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On Riesz Homomorphisms in von Neumann Regular *f*-Algebra

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1. Introduction

Over the last decade, the importance of f-algebras in the theory of Riesz spaces has steadily grown. It is only recently that other various lattice-ordered algebraic structures have been getting more attention. For instance, d-algebras, which were not so much studied previously, are the main purpose of this paper.

This paper deals with two main results. In the first we present a result about the connection between Riesz homomorphisms and algebra homomorphisms on f-algebras. Perhaps the most striking theorem in this direction, due to Hager and Robertson [10], is that a Riesz homomorphism between two Archimedean unital f-algebras that preserves identity is an algebra homomorphism. There are many results of this kind in the literature. We mention other results in that direction. It is proven by van Putten in his thesis [16] that the set of all Riesz homomorphisms between two Archimedean unital f-algebras that preserve identity coincides with the set of extreme points of the convex set of all Markov linear operators (i.e., positive linear operators that preserve identity). We also mention the paper of Huijsmans and de Pagter [13], in which the connection between Riesz homomorphisms and algebra homomorphisms on f-algebras is considered in great detail. One of the major results presented in [13] is that a Riesz homomorphism T from an Archimedean f-algebra A with unit element e into an Archimedean semiprime f-algebra B is an algebra homomorphism if and only if Te is idempotent (i.e., $(Te)^2 = Te$) (see [13, Theorem 5.4]).

In the present paper we intend to make some contributions to this area.

It is known that every Riesz space A has a universal completion A^u, i.e., there exists a unique (up to a Riesz homomorphism) universally complete Riesz space A^u so that A can be identified with an order dense Riesz subspace of A^u. Moreover, A^u is furnished with a multiplication, under which A^u is an f-algebra with a unit element (see [1, Section 8]). In the second part of this paper, we present a relationship between structure of a d-algebra and the that of its universal completion. By means of the well-known features of the multiplication of an f-algebra, we aim at obtaining more information about the structure of a d-algebra.

We assume that the reader is familiar with the notion of Riesz spaces (or vector lattices).

2. Preliminaries

For further details of terminology and properties of Riesz spaces and order bounded operators not explained or proved in this paper, we refer the reader to [1, 14, 17], to [2] for elementary ℓ-algebra and d-algebra, and to [1, 17] for f-algebra and orthomorphism theories.

A lattice ordered group (briefly an ℓ-group) A is called *Archimedean* if for each non zero $f \in A$ the set $\{nf : n = \pm 1, \pm 2, \dots\}$ has no upper bound in A. *In order to avoid unnecessary repetition we will assume throughout that all ℓ-groups under consideration are Archimedean.* An ℓ-group A which is simultaneously a ring with the property that $fg \in A^+$ for all $f, g \in A^+$ (equivalently, $|fg| \leq |f||g|$ for all $f, g \in A$), where A^+ is the positive cone of A, is called a *lattice ordered ring* (briefly, an *ℓ-ring*). In addition, if A is a real Riesz space, then A is called an *ℓ-algebra*.

Abstract f-ring theory and f-algebra theory have been studied by many authors (see e.g. [1, 5, 7, 17]). Some of these authors (see e.g. §8 in the paper [7] by Birkhoff and Pierce) define an *f-ring* as a lattice ordered ring with the property that $f \wedge g = 0$ and $h \in A^+$ implies $fh \wedge g = hf \wedge g = 0$. Others (see e.g. Definition 9.1.1 in the book [5] by Bigard, Keimel and Wolfenstein) define an f-ring as a lattice ordered ring which is isomorphic to a subdirect union of totally ordered rings. It is often desirable to have the equivalence of the two definitions available. However, any known equivalence proof is based on arguments using Zorn's lemma. If one uses the second definition, it will be possible to prove a certain number of standard theorems on f-rings by means of the "metamathematical" observation that any identity holding in every totally ordered ring holds in every f-ring.

