# On the Structure of Rank One Elements
# in Banach Algebras

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## 1. Introduction

Throughout this paper $A$ denotes a complex, unital Banach algebra with the minimum requirement that $A$ be semiprime, that is $xAx = \{0\}$ implies $x = 0$ holds for all $x \in A$. It is obvious that any semisimple Banach algebra is also semiprime. We call an element $a \in A$ spatially rank one if $a$ is rank one in the sense of J. Puhl ([12], Definition 2.2):

(1.1)     $a \in A$ *is spatially rank one if and only if* $aAa \subseteq \mathbb{C}a$, *and* $a \neq 0$.

One motivation for Puhl's definition is the fact that (and this is easy to prove) for $T \in A = B(X)$, the algebra of bounded linear operators on a Banach space $X$, we have that $\dim \mathcal{R}(T) = 1$ if and only if (1.1) holds.

For further examples of Banach algebras with spatially rank one elements see [12]. We shall denote the collection of spatially rank one elements by $\mathcal{F}_1$. Since Puhl's paper, other authors (see list of references) have contributed to the topics of rank one and finite elements (for which rank one elements are the building blocks) yielding some characterizations and at least one different definition of rank one elements ([8], Definition 2.2):

(1.2)

$a \in A$ *is said to be spectrally rank one if and only if*

$a \neq 0$ *and* $x \in A \Rightarrow \# \sigma(xa) \setminus \{0\} \leq 1$,

where $\sigma$ denotes the spectrum and $\#$ the number of elements in the given set. If $\mathcal{G}_1$ is the collection of spectrally rank one elements then, since $\mathcal{F}_1$ absorbs nonzero products in $A$ and the elements of $\mathcal{F}_1$ have at most one nonzero point in their spectra ([12], Lemma 2.7, Lemma 2.8), it follows that $\mathcal{F}_1 \subseteq \mathcal{G}_1$. In general this inclusion may be strict; if $A$ is semiprime but not semisimple let $a \in \mathrm{Rad}\,(A)$, $a \neq 0$. If $axa = \lambda a$ for some $0 \neq \lambda \in \mathbb{C}$, then $\frac{ax}{\lambda}$ is an idempotent whence $\sigma\left(\frac{ax}{\lambda}\right) = \{0, 1\}$ which is impossible. So we cannot have $a \in \mathcal{F}_1$. But clearly $\mathrm{Rad}\,(A) \subseteq \mathcal{G}_1$ and hence it follows that $\mathcal{F}_1 \neq \mathcal{G}_1$. Harte ([8], Theorem 4) now continues to show that $\mathcal{F}_1 = \mathcal{G}_1$ precisely when $A$ is semisimple. An earlier paper by T. Mouton and H. Raubenheimer [9] also gives a spectral characterisation of $\mathcal{F}_1$ in the case where $A$ is semisimple ([9], Theorem 2.2):

$$(1.3) \qquad \left.\begin{array}{l} a \in \mathcal{F}_1 \text{ if and only if} \\ \text{the conditions } b \in A \text{ and} \\ s_0,\, s_1 \in \mathbb{C},\ 0 \neq s_0 \neq s_1 \neq 0 \end{array}\right\} \Rightarrow \sigma\,(b + s_0 a) \cap \sigma\,(b + s_1 a) \subseteq \sigma\,(b)\,.$$

In fact, Mouton and Raubenheimer's arguments actually imply Harte's result, but their method uses an analytical property of the spectrum ([1], Theorem 3.4.25) as opposed to Harte's essentially algebraic technique. The set $\mathcal{G}_1$ was also studied in [13] where it was shown ([13], Theorem 1) that $\mathcal{G}_1$ is a closed multiplicative ideal in $A$. In yet another paper, by J.A. Erdos, S. Giotopoulos and M.S. Lambrou, incidentally completed in the same year as Puhl's article, another notion of rank one is introduced [6]:

(1.4)     *An element $u \neq 0$ of a semisimple Banach algebra, is said to be rank one if and only if $u$ has an image of rank one in some faithful representation of $A$.*

They showed ([6], Theorem 6) that this definition is equivalent to requiring that $u$ be a single element of $A$, that is, $aub = 0$ implies $au = 0$ or $ub = 0$, and that $u$ acts compactly on $A$, that is, the map $a \longmapsto uau$ is compact. Rather surprisingly, Erdos shows in [5] that the condition, $u$ being single alone, is enough to ensure the above-mentioned equivalence to (1.4) in the $C^*$-algebra case. Using a topological idea to derive a simplified form of minimal ideals, we give, as a corollary, a particularly easy proof of the equivalence of (1.1) and (1.4). A minimal left ideal $J \neq \{0\}$ of $A$ is a left ideal such that $\{0\}$ and $J$ are the only left ideals contained in $J$. Minimal right ideals are defined in a similar way. Minimal ideals are important in the study of rank one elements because of their relationship to minimal idempotents, which are defined to be idempotents belonging to $\mathcal{F}_1$. For a discussion on minimal ideals see [3],

