t)) \, , \; t \in I \, . \tag{7}$$

In what follows we shall assume that

$$\alpha(t) \leq \beta(t) \, , \; t \in I \, , \tag{8}$$

or either

$$\beta(t) \leq \alpha(t) \, , \; t \in I \, . \tag{9}$$

For $u, v \in C(I)$, $u \leq v$ we define the set

$$[u, v] = \{ w \in C(I) \, : \, u(t) \leq w(t) \leq v(t) \, , \; t \in I \} \, .$$

Of course, to obtain a solution satisfying some initial or boundary condition and lying between a subsolution and a supersolution we need additional conditions. For example, in the periodic case (4) it suffices that (see [8, 10])

$$\alpha(0) \leq \alpha(T), \qquad \beta(0) \geq \beta(T) \tag{10}$$

and in the antiperiodic case it suffices that (for more details see [6])

$$\alpha(0) \leq -\beta(T), \qquad \beta(0) \geq -\alpha(T). \tag{11}$$

The purpose of this paper is to present new existence results for equation (1) with the nonlinear boundary condition (2) that includes, among others, the case of the initial value condition (3), the periodic condition (4) and the antiperiodic boundary condition (5). To this end, we introduce a new concept of coupled lower and upper solutions that allow us to obtain a solution in the sector $[\alpha,\beta]$ or $[\beta,\alpha]$. We point out that our method, being new, unifies the treatment of many different first order problems.

We finish this introduction with a lemma

LEMMA 1. *Let $L : C(I) \to C_0(I) \times \mathbb{R}$ be defined by*

$$[Lu](t) = \left(u(t) - u(0) + \lambda \int_0^t u(s)\,ds,\ au(0) + bu(T) \right)$$

where λ, a and b are real constants such that

$$a + be^{-\lambda T} \neq 0,$$

and here

$$C_0(I) = \{u \in C(I) : u(0) = 0\}.$$

Then L^{-1} exists and it is continuous and defined by

$$[L^{-1}(y,\gamma)](t) = e^{-\lambda t}A + y(t) - \lambda \int_0^t e^{-\lambda(t-s)} y(s)\,ds$$

with

$$A = \frac{\gamma + b\lambda \int_0^T e^{-\lambda(T-s)} y(s)\,ds - by(T)}{a + be^{-\lambda T}}.$$

2. COUPLED LOWER AND UPPER SOLUTIONS

To cover different possibilities for the nonlinear boundary function g we introduce the following concept.

DEFINITION 2. We say that α, $\beta \in C^1(I)$ are coupled lower and upper solutions for the problem (1)-(2) if α is a subsolution and β a supersolution for the equation (1), condition (8) holds, and

$$\begin{aligned} &\max\{g(\alpha(0),\alpha(T)), g(\alpha(0),\beta(T))\} \leq 0 \\ &\qquad \leq \min\{g(\beta(0),\beta(T)), g(\beta(0),\alpha(T))\}\ . \end{aligned} \tag{12}$$

We note that this definition generalizes the classical concepts. For instance, for the periodic case we obtain from (12) that (10) holds; and for the antiperiodic case, (12) implies (11).

Also note that in some cases (periodic, initial) it is possible to define a lower or an upper solution independently but in others (antiperiodic) it is necessary to define both together. More precisely, if g is monotone decreasing in the second variable, Definition 2 allow us to consider a lower and an upper solution independently.

THEOREM 1. *Assume that* α, β *are coupled lower and upper solutions for the problem* (1)-(2). *In addition, suppose that the functions*

$$\begin{aligned} h_\alpha(x) &:= g(\alpha(0), x) \\ h_\beta(x) &:= g(\beta(0), x) \end{aligned}$$

are monotone (either nonincreasing or nondecreasing) in $[\alpha(T), \beta(T)]$.

Then there exists at least one solution of the problem (1)-(2) *between the lower and the upper solution.*

Proof. Let $\lambda > 0$ and consider the modified problem

$$\begin{aligned} u'(t) + \lambda u(t) &= F^*(t, u(t)), \quad t \in I, \\ u(0) &= g^*(u(0), u(T)), \end{aligned} \tag{13}$$

with

$$F^*(t,u) = \begin{cases} f(t,\beta(t)) + \lambda\beta(t), & \text{if } \beta(t) < u \\ f(t,u) + \lambda u, & \text{if } \alpha(t) \le u \le \beta(t) \\ f(t,\alpha(t)) + \lambda\alpha(t), & \text{if } u < \alpha(t), \end{cases}$$

and

$$g^*(x,y) = p(0,x) - g(p(0,x),\ p(T,y))$$

and

$$p(t,x) = \max\{\alpha(t), \min\{x, \beta(t)\}\}.$$

Note that if u is a solution of (13) between α and β, then u is a solution of (1)-(2).