We adopt in this paper the original Birkhoff-Pierce definition of an *f-ring* (i.e., an ℓ-ring with the additional property that $f \wedge g = 0$ and $h \in A^+$ implies $fh \wedge g = hf \wedge g = 0$) as our starting point. An f-ring A is said to be an f-algebra if A is also a real Riesz space. It was shown by Birkhoff and Pierce in [7, Section 8] that any Archimedean f-ring is commutative, but for a more elementary proof, due to Zaanen, we refer to [17, Theorem 140.10] or [5, Theorem 12.3.2]. If A is an f-ring then A has positive squares and $|fg| = |f||g|$ for all $f, g \in A$. Every f-ring A with identity element is *semiprime* (or *reduced*), that is, 0 is the only nilpotent in A. If f, g are elements in a semiprime f-ring A, then $|f| \wedge |g| = 0$ if and only if $fg = 0$. All f-ring properties listed above are satisfied by any f-algebra. The ℓ-algebra A is called a *d-algebra* whenever $f \wedge g = 0$ and $h \in A^+$ imply $fh \wedge gh = hf \wedge hg = 0$ (equivalently, $|fg| = |f||g|$ for all $f, g \in A$). Any f-algebra is a d-algebra, but not conversely (see [2]). Archimedean d-algebras neither need to be commutative nor have positive squares (see [2]).

The relatively uniform topology on Riesz spaces plays a key role in the context of this work. Let us therefore recall the definition and some elementary properties of this topology. By $\mathbb{N}$ we mean the set $\{0, 1, 2, \dots\}$. Let A be an archimedean Riesz space and $u \in A^+$. A sequence $(f_n)_{n\in\mathbb{N}}$ of elements of A is said to *converge u-uniformly* to $f \in A$ whenever, for every $\epsilon > 0$, there exists a natural number N_ϵ such that $|f - f_n| \leq \epsilon u$ for all $n \geq N_\epsilon$. This is denoted by $f_n \to f(u)$. The element u is called the *regulator of convergence*. The sequence $(f_n)_{n\in\mathbb{N}}$ is said to *converge relatively uniformly* to $f \in A$ if $f_n \to f(u)$ for some $u \in A^+$. We shall write $f_n \to f\,(\text{r.u.})$ if we do not want to specify the regulator. Relatively uniform limits are unique if and only if A is Archimedean [14, Theorem 63.2].

The non empty subset D of A is called *relatively uniformly closed* whenever it follows that if $(f_n)_{n\in\mathbb{N}} \in D$ and $f_n \to f\,(\text{r.u.})$ then $f \in D$. We emphasize that the regulator does not need to be an element of D. The relatively uniformly closed subsets are the closed sets for a topology in A, *the relatively uniform topology*.

The notion of relatively uniform Cauchy sequence is defined in the obvious way. The Archimedean Riesz space A is called *relatively uniformly complete* whenever every relatively uniform Cauchy sequence has a (unique) limit. We refer to [14] for the relatively uniform topology. In the end of this paragraph, we recall an important fact about unital f-algebras. Let A be an Archimedean f-algebra with unit element $e > 0$. For every $0 \leq f \in A$, the increasing sequence $0 \leq f_n = f \wedge ne$ converges relatively uniformly to f in A (for details on this see, e.g., [1, Theorem 8.22]).

Let A and B be Riesz spaces. The operator $\pi : A \to B$ is called *order bounded* if the image under π of an order bounded set in A is again an order bounded set in B. The operator π is called *positive* if $\pi(A^+) \subset B^+$. The operator π is called *Riesz homomorphism* (or *lattice homomorphism*) whenever $f \wedge g = 0$ implies $\pi(f) \wedge \pi(g) = 0$. Obviously, every Riesz homomorphism is positive. The order bounded operator $\pi : A \to A$ is called *orthomorphism* if $|f| \wedge |g| = 0$ implies $|\pi(f)| \wedge |g| = 0$. The collection $\mathrm{Orth}(A)$ of all orthomorphisms on A is, with respect to the usual Riesz spaces and composition as multiplication, an Archimedean f-algebra with the identity mapping I_A on A as a unit element (for details on this see, e.g., [1, Theorem 8.24]). Every orthomorphism π of A is order continuous [1, Theorem 8.10]. If A is supposed to be, in addition, an f-algebra, then for every $f \in A$ the map π_f, defined by $\pi_f(g) = fg$ for all $g \in A$, is an orthomorphism of A. Furthermore, if A is an f-algebra with unit element, then the mapping $f \to \pi_f$ from A into $\mathrm{Orth}(A)$ is a Riesz and algebra isomorphism. Therefore, for every $\pi \in \mathrm{Orth}(A)$ there exists a unique element $f \in A$ such that $\pi = \pi_f$ and π is a positive orthomorphism if and only if $f \geq 0$ (for details on this see, e.g., [1, Theorem 8.27]).