Chapter IV. With respect to spatially rank one elements, it is easy to see that $a \in \mathcal{F}_1$ if and only if there exists a unique linear functional $f_a$ on $A$ such that $f_a(x)\, a = axa$ for all $x \in A$. Moreover, it follows easily that $\sigma(a) = \{0, f_a(1)\}$ ([12], Lemma 2.8). The complex number $f_a(1)$ is called the trace of $a$ and is denoted $t_r(a)$. Note that for $a \in \mathcal{F}_1$ it might very well happen that $t_r(a) = 0$ and, in fact, every quasinilpotent $a \in \mathcal{F}_1$ is nilpotent with $a^2 = 0$. The above scenario is, however, not possible for commutative algebras. Puhl extends the trace to finite elements (all finite sums of elements in $\mathcal{F}_1$) in a natural way such that the trace has the expected properties ([12], Theorem 4.5). For more and recent results on these matters see [2] and [7]. We define $\mathrm{Exp}(A)$, the group of generalised exponentials in $A$, by

$$\mathrm{Exp}\,(A) = \left\{ \prod_{i=1}^{n} e^{x_i} : x_i \in A, \quad n \in \mathbb{N} \right\}.$$

It can be shown ([1], Theorem 3.3.7) that $\mathrm{Exp}(A)$ is the connected component of $A^{-1}$ (the invertible elements of $A$) containing the identity. By $\exp(A)$ we mean the set of exponentials of $A$, that is $\exp(A) = \{e^x : x \in A\}$. If $A$ is not commutative then it is possible ([10], p. 230–231) that $\exp(A) \neq \mathrm{Exp}(A)$. We use $r_\sigma(x)$ to denote the spectral radius of $x \in A$. By an idempotent we always mean a nontrivial idempotent. If $A$ is a Banach algebra and $B$ is a subset of $A$ then, as usual, $\overline{B}$ denotes the closure of $B$ in $A$.

## 2. The structure of rank one elements

For semiprime $A$ with $\mathcal{F}_1 \neq \emptyset$ we partition $\mathcal{F}_1$ into two disjoint subsets; $\mathcal{F}_1^0$ denotes the nilpotent elements of $\mathcal{F}_1$ and $\mathcal{F}_1^1$ the non–nilpotent elements of $\mathcal{F}_1$. Note that in the commutative case $\mathcal{F}_1^0 = \emptyset$. For if $u \in \mathcal{F}_1$ with $u^2 = 0$ then $u\,A\,u = \{0\}$ which contradicts the semiprime condition imposed on $A$. On the other hand, if $\mathcal{F}_1 \neq \emptyset$ then the semiprime condition guarantees that $\mathcal{F}_1^1$ is never empty. The following lemma is instrumental in many of the forthcoming results.

LEMMA 2.1. *Let $A$ be a semiprime Banach algebra with $\mathcal{F}_1 \neq \emptyset$.*

(i) *If $u \in \mathcal{F}_1^1$, $v \in A$ with $uv \in \mathcal{F}_1^1$ then $uv$ can be written as $uv = \alpha e^x u e^{-x}$ where $0 \neq \alpha \in \mathbb{C}$ and $x \in A$. A similar statement holds for $vu \in \mathcal{F}_1^1$.*

(ii) *If $u \in \mathcal{F}_1^0$ then there exists $v \in \exp(A)$ such that $uv$ and $vu \in \mathcal{F}_1^1$.*

*Proof.* (i) The rank 1 elements $\frac{u}{t_r(u)}$ and $\frac{uv}{t_r(uv)}$ are both idempotents satisfying

$$\frac{u}{t_r\left(u\right)} = \frac{uvu}{t_r\left(uv\right)t_r\left(u\right)}, \qquad \frac{uv}{t_r\left(uv\right)} = \frac{u^2v}{t_r\left(u\right)t_r\left(uv\right)}$$

and hence

$$\left(\frac{u}{t_r\left(u\right)} - \frac{uv}{t_r\left(uv\right)}\right)^2 = 0$$

so that

$$r_\sigma\left(\frac{u}{t_r\left(u\right)} - \frac{uv}{t_r\left(uv\right)}\right) = 0.$$

It follows from ([14], Lemma 3.1) that

$$uv = \alpha\left(e^x u e^{-x}\right)$$

for some $x \in A$ and $\alpha = \frac{t_r(uv)}{t_r(u)}$.

(ii) If $u \in \mathcal{F}_1^0$ then $u^2 = 0$ but, since $A$ is semiprime, there is $x \in A$ such that $uxu = u$. If we choose $\lambda \in \mathbb{C}$ with $|\lambda| > r_\sigma\left(x\right)$ then it follows by the holomorphic functional calculus that $-\lambda + x \in \exp\left(A\right)$. Thus

$$0 \neq \left[u\left(-\lambda + x\right)\right]^2 = u\left(-\lambda + x\right) \in \mathcal{F}_1^1.$$

∎

We are now in a position to identify the connected components of $\mathcal{F}_1$ in terms of $\mathrm{Exp}\left(A\right)$.