We define the mappings

$$L\colon C(I) \to C_0(I) \times \mathbb{R}$$

and

$$N\colon C(I) \to C_0(I) \times \mathbb{R}$$

by

$$[Lu](t) = \left(u(t) - u(0) + \lambda \int_0^t u(s)\, ds, u(0) \right)$$

and

$$[Nu](t) = \left(\int_0^t F^*(t, u(t)),\ g^*(u(0), u(T)) \right).$$

Clearly N is continuous and compact (by the Arzelá-Ascoli theorem). Also from Lemma 1 with $a = 1$ and $b = 0$, L^{-1} exists and is continuous.

On the other hand, solving (13) is equivalent to find a fixed point of

$$L^{-1}N \colon C(I) \to C(I).$$

Now, Schauder's fixed point theorem guarantees the existence of at least a fixed point since $L^{-1}N$ is continuous and compact.

It remains to show that u satisfies

$$\alpha(t) \le u(t) \le \beta(t), \quad t \in [0, T].$$

Assume that $u - \beta$ attains a positive maximum on $[0, T]$ at s_0. We shall consider three cases:

Case 1. $s_0 \in (0, T]$.

Then there exists $\tau \in (0, s_0)$ such that

$$0 \le u(t) - \beta(t) \le u(s_0) - \beta(s_0), \qquad \text{for all } \ t \in [\tau, s_0].$$

This yields a contradiction, since

$$\begin{aligned} \beta(s_0) - \beta(\tau) \quad &\le \quad u(s_0) - u(\tau) = \int_\tau^{s_0} [f(s, \beta(s)) - \lambda(u(s) - \beta(s))] ds \\ &< \quad \int_\tau^{s_0} \beta'(s) ds = \beta(s_0) - \beta(\tau). \end{aligned}$$

Case 2. $s_0 = 0$ and h_β monotone nonincreasing.

Then $0 < u(0) - \beta(0)$ and from (12) we obtain that

$$g(\alpha(0), \alpha(T)) \le 0 \le g(\beta(0), \beta(T)).$$

Now we have

$$\begin{aligned} u(0) = g^*(u(0), u(T)) &= \beta(0) - g(\beta(0), p(T, u(T))) \\ &\le \beta(0) - g(\beta(0), \beta(T)) \le \beta(0) \end{aligned}$$

which contradicts $0 < u(0) - \beta(0)$.

Case 3. $s_0 = 0$ and h_β monotone nondecreasing.
In this case we have $0 < u(0) - \beta(0)$ and

$$g(\alpha(0), \beta(T)) \le 0 \le g(\beta(0), \alpha(T)).$$

Now we get the contradiction

$$\begin{aligned} u(0) = g^*(u(0), u(T)) &= \beta(0) - g(\beta(0), p(T, u(T))) \\ &\le \beta(0) - g(\beta(0), \alpha(T)) \le \beta(0). \end{aligned}$$

Consequently, $u(t) \le \beta(t)$ for all $t \in I$. Similarly, one can show that $\alpha \le u$ on I. ∎

3. Coupled lower and upper solutions in reverse order

Now we consider the case when (9) holds.

Definition 3. We say that $\alpha\,,\, \beta \in C^1(I)$ are coupled lower and upper solutions for the problem (1)-(2) in reverse order if α is a subsolution and β a supersolution for the equation (1), condition (9) holds, and

$$\begin{aligned} \max\{g(\alpha(0), \alpha(T)), g(\beta(0), \alpha(T))\} &\le 0 \\ \le \min\{g(\beta(0), \beta(T)), g(\alpha(0), \beta(T))\}&\,. \end{aligned} \tag{14}$$

Theorem 2. *Assume that $\alpha\,,\, \beta$ are coupled lower and upper solutions in reverse order for the problem* (1)-(2)*. In addition, suppose that the functions*

$$\begin{aligned} h_\alpha(x) &:= g(x, \alpha(T)) \\ h_\beta(x) &:= g(x, \beta(T)) \end{aligned}$$

are monotone (either nonincreasing or nondecreasing) in $[\beta(0), \alpha(0)]$.
Then there exists at least one solution of the problem (1)-(2) *in* $[\beta, \alpha]$.

Proof. Let $\lambda > 0$ and consider the modified problem

$$\begin{aligned} &u'(t) - \lambda u(t) = F^*(t, u(t))\,, \quad t \in I\,, \\ &u(T) = g^*(u(0), u(T))\,, \end{aligned}$$

with

$$F^*(t,u) = \begin{cases} f(t,\beta(t)) - \lambda\beta(t), & \text{if } \beta(t) < u \\ f(t,u) - \lambda u, & \text{if } \beta(t) \le u \le \alpha(t) \\ f(t,\alpha(t)) - \lambda\alpha(t), & \text{if } u > \alpha(t), \end{cases}$$

and

$$g^*(x,y) = p(T,x) + g(p(0,x),\ p(T,y))$$

and

$$p(t,x) = \max\{\beta(t), \min\{x, \alpha(t)\}\}.$$

Now the proof is analogous to the proof of Theorem 1 using Lemma 1 with $a = 0$ and $b = 1$. ∎

It is possible to replace f continuous by f L^1–Carathéodory, and (6) and (7) by the integral conditions in [7] and the results in this paper are also true.

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