A Dedekind complete Riesz space is called *universally complete* whenever every set of pairwise disjoint positive elements has a supremum. Every Riesz space A has a *universally completion* A^u, i.e., there exists a unique (up to a Riesz homomorphism) universally complete (and therefore Dedekind complete) Riesz space A^u so that A can be identified with an order dense Riesz subspace of A^u. Moreover, A^u is furnished with a multiplication, under which A^u is an f-algebra with unit element.

An ℓ-algebra A is called *von Neumann regular* if for every $f \in A$, there exists $g \in A$ such that $f = f^2 g$. It is proved in [15] that A^u is a von Neumann regular f-algebra. More about von Neumann regular algebras can be found in [12, 15].

Let B be a commutative ℓ-algebra with positive squares, A an Archimedean ℓ-algebra, and T a positive operator from B into A. We say that the Cauchy-Schwarz inequality is valid in A if

$$T(fg)^2 \leq T(f^2)T(g^2)$$

holds in A for all $f, g \in B$. The Cauchy-Schwarz inequality has been established for Archimedean f-algebras. We give a short historical account.

In 1986, Huijsmans and de Pagter proved the Cauchy-Schwarz inequality in Archimedean semiprime f-algebras [11]. Some years after, Bernau and

Huijsmans in [3] generalized the Cauchy-Schwarz inequality to the case where A is an arbitrary Archimedean f-algebra.

Let A be a Riesz space and $f \in A$. We denote by $\{f\}$ the principal band generated by f, and by $\{f\}^d$ its disjoint complement.

We end this section with an important proposition, which turns out to be useful for later purposes.

PROPOSITION 1. *Let A be an Archimedean von Neumann regular f-algebra and $f \in A^+$. Then there exists $\xi \in A^+$ such that the principal band $\{f\}$ of A generated by f is a sub-f-algebra of A with ξf as its unit element.*

Proof. Let $\xi \in A$ such that $f = \xi f^2$. Obviously, the band $\{f\}$ in A is an f-algebra on its own. We claim that $f\xi$ is the unit element of $\{f\}$. Indeed, if $g \in \{f\}$, then the fact that $\{f\}$ is a ring ideal in A implies that $(g - g\xi f) \in \{f\}$ for all $g \in \{f\}$. On the other hand, $f(g - g\xi f) = 0$ and thus $(g - g\xi f) \in \{f\}^d$ (because A is semiprime). Consequently, $g = g\xi f$ for all $g \in \{f\}$. Therefore, ξf is the unit element of $\{f\}$ and we are done. ∎

3. A relationship between Riesz homomorphisms and algebra homomorphisms

The next result is a generalization of a theorem due to Huijsmans and de Pagter about the connection between Riesz homomorphisms and algebra homomorphisms of f-algebras. The theorem in question is that if A is an Archimedean f-algebra with unit element e and B is an Archimedean semiprime f-algebra, then the Riesz homomorphism T from A into B is an algebra homomorphism if and only if $(Te)^2 = Te$ (see [13, Theorem 5.4]). We give a generalization of this result.

THEOREM 1. *Let A be an Archimedean f-algebra with unit element e, B an Archimedean von Neumann regular f-algebra, and $T : A \to B$ a Riesz homomorphism. Then there exists $\xi \in B^+$ such that*

$$T(fg) = \xi T(f)T(g) \qquad \textit{for all } f, g \in A\,.$$

Proof. In order to prove this theorem, it is obviously sufficient to show that there exists $\xi \in B^+$ such that

$$T(f2) = \xi(Tf)^2 \qquad \text{for all } f \in A^+\,.$$

First, suppose that $0 \leq f \leq e$. Since B is a von Neumann regular f-algebra, there exists $\xi \in B^+$ such that $Te = \xi(Te)^2$. It results from the preceding proposition that $\{Te\}$ is a sub-f-algebra of B with ξTe as its unit element.