THEOREM 2.2. *Let $A$ be a semiprime Banach algebra with $\mathcal{F}_1 \neq \emptyset$.*

(i) *If $u \in \mathcal{F}_1$ then the connected component of $\mathcal{F}_1$ containing $u$ is the set*

$$K_u^{\mathcal{F}_1} = \mathrm{Exp}\left(A\right) u \, \mathrm{Exp}\left(A\right).$$

(ii) *If $u \in \mathcal{F}_1$ then $u$ belongs to the centre of $A$ if and only if*

$$K_u^{\mathcal{F}_1} = \mathbb{C}u \setminus \{0\}.$$

*Proof.* (i) Using the idea that for $a, b \in \mathrm{Exp}(A)$ with

$$a = \prod_{i=1}^{k} e^{x_i}, \qquad b = \prod_{i=1}^{n} e^{y_i}, \quad n \geq k$$

$$g\left(t\right) = \prod_{i=1}^{n} e^{tx_i + (1-t)y_i}, \qquad x_i = 0 \quad \text{for } i > k, t \in [0,1]$$

is a path in $\mathrm{Exp}(A)$ connecting $a$ and $b$, it follows that $K_u^{\mathcal{F}_1}$ is arcwise connected. We show $K_u^{\mathcal{F}_1}$ is maximal connected in $\mathcal{F}_1$. Let $u \in \mathcal{F}_1$ and let $(r_n\, u\, s_n)$ be a sequence in $K_u^{\mathcal{F}_1}$ such that $r_n\, u\, s_n \to v \in \mathcal{F}_1$. If $v \in \mathcal{F}_1^1$ then, by continuity of the trace, we may assume $(r_n\, u\, s_n) \subseteq \mathcal{F}_1^1$. So for $n$ sufficiently large

$$r_\sigma \left( \frac{r_n\, u\, s_n}{t_r\,(r_n\, u\, s_n)} - \frac{v}{t_r\,(v)} \right) < 1$$

and hence by ([14], Lemma 3.1) we have $v \in K_u^{\mathcal{F}_1}$. If $v \in \mathcal{F}_1^0$ then the above argument does not apply directly. Using Lemma 1(ii), there is $a \in \exp\,(A)$ such that $va \in \mathcal{F}_1^1$. Since $r_n\, u\, s_n a \to va$ it follows by the first argument that $va \in K_u^{\mathcal{F}_1}$ and hence that $v \in K_u^{\mathcal{F}_1}$. This proves $K_u^{\mathcal{F}_1}$ closed in $\mathcal{F}_1$. On the other hand, if $(v_n)$ is a sequence in $\mathcal{F}_1 \backslash K_u^{\mathcal{F}_1}$ and $v_n \to r\, u\, s \in K_u^{\mathcal{F}_1}$ then $r^{-1} v_n s^{-1} \to u$ and, using the same argument above, it follows that $v_n \in K_u^{\mathcal{F}_1}$ for $n$ sufficiently large, which is a contradiction. This proves that $K_u^{\mathcal{F}_1}$ is open in $\mathcal{F}_1$ and hence $K_u^{\mathcal{F}_1}$ is maximal connected in $\mathcal{F}_1$.

(ii) Suppose $u \in \mathcal{F}_1$ belongs to the centre of $A$. From (i) it follows that $K_u^{\mathcal{F}_1} = u\,\mathrm{Exp}(A)$. By the observation preceding Lemma 2.1, $u \in \mathcal{F}_1^1$. Hence if $uv \in u\,\mathrm{Exp}(A)$ then also $uv \in \mathcal{F}_1^1$ and so in view of Lemma 2.1(i) we have that

$$uv = \alpha w^{-1} uw = \alpha u \quad \text{where} \quad v, w \in \mathrm{Exp}\,(A) \quad \text{and} \quad 0 \neq \alpha \in \mathbb{C}.$$

By homogenity in $v$ we have

$$K_u^{\mathcal{F}_1} = \mathbb{C}\, u \setminus \{0\}.$$

Suppose, on the other hand, that $K_u^{\mathcal{F}_1}$ reduces to $\mathbb{C}\, u \setminus \{0\}$. If $u \in \mathcal{F}_1^0$ then by Lemma 2.1(ii) there is $v \in \exp\,(A)$ such that $uv \in \mathcal{F}_1^1$. But also, $uv \in K_u^{\mathcal{F}_1}$ and thus $uv = \alpha u$, $0 \neq \alpha \in \mathbb{C}$. This is a contradiction since $(uv)^2 = \alpha^2 u^2 = 0$ implies $uv \in \mathcal{F}_1^0$. So we must have $u \in \mathcal{F}_1^1$. Now since

$$\frac{w^{-1} uw}{t_r\,(u)} \in K_u^{\mathcal{F}_1}, \quad w \in \mathrm{Exp}\,(A)$$

it follows that for each $w \in \mathrm{Exp}(A)$ there is $0 \neq \alpha \in \mathbb{C}$ such that

$$\frac{w^{-1} uw}{t_r\,(u)} = \alpha u.$$

Moreover, since the left hand side is an idempotent, we must have $\alpha = \frac{1}{t_r(u)}$ for each $w$. From ([14], Corollary 3.5) it follows that $\frac{u}{t_r(u)}$ and hence $u$, is central.  ∎

Recall that a left ideal $J$ is a minimal left ideal in $A$ if and only if $J = Au$ where $u$ is some element of $\mathcal{F}_1$. The following corollary may effect a significant simplification of minimal ideals in view of the fact that $\mathrm{Exp}(A) \subseteq A^{-1}$.