As shown in Corollary 3.3(i) in [13], the elements u_n defined by

$$u_n = \sup\left\{2\alpha f - \alpha^2 e \,:\, \alpha = \frac{k}{n},\, k \in \{0, 1, \ldots, n\}\right\}$$

satisfy

$$0 \leq f^2 - u_n \leq \frac{1}{n^2} e\,,$$

hence

$$0 \leq T(f^2) - Tu_n \leq \frac{1}{n^2} Te \qquad (n \in \{1, 2, \ldots\})\,.$$

So, the sequence $(Tu_n)_n$ converges uniformly to $T(f^2)$ and thus $\xi Tu_n \to \xi T(f^2)$ (r.u.). On the other hand T is a Riesz homomorphism, so we have

$$Tu_n = \sup\left\{2\alpha Tf - \alpha^2 Te \,:\, \alpha = \frac{k}{n},\, k \in \{0, 1, \ldots, n\}\right\},$$

hence

$$\xi Tu_n = \sup\left\{2\alpha \xi Tf - \alpha^2 \xi Te \,:\, \alpha = \frac{k}{n},\, k \in \{0, 1, \ldots, n\}\right\}. \tag{3.1}$$

Now, we can prove that $T(A) \subset \{Te\}$. For any $g \in A^+$, it follows that $Tg \wedge nTe \to Tg$ (r.u.) since $g \wedge ne \to g$ (r.u.) which implies that $Tg \in \{Te\}$, hence $T(A) \subset \{Te\}$. Since $\{Te\}$ is an archimedean f-algebra with unit ξTe, and $T(A) \subset \{Te\}$, then by applying Corollary 3.3(i) in [13] to (3.1) we have

$$0 \leq (\xi Tf)^2 - \xi Tu_n \leq \frac{1}{n^2} \xi Te\,,$$

so

$$\xi Tu_n \to (\xi Tf)^2 \text{ (r.u.)}\,.$$

We conclude by uniqueness of relatively uniform limits that

$$\xi T(f^2) = (\xi Tf)^2\,.$$

Therefore $\xi T(f^2) = \xi^2 (Tf)^2$. Since $T(f^2), (Tf)^2 \in \{Te\}$, multiplying by Te the preceding equality we obtain $T(f^2) = \xi^2 Te (Tf)^2 = \xi (Tf)^2$, so

$$T(f^2) = \xi (Tf)^2\,.$$

Now, let $f \in A^+$. From $f \wedge ne \to f\,(\text{r.u.})$ it follows by an easy limit process that $T(f^2) = \xi(Tf)^2$, which implies

$$T(f^2) = \xi(Tf)^2 \qquad \text{for all } f \in A\,.$$

Let $f, g \in A$. From $fg = \frac{(f+g)^2-(f-g)^2}{4}$ it follows that

$$T(fg) = \xi TfTg\,.$$

This is the desired result. ∎

As an immediate application of the theorem above we obtain the following result.

Corollary 1. *Let A be an Archimedean unital f-algebra, B an Archimedean vector lattice, and $T : A \to B$ a Riesz homomorphism. The range $T(A)$ of T, which is a Riesz subspace of B, is an f-algebra with respect to the multiplication $*$ defined in $T(A)$ by*

$$Tf * Tg = T(fg)$$

for all $f, g \in A$. Moreover, if e is the unit element of A, then Te is the unit element of $T(A)$.

Proof. We have already mentioned in the preliminaries that B^u is provided with a multiplication, under which B^u is an Archimedean von Neumann regular f-algebra with unit element. This multiplication will be denoted again by juxtaposition.

Let $u, v, w \in T(A)$ such that $u \wedge v = 0$ and $w \geq 0$. Since T is a Riesz homomorphism, there exists $0 \leq f, g, h \in A$ such that $u = Tf$, $v = Tg$, and $w = Th$. On the other hand, from the preceding theorem there exists $\xi \in (B^u)^+$ such that $T(hf) = \xi T(h)T(f)$. Now from $u \wedge v = 0$, it follows that $Tf \wedge Tg = 0$ and, in view of the fact that B^u is an f-algebra, $\xi T(h)T(f) \wedge Tg = 0$. So, we can write

$$T(hf) \wedge Tg = T(fh) \wedge Tg = 0\,.$$

Thus

$$(w * u) \wedge v = (u * w) \wedge v = 0.$$

We deduce that $T(A)$ is an f-algebra. Suppose now that e is the unit element of A. So $Tf * Te = T(fe) = Tf$ for all $f \in A$; it follows that Te is the unit element of $T(A)$ and the proof of the corollary is complete. ∎

4. Some results in Archimedean d-algebras

Let A be an Archimedean vector lattice with universal completion A^u. We have already mentioned in the preliminaries that A^u is provided with a multiplication, under which A^u is an f-algebra with unit element. This multiplication will be denoted by juxtaposition.

Assume now that A is a d-algebra. The main topic of the following result is to establish a relationship between the structure of d-algebra on A and the structure of f-algebra in A^u.

The principal order ideal generated by $0 < e \in A$ is denoted by A_e. Moreover,

$$A_e = \{f \in A : \exists \lambda \in \mathbb{R} \text{ with } |f| \leq \lambda e\}.$$

Theorem 2. *Let A be an Archimedean Riesz space, assume that A is a d-algebra under $*$, and let $0 < e \in A$. Then, there exists $\xi \in A^u$ such that*

$$f * g = \xi(f * e)(e * g) \qquad \textit{for all } f, g \in A_e\,.$$

Proof. We can assume without loss of generality that $0 \leq g \leq e$. Let

$$T(f) = f * e \qquad \text{and} \qquad S(f) = f * g\,.$$

So $T : A \to A^u$ is a Riesz homomorphism and $0 \leq S \leq T$. So it follows from [1, Theorem 8.16] that there exists a positive orthomorphism $R \in \mathrm{Orth}(A^u)$ satisfying $S = RT$. Hence, A^u is an f-algebra with unit element, so there exists $\omega \in A^u$ such that

$$R(f) = \omega f \qquad \text{for all } f \in A^u\,.$$

This implies that $S(f) = \omega T(f)$ for all $f \in A$. So

$$f * g = \omega(f * e) \qquad \text{for all } f \in A\,.$$

Observe now that for $f \in A_e$, $f * e \in \{Te\}$. Then we can assume that $\omega \in \{Te\}$.

Now, by Proposition 1, there exists $0 < \xi \in A^u$ such that $\xi T(e)$ is the unit element of $\{Te\}$. Since $S(e) = \omega T(e)$, we have $S(e)\xi = \omega$. Therefore

$$f * g = \xi(e * g)(f * e) \qquad \text{for all } f \in A_e\,.$$

It is an easy task to verify that

$$f * g = \xi(e * g)(f * e) \qquad \text{for all } f, g \in A_e\,.$$

The proof is complete. ∎

We end this paper by showing an inequality in an Archimedean d-algebra which replaced the Cauchy-Schwarz inequality. Note that the Cauchy-Schwarz inequality fails for Archimedean d-algebra as it is shown in the following example.

EXAMPLE 1. Consider $A = \mathbb{R}^2$ with the coordinatewise partial ordering, for which A is an Archimedean Riesz space. Equip A with the following multiplication (see [3, Example 2.8]):

$$\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} = \begin{pmatrix}\alpha_1\beta_1\\ \alpha_1\beta_2\end{pmatrix}.$$

Then A becomes a d-algebra. For

$$f = \begin{pmatrix}1\\ 4\end{pmatrix}, \quad g = \begin{pmatrix}2\\ 2\end{pmatrix}, \quad h = \begin{pmatrix}4\\ 1\end{pmatrix}$$

we have

$$\lambda^2 f + 2\lambda g + h = \begin{pmatrix}(\lambda+2)^2\\ (2\lambda+1)^2\end{pmatrix} \geq 0$$

for all $\lambda \in \mathbb{R}$, but

$$g^2 = \begin{pmatrix}4\\ 4\end{pmatrix}, \quad fh = \begin{pmatrix}4\\ 1\end{pmatrix},$$

so $g^2 \leq fh$ does not hold.