COROLLARY 2.3. *Let $A$ be a semiprime Banach algebra with $\mathcal{F}_1 \neq \emptyset$. Every minimal left ideal $J$ of $A$ has the form*

$$J = \mathrm{Exp}\,(A)\,u \cup \{0\} \quad with \ \ u \in \mathcal{F}_1.$$

*Similarly, every minimal right ideal has the form*

$$J = u\,\mathrm{Exp}\,(A) \cup \{0\} \quad with \ \ u \in \mathcal{F}_1.$$

*Proof.* Since a left ideal $J \subseteq A$ is minimal if and only if $J = Au$ where $u \in \mathcal{F}_1$ it suffices to prove that $Au = \mathrm{Exp}(A)\,u \cup \{0\}$ for $u \in \mathcal{F}_1$. Since $A$ is connected it follows from Theorem 2.2 that

$$(2.1) \qquad\qquad \mathrm{Exp}\,(A)\,u\,\mathrm{Exp}\,(A) = A\,u\,A - \{0\}$$

and hence that

$$(*) \qquad\qquad \mathrm{Exp}\,(A)\,u\,\mathrm{Exp}\,(A)\,u \cup \{0\} = A\,u\,A\,u.$$

Since $A$ is semiprime, $u \in \mathcal{F}_1$ and $\mathrm{Exp}(A) + \mathrm{Exp}(A) = A$ we have that

$$(**) \qquad\qquad u\,\mathrm{Exp}\,(A)\,u \cup \{0\} = u\,A\,u = \mathbb{C}u.$$

Combining $(*)$ and $(**)$ we obtain $\mathrm{Exp}(A)\,u \cup \{0\} = Au$. The proof for a minimal right ideal follows along the same lines. ∎

The equivalence of (1.1) and (1.4) for semisimple $A$ follows almost trivially.

COROLLARY 2.4. *Let $A$ be a semisimple Banach algebra with $\mathcal{F}_1 \neq \emptyset$. Then $u \in \mathcal{F}_1$ if and only if $u$ is a single element of $A$ and $u$ acts compactly on $A$. Thus, for semisimple $A$, the rank one definitions (1.1), (1.2) and (1.4) are equivalent.*

*Proof.* Because of ([6], Theorem 4) we have that $u$ single and $u$ acting compactly imply $u \in \mathcal{F}_1$. If $u \in \mathcal{F}_1$ then the Bolzano–Weierstrass Theorem implies $u$ acts compactly on $A$. Also, if $aub = 0$ for some $a, b \in A$ and $au \neq 0$ then Corollary 2.3 implies $au = vu$ for some $v \in \mathrm{Exp}(A)$. This means that $vub = 0$ and hence $ub = 0$. So $u$ is single. ∎

*Remark.* For $a \in A^{-1}$ the connected component of $A^{-1}$ containing $a$ is given by

$$K_a^{A^{-1}} = \mathrm{Exp}\,(A)\,a.$$

But since $\mathrm{Exp}(A)$ is a normal subgroup of $A^{-1}$ we may write

$$K_a^{A^{-1}} = \mathrm{Exp}\,(A)\,a = a\,\mathrm{Exp}\,(A) = \mathrm{Exp}\,(A)\,a\,\mathrm{Exp}\,(A).$$

Consideration of algebras of matrices shows that even in the simplest of non-commutative cases we may have

$$K_u^{\mathcal{F}_1} \neq \mathrm{Exp}\,(A)\,u \neq u\,\mathrm{Exp}\,(A) \neq K_u^{\mathcal{F}_1} \quad \text{for} \quad u \in \mathcal{F}_1.$$

Proposition 2.5 is Theorem 2.2 in [2] for rank one elements in the setting of semiprime Banach algebras. We provide a simple direct proof in this case.

PROPOSITION 2.5. *If $A$ is a semiprime Banach algebra with $u \in \mathcal{F}_1$ then $K_u^{\mathcal{F}_1} \cap \mathcal{F}_1^1$ is dense in $K_u^{\mathcal{F}_1}$.*

*Proof.* Let $x \in K_u^{\mathcal{F}_1}$ be nilpotent. Since $A$ is semiprime we may choose $a \in A$ such that $ax \in K_u^{\mathcal{F}_1}$ and $ax \in \mathcal{F}_1^1$. If $(\lambda_n)$ is a sequence in $\mathbb{C}$ with $\lambda_n \neq 0$ and $\lambda_n \to 0$, then $(\lambda_n a + 1)\,x \to x$. Since $(\lambda_n a + 1)\,x \neq 0$ we have $(\lambda_n a + 1)\,x \in K_u^{\mathcal{F}_1}$ (Corollary 2.3) for each $n$ and since $[(\lambda_n a + 1)\,x]^2 = \lambda_n^2\,(ax)^2 + \lambda_n xax \neq 0$, it follows that $((\lambda_n a + 1)\,x) \subseteq K_u^{\mathcal{F}_1} \cap \mathcal{F}_1^1$. ∎