PROPOSITION 2. (NEGATIVE DISCRIMINANT) *Let A be an Archimedean Riesz space, assume that A is a d-algebra under $*$ and let $f, g, h \in A$ such that*

$$\lambda^2 f + 2\lambda g + h \geq 0 \qquad \text{for all } \lambda \in \mathbb{R}.$$

Then

$$g * g \leq \frac{f * h + h * f}{2}.$$

Proof. Observe that if we take $\lambda = 0$, we find $h \geq 0$. For $\lambda > 0$ and dividing $\lambda^2 f + 2\lambda g + h \geq 0$ by λ^2 we get

$$f + \frac{2}{\lambda} g + \frac{1}{\lambda^2} h \geq 0.$$

Let $\lambda \to +\infty$ and use the Archimedean property to obtain $f \geq 0$. Put

$$e = f + |g| + h.$$

Using the preceding theorem, there exists $0 \leq \xi \in A^u$ such that

$$f * h = \xi(e * h)(f * e),$$
$$h * f = \xi(e * f)(h * e),$$
$$g * g = \xi(e * g)(g * e).$$

On the other hand, A is a Riesz subspace of A^u. Then $\lambda^2 f + 2\lambda g + h \geq 0$ in A^u for all $\lambda \in \mathbb{R}$, which implies that

$$\lambda^2(f * e) + 2\lambda(g * e) + (h * e) \geq 0,$$
$$\lambda^2(e * f) + 2\lambda(e * g) + (e * h) \geq 0$$

in A^u for all $\lambda \in \mathbb{R}$. Since A^u is an f-algebra, so by the Cauchy-Schwarz inequality, we get

$$(g * e)^2 \leq (f * e)(h * e),$$
$$(e * g)^2 \leq (e * f)(e * h).$$

Multiplying the above inequalities we obtain

$$(g * e)^2(e * g)^2 \leq (f * e)(h * e)(e * f)(e * h);$$

multiplying by ξ^2 the two members of the obtained inequality, it comes that

$$(\xi(g * e)(e * g))^2 \leq \xi(f * e)(e * h)\xi(e * f)(h * e);$$

therefore

$$(g * g)^2 \leq (f * h)(h * f).$$

It is not hard to prove that

$$(f * h)(h * f) \leq \left(\frac{(f * h) + (h * f)}{2}\right)^2;$$

therefore

$$(g * g)^2 \leq \left(\frac{(f * h) + (h * f)}{2}\right)^2.$$

Since A^u is a semiprime f-algebra, then we get

$$|g * g| \leq \left|\frac{(f * h) + (h * f)}{2}\right|$$

and consequently

$$g * g \leq \frac{(f * h) + (h * f)}{2}.$$

This is the desired result. ∎

Standard arguments lead us to the Cauchy-Schwarz inequality in Archimedean d-algebras.

Theorem 3. *Let B be a commutative ℓ-algebra with positive squares, A an Archimedean d-algebra, and T a positive operator from B into A. Then*

$$T(fg)^2 \leq \frac{1}{2}\Big(T(f^2)T(g^2) + T(g^2)T(f^2)\Big)$$

holds in A for all $f, g \in B$.

Proof. Let $\lambda \in \mathbb{R}$ and $f, g \in B$. Since squares in B are positive, we get

$$T((\lambda f + g)^2) = \lambda^2 T(f^2) + 2\lambda T(fg) + T(g^2) \geq 0$$

in A. According to the preceding proposition, the inequality

$$T(fg)^2 \leq \frac{1}{2}\Big(T(f^2)T(g^2) + T(g^2)T(f^2)\Big)$$

holds in A, which completes the proof. ∎

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