In view of Theorem 2.2, we observe that if we use the idea in Proposition 2.5 then we even have:

PROPOSITION 2.6. *Let $A$ be semiprime with $\mathcal{F}_1 \neq \emptyset$ and let $u \in \mathcal{F}_1^1$. Then*

$$K_u^{\mathcal{F}_1} = \overline{\exp\,(A)\,u\,\exp\,(A)} \setminus \{0\}.$$

*In particular if $K_u^{\mathcal{F}_1}$ contains no nilpotent elements then*

$$K_u^{\mathcal{F}_1} = \exp\,(A)\,u\,\exp\,(A).$$

*Proof.* We may assume $u$ is idempotent for otherwise, replace $u$ by $\frac{u}{t_r(u)}$. Let $aub \in K_u^{\mathcal{F}_1}$. If $au, ub \in \mathcal{F}_1^1$ then by Lemma 2.1(i) $au = \alpha e^x u e^{-x}$ and $ub = \beta e^y u e^{-y}$, $\alpha, \beta \in \mathbb{C}$ and $x, y \in A$. So, since $u \in \mathcal{F}_1$, $aub = \gamma e^x u e^{-y}$ for some $0 \neq \gamma \in \mathbb{C}$. Thus

$$aub = e^{x+\frac{\lambda}{2}} u \, e^{-y+\frac{\lambda}{2}} \quad \text{where} \quad e^\lambda = \gamma,$$

and hence

$$aub \in \exp(A) \, u \, \exp(A).$$

If $au \in \mathcal{F}_1^0$ let $(\lambda_n) \subseteq \mathbb{C}$, $\lambda_n \neq 0$ be a sequence with $\lambda_n \to 0$. Now for each $n$

$$[(\lambda_n + a) \, u]^2 = \lambda_n^2 u + \lambda_n au$$

since $au \in \mathcal{F}_1^0$ and $a \in \mathrm{Exp}(A)$. Note that $\lambda_n^2 u + \lambda_n au \neq 0$ for each $n$ since $u^2 \neq 0$. Thus we have

$$(\lambda_n + a) \, u \to au \quad \text{with each} \quad (\lambda_n + a) \, u \in \mathcal{F}_1^1.$$

Appealing to Lemma 2.1(i) again

$$(\lambda_n + a) \, u = \alpha_n e^{x_n} u \, e^{-x_n} \quad \text{with} \quad \alpha_n \in \mathbb{C} \text{ and } x_n \in A.$$

Similarly if $ub \in \mathcal{F}_1^0$ then

$$u \, (\lambda_n + b) = \beta_n e^{y_n} u \, e^{-y_n} \quad \text{with} \quad \beta_n \in \mathbb{C} \text{ and } y_n \in A.$$

As in the first part of the proof

$$aub = \lim_{n \to \infty} e^{x_n + \frac{\gamma_n}{2}} u e^{-y_n + \frac{\gamma_n}{2}}$$

where $(\gamma_n)$ is some sequence in $\mathbb{C}$. So we have

$$aub \in \overline{\exp(A) \, u \exp(A)} \setminus \{0\}.$$

$\blacksquare$

If $A$ is a semiprime Banach algebra with $\mathcal{F}_1 \neq \emptyset$ is it possible that every idempotent belongs to $\mathcal{F}_1$? Obviously this holds for $\mathbb{C}^2$ and a few calculations show that also $M_2(\mathbb{C})$ possesses this property. We can use the component structure, and in particular the fact that distinct components are orthogonal, to show that these examples are the only semiprime Banach algebras with this property.

THEOREM 2.7. *Let $A$ be a semiprime Banach algebra. Then the following are equivalent:*

  (i) *$A$ contains a minimal idempotent $p$ such that $1 - p$ is also minimal.*

 (ii) *Every idempotent $p \in A$ is minimal.*

(iii) *$A$ is isomorphic to $\mathbb{C}^2$ or $A$ is isomorphic to $M_2 (\mathbb{C})$.*

*Proof.* (iii) $\Rightarrow$ (ii). A standard computation shows $\mathbb{C}^2$ and $M_2 (\mathbb{C})$ have every idempotent minimal.

(ii) $\Rightarrow$ (i). Trivial.

(i) $\Rightarrow$ (iii). Let $p$ and $1 - p$ both belong to $\mathcal{F}_1$. Then we have that ([3], §30.6 and §30.7) implies that $Ap$ is a maximal and a minimal left ideal of $A$. So if $A$ is not semisimple then $\mathrm{Rad}(A) \subseteq Ap = \mathrm{Exp}(A) \, p \cup \{0\}$, which implies $p \in \mathrm{Rad}(A)$. But this is absurd and we conclude that $A$ is semisimple. If $p$ is central then we may write

$$A = pAp \oplus (1 - p) A (1 - p) = \mathbb{C}p \oplus \mathbb{C}(1 - p)$$

which proves $\dim(A) = 2$. From the Wedderburn–Artin Theorem it follows that $A$ is isomorphic to $\mathbb{C}^2$. Suppose $p$ is not central. We show that $\dim(Ap) = 2 = \dim(A(1 - p))$. If $K_p^{\mathcal{F}_1} \neq K_{1-p}^{\mathcal{F}_1}$ then for any $v, w \in \mathrm{Exp}(A)$ we have

$$(1 - p) \, vp = 0 = vp - pvp \ \ \text{and} \ \ pw(1 - p) = 0 = pw - pwp.$$

Since $v, w \in \mathrm{Exp}(A)$ were arbitrary in $\mathrm{Exp}(A)$ it follows that $K_p^{\mathcal{F}_1} = \mathbb{C}p \setminus \{0\}$. But, using Theorem 2.2(ii), this means that $p$ is central, which contradicts our assumption. So we must have $K_p^{\mathcal{F}_1} = K_{1-p}^{\mathcal{F}_1}$. Let $x \in Ap$, $x \neq 0$. From Corollary 2.3 we have that $x = vp$ for some $v \in \mathrm{Exp}(A)$. We can now write $vp = w(1 - p) u$ where $w, u \in \mathrm{Exp}(A)$. Multiplying the above equation throughout by $1 - p$ on the left and by $p$ on the right we obtain $vp = \lambda \, up + \gamma p$ where $\lambda, \gamma \in \mathbb{C}$. Since $p$ is not central we may assume that $\{up, p\}$ is linearly independent, for otherwise we may prove there is a linearly independent set $\{pw, p\}$ ($w \in \mathrm{Exp}(A)$) in $pA$. We claim $\{up, p\}$ is a basis for $Ap = \mathrm{Exp}(A) \, p \cup \{0\}$. Let $v' \in \mathrm{Exp}(A)$. Then since $v'p = v'v^{-1}vp$ we have that

$$Ap = \mathrm{Exp}(A) \, w(1 - p) \, u \cup \{0\}.$$

So, for nonzero $xp = aw(1 - p) u$, $a \in \mathrm{Exp}(A)$ it follows

$$(1 - p) \, xp = (1 - p) \, aw(1 - p) \, up$$

implies that $xp = \alpha_1 up + \alpha_2 p$ for some $\alpha_1, \alpha_2 \in \mathbb{C}$. Thus $\dim(Ap) = 2$. Now, using a similar idea, we may prove that $1 \leq \dim(A(1-p)) \leq 2$. If $\dim(A(1-p)) = 1$ then $\dim(A) = 3$ which is impossible since $A$ is semisimple and noncommutative. Thus $\dim(A(1-p)) = 2$. Since $A = Ap \oplus A(1-p)$ it follows from the Wedderburn–Artin Theorem that $A$ is isomorphic to $M_2(\mathbb{C})$.
∎

Although it is constructive in the second part of the above proof to find a basis for $Ap$, one can directly conclude that $\dim A \leq 4$; if $x$ is invertible, then in view of $xp$ and $x(1-p)$ being rank one and

$$A \subseteq xAx = xpAxp + xpAx(1-p) + x(1-p)Axp + x(1-p)Ax(1-p),$$

it follows from ([6], Theorem 4) that $\dim A \leq 4$.

The last part of the proof of Theorem 2.7 suggests that the semiprime condition cannot be omitted. Indeed, if $A$ is the nonsemiprime unital subalgebra of $M_2(\mathbb{C})$ consisting of matrices with the lower left entries equal to zero then $A$ contains a minimal idempotent $p$ such that $1 - p$ is also minimal, but $\dim(A) = 3$.

Let us suppose now that $A$ is semiprime with $\mathcal{F}_1 \neq \emptyset$ but that $A$ is not semisimple. As we have mentioned earlier, $\mathcal{F}_1 \subsetneq \mathcal{G}_1$, since $\text{Rad}(A) \setminus \{0\} \subseteq \mathcal{G}_1$ and $\text{Rad}(A) \cap \mathcal{F}_1 = \emptyset$. Of course there may be other elements (besides $\text{Rad}(A)$) that belong to $\mathcal{G}_1$ but not to $\mathcal{F}_1$. In fact, as we shall see (Theorem 2.10(v)), these elements constitute an important subset of $\mathcal{G}_1$.

Lemma 2.8. *Let $A$ be a semiprime Banach algebra such that $A$ is not semisimple and suppose $a \notin \text{Rad}(A)$. Then $a \in \mathcal{G}_1$ if and only if, given any $x \in A$, there is a unique $\lambda \in \mathbb{C}$ such that $axa - \lambda a \in \text{Rad}(A)$.*

*Proof.* Let $a \in \mathcal{G}_1$ with $a \notin \text{Rad}(A)$. If $B = A/\text{Rad}(A)$ and $\dot{x}$ denotes the coset element of $B$ under the canonical homomorphism then, since $\sigma_A(x) = \sigma_B(\dot{x})$ for each $x$ in $A$ and $\mathcal{F}_1 = \mathcal{G}_1$ in $B$ ([8], Theorem 4), the existence of $\lambda$ follows. If $\lambda_1 \neq \lambda$ also satisfies $axa - \lambda_1 a \in \text{Rad}(A)$, then $(\lambda - \lambda_1)a \in \text{Rad}(A)$ implies $a \in \text{Rad}(A)$ which contradicts our assumption. On the other hand, if for $x \in A$ arbitrary, there is a unique $\lambda \in \mathbb{C}$ such that $axa - \lambda a \in \text{Rad}(A)$, then we have $\sigma\big((xa)^2 - \lambda xa\big) = \{0\}$. By the Spectral Mapping Theorem $\alpha \in \sigma(xa)$ must satisfy $\alpha^2 - \lambda\alpha = 0$ and hence $\#\sigma(xa) \setminus \{0\} \leq 1$. Thus $a \in \mathcal{G}_1$. ∎

Thus, the elements of $\mathcal{G}_1$ that belong to neither $\mathcal{F}_1$ nor $\text{Rad}(A)$ are exactly those $a \in A$ for which, given $x \in A$, there exist a unique $\lambda \in \mathbb{C}$ such that

$axa - \lambda a \in \operatorname{Rad}(A)$ with at least one $x_0 \in A$ satisfying $ax_0 a - \lambda_0 a \neq 0$. Note that, using Lemma 2.8, we have $A$ contains an invertible $a \in \mathcal{G}_1$ iff $A/\operatorname{Rad}(A) \cong \mathbb{C}$. Of course, by the Gelfand-Mazur Theorem, $A$ contains an invertible $a \in \mathcal{F}_1$ iff $A \cong \mathbb{C}$. If $A$ is semisimple then $\mathcal{F}_1$ and $\mathcal{G}_1$ coincide so that the component of $\mathcal{G}_1$ containing $u \in \mathcal{G}_1$ is the set

$$K_u^{\mathcal{G}_1} = \operatorname{Exp}(A) \, u \, \operatorname{Exp}(A).$$

Suppose again $A$ is semiprime but not semisimple. Let $a \neq 0$, $a \in \operatorname{Rad}(A)$. Now if $u \in A$ is any other element of $\mathcal{G}_1$, then for $t \in [0,1]$ and $x \in A$ arbitrary we have from ([15], p. 3) that

$$\sigma\big(x\,(tu + (1-t)\,a)\big) = \sigma\,(txu)$$

which shows that the line with end points $u$ and $a$ belongs to $\mathcal{G}_1$. We have the following:

PROPOSITION 2.9. *If $A$ is semiprime but not semisimple and $u \in \mathcal{G}_1$ then $K_u^{\mathcal{G}_1} = \mathcal{G}_1$, so that $\mathcal{G}_1$ is connected.*

For a semiprime $A$ which is not semisimple we call an element of $\mathcal{G}_1$ that neither belongs to $\mathcal{F}_1$ nor to $\operatorname{Rad}(A)$, quasispatially rank one. The set of quasispatially rank one elements is denoted by $\mathcal{H}_1$.

Theorem 2.10 gives the relationships between the disjoint constituent parts $\mathcal{F}_1$, $\mathcal{H}_1$ and $\operatorname{Rad}(A)\backslash\{0\}$ of $\mathcal{G}_1$. We first give an example of a semiprime Banach algebra which is not semisimple such that $\mathcal{F}_1 = \emptyset$ but $\mathcal{H}_1 \neq \emptyset$.

Let $A$ be the Banach space of all complex functions that are defined and continuous on the closed unit disc, $\mathbb{D}$, in $\mathbb{C}$ and analytic on its interior. The norm on $A$ is the usual sup norm, $||f|| = \sup_{\lambda \in \mathbb{D}} |f(\lambda)|$, but multiplication is defined by convolution

$$f * g\,(\lambda) = \int_0^\lambda f\,(\lambda - \mu)\,g\,(\mu)\,d\mu$$

$$= \int_0^1 f\,(\lambda\,(1-s))\,g\,(\lambda s)\,\lambda\,ds.$$

Now $A$ is a commutative Banach algebra which is semiprime but not semisimple ([11], p. 507–508), and in fact $A = \operatorname{Rad}(A)$. If $A_1$ is the unital Banach algebra obtained as the result of formally adjoining a unit element to $A$, then it's easy to see that $\mathcal{F}_1 = \emptyset$ but $\mathcal{H}_1 \neq \emptyset$ in $A_1$.

THEOREM 2.10. *Let $A$ be a semiprime Banach algebra such that $A$ is not semisimple and suppose $\mathcal{F}_1 \neq \emptyset$. Then*

(i) $\overline{\mathcal{F}}_1 \cap \mathrm{Rad}\,(A) = \{0\}$.

(ii) $\overline{\mathcal{F}}_1 \cdot \mathrm{Rad}\,(A) = \mathrm{Rad}\,(A) \cdot \overline{\mathcal{F}}_1 = \{0\}$.

(iii) $\mathcal{F}_1$ *and* $\mathrm{Rad}\,(A) \setminus \{0\}$ *are closed in* $\mathcal{G}_1$ *and hence* $\mathcal{H}_1$ *is open in* $\mathcal{G}_1$ *with* $\mathcal{H}_1 \neq \emptyset$.

(iv) $\mathrm{Rad}\,(A) \setminus \{0\} + \mathcal{F}_1 \subseteq \mathcal{H}_1$.

(v) $\mathcal{H}_1$ *is dense in* $\mathcal{G}_1$.

*Proof.* (i) We already have $\mathcal{F}_1 \cap \mathrm{Rad}\,(A) = \emptyset$. If $(u_n)$ is a sequence in $\mathcal{F}_1$ converging to an element $u \in \mathrm{Rad}\,(A)$, then, since $A$ is semiprime, we can find a sequence of idempotents converging to a radical element. But this contradicts the continuity of the spectrum on $\mathrm{Rad}\,(A)$, hence $\overline{\mathcal{F}}_1 \cap \mathrm{Rad}\,(A) = \emptyset$.

(ii) It suffices to prove $\mathcal{F}_1 \cdot \mathrm{Rad}\,(A) = \{0\}$. Let $u \in \mathcal{F}_1$ and $r \in \mathrm{Rad}\,(A)$, $r \neq 0$. If $ur \neq 0$ then $ur \in \mathcal{F}_1$ and $ur \in \mathrm{Rad}\,(A)$ which is impossible by (i).

(iii) Obviously $\mathrm{Rad}\,(A) \setminus \{0\}$ is closed in $\mathcal{G}_1$. Let $(u_n)$ be a sequence in $\mathcal{F}_1$ such that $u_n \to u \in \mathcal{G}_1$. It follows from (ii) that $u \notin \mathrm{Rad}\,(A)$. So if $u \notin \mathcal{F}_1$ then $u \in \mathcal{H}_1$ and by the comments following Lemma 2.8 there are $x_0 \in A$ and $\lambda_0 \in \mathbb{C}$ such that $ux_0u - \lambda_0 u = r \in \mathrm{Rad}\,(A)$, $r \neq 0$. Now $u_n x_0 u_n = \lambda_n u_n$ where $(\lambda_n)$ is a convergent sequence in $\mathbb{C}$, say $\lambda_n \to \lambda \in \mathbb{C}$. It follows that $\lambda u - \lambda_0 u = r$. Since $r \neq 0$, $\lambda \neq \lambda_0$ which means $u \in \mathrm{Rad}\,(A)$. But this is impossible by (i) so $\mathcal{F}_1$ is closed in $\mathcal{G}_1$. Clearly we have now that $\mathcal{H}_1$ is open in $\mathcal{G}_1$ and moreover, if $\mathcal{H}_1 = \emptyset$ then this would contradict Proposition 2.9, so $\mathcal{H}_1 \neq \emptyset$.

(iv) Let $r \in \mathrm{Rad}\,(A) \setminus \{0\}$ and $u \in \mathcal{F}_1$. Now $u + r \in \mathcal{G}_1$ ([15], p. 3) and certainly $u + r \notin \mathrm{Rad}\,(A)$. If $u + r \in \mathcal{F}_1$ then, (using (ii)), given any $x \in A$ there is $\lambda$ such that $uxu + rxr = \lambda u + \lambda r$. But corresponding to $x$ there is $\lambda_x \in \mathbb{C}$ such that $uxu = \lambda_x u$. If $\lambda_x \neq \lambda$ then $u \in \mathrm{Rad}\,(A)$ which is impossible and if $\lambda_x = \lambda$ then $rxr = \lambda r$. But if the latter instance occurs for every $x \in A$ then we would have $r \in \mathcal{F}_1$ which is absurd. So we conclude $u + r \in \mathcal{H}_1$.

(v) If $u \in \mathcal{F}_1$ and $r \in \mathrm{Rad}\,(A) \setminus \{0\}$, then by (iv) the interior of the line segment joining $u$ and $r$ belongs to $\mathcal{H}_1$. Thus $\mathcal{H}_1$ is dense in $\mathcal{G}_1$. ∎

With respect to Theorem 2.10 it would be interesting to know in which cases (if any) one can have equality in (iv). Also note that if $\mathcal{F}_1 \neq \emptyset$ is replaced by $\mathcal{F}_1 = \emptyset$ in Theorem 2.10, then either $\mathcal{H}_1 = \emptyset$ or $\mathcal{H}_1$ is dense in $\mathcal{G}_1$.

We finally observe that the perturbation theory for $\mathcal{F}_1$ ([8], Theorem 5) carries verbatimly over to $\mathcal{G}_1$.

PROPOSITION 2.11. *If $A$ is semiprime, if $a \in A$ and if $d \in \mathcal{G}_1$, then*

$$acc\, \sigma\, (a + d) \subseteq \sigma\, (a)^{\hat{}}$$

*where $acc\, \sigma\, (a + d)$ denotes the set of accumulation points of $\sigma\, (a + d)$ and $\sigma\, (a)^{\hat{}}$ denotes the polynomially convex hull of $\sigma\, (a)$.*

*Proof.* If $d \in \mathcal{G}_1$ in $A$ then the coset element $\dot{d}$ in $A/\operatorname{Rad}(A)$ belongs to $\mathcal{F}_1$. Since the spectrum is invariant under the canonical homomorphism $x \longmapsto \dot{x}$ the result follows by consideration of ([8], Theorem 5). ∎

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