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Quadratic Estimation from Non-Independent Uncertain Observations with Coloured Noise*

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1. Introduction

An usual hypothesis to address the estimation problem in linear stochastic systems is that the involved processes are gaussian; in this case, the least-squares (LS) estimator of the signal is a linear function of the observations and it can be easily obtained as the LS linear estimator. However, there exist many practical situations where this gaussianity assumption is not realistic and the LS estimator is not easily obtainable. This difficulty motivates the necessity of looking for suboptimal estimators which improve the extensively used linear ones. In this context, some authors as De Santis et al. [5] and Carraveta et al. [3] have focused the study of the estimation problem in non-gaussian systems on the search of polynomial estimators. Under a state-space approach, a recursive algorithm for the LS second-order filter is derived in [5] and generalized in [3] to an algorithm for the LS polynomial filter with arbitrary degree; this estimation theory has been used, among others, by Dalla-Mora et al. [4] for restoration of an image corrupted by additive non-gaussian noise. In the last years, some other signal processing problems have

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been successfully solved by using polynomial estimators (see, for example, [1] and [12]), and this fact justify the study of these estimators.

Systems with uncertain observations are characterized by including an observation multiplicative noise described by a sequence of Bernoulli random variables whose values -one or zero- indicate the presence or absence of signal in the observation, respectively. These systems constitute an appropriate model for analyzing those situations where the observation may not contain the signal to be estimated, in which case it consists only of noise (for example, situations of fading or reflection of transmitted signals from the ionosphere).

Due to the multiplicative noise component, even if the additive noises are gaussian, the optimal estimators are not easily derived and the estimation problem in systems with uncertain observations must be addressed under a suboptimal approach. Particularly, linear and polynomial estimation problems from uncertain observations have been treated by several authors, as NaNacara and Yaz [11] and Caballero et al. [2], for different state-space models.

Nevertheless, usually the state-space model for the signal to be estimated is unavailable and only covariance information about the processes involved in the observation equation is often accessible. The linear estimation problem using that information has been considered in Nakamori et al. [8] under the assumption that the uncertainty in the observations is modeled by independent Bernoulli variables; however, there exist many situations, such as data transmission in multichannel systems, in which this independence assumption is not appropriate.

Under the assumption of non-independence of the Bernoulli variables modelling the uncertainty and using the state-space model, Hadidi and Schwartz [6] studied the LS linear estimation problem; they proved that the linear estimators are not recursive in general, and they found a necessary and sufficient condition for the recursivity. Under this condition, without using the state-space model but only covariance information, recursive algorithms for the linear estimation problem are obtained in Nakamori et al. [9] considering that the observations are perturbed by additive white plus coloured noises; in Nakamori et al. [10] non-independent uncertain observations perturbed only by white noise are considered and algorithms for the quadratic estimators, which improve the linear ones, are proposed.

In this paper the results in [9] and [10] are generalized; specifically, we address the LS quadratic filtering and fixed-point smoothing problems from uncertain observations perturbed by white and coloured noises, when the uncertainty is modeled by non-independent Bernoulli random variables and the condition in [6] on the conditional probability matrices of these variables is satisfied. Apart from the $(2,2)$-element of these matrices and the probability that the signal exists in the observations, only the moments up to the fourth one of the signal and additive noises (in a specific form which is satisfied for a general kind of processes) are required for the estimation. The technique proposed by De Santis [5], consisting in augmenting the signal and observation vectors with their second-order Kronecker powers, is used to obtain the quadratic estimators.

2. Hypotheses on the model

Let $z(k)$ and $y(k)$ be $n \times 1$ vectors which describe the signal and the observation at time k, respectively. Let us suppose that the observation equation is given by

$$y(k) = U(k)z(k) + v(k) + v_0(k) \tag{1}$$

where the additive noises, $\{v(k);\ k \geq 0\}$ and $\{v_0(k);\ k \geq 0\}$, are white and coloured sequences, respectively, and the multiplicative noise, $\{U(k);\ k \geq 0\}$, is a sequence of Bernoulli random variables with $P[U(k) = 1] = p(k) \neq 0$. Hence, in each instant of time k, the observation $y(k)$ may not contain the signal ($U(k) = 0$), in which case it will only consist of noise. The probability $1 - p(k)$ that the observation at time k is only noise is named *false alarm probability.*

We assume the following hypotheses on the signal and the noise processes of equation (1):

(I) The signal process $\{z(k);\ k \geq 0\}$ has zero mean and its autocovariance function, $K_z(k,s) = E[z(k)z^T(s)]$, as well as the autocovariance function of its second-order powers,

$$K_{z^{[2]}}(k,s) = E\Big[(z^{[2]}(k) - E[z^{[2]}(k)])(z^{[2]}(s) - E[z^{[2]}(s)])^T\Big],$$

($z^{[2]}(k) = z(k) \otimes z(k)$, where $\otimes$ denotes the Kronecker product [7]), are expressed in a semi-degenerate kernel form,

$$K_z(k,s) = \begin{cases} A(k)B^T(s), & 0 \leq s \leq k, \\ B(k)A^T(s), & 0 \leq k \leq s, \end{cases}$$

$$K_{z^{[2]}}(k,s) = \begin{cases} a(k)b^T(s), & 0 \leq s \leq k, \\ b(k)a^T(s), & 0 \leq k \leq s, \end{cases}$$

where the $n \times M'$ matrix functions A, B and the $n^2 \times L'$ matrix functions a, b are known.
Moreover, let us suppose that the covariance of the signal and its second-order powers, $K_{zz^{[2]}}(k,s) = E\left[z(k)z^{[2]T}(s)\right]$, can be also expressed as

$$K_{zz^{[2]}}(k,s) = \begin{cases} c_1(k)c_2^T(s), & 0 \le s \le k, \\ d_1(k)d_2^T(s), & 0 \le k \le s, \end{cases}$$

where c_1, c_2, d_1 and d_2 are $n \times N'$, $n^2 \times N'$, $n \times P'$ and $n^2 \times P'$ known matrix functions, respectively.
(II) The noise process $\{v(k);\ k \ge 0\}$ is a zero-mean white sequence and its moments, up to the fourth one, are also known and will be denoted as follows

$$R_v(k) = E[v(k)v^T(k)], \quad R_{vv^{[2]}}(k) = E[v(k)v^{[2]T}(k)],$$

$$R_{v^{[2]}}(k) = E\Big[(v^{[2]}(k) - E[v^{[2]}(k)])(v^{[2]}(k) - E[v^{[2]}(k)])^T\Big].$$

(III) The coloured noise $\{v_0(k);\ k \ge 0\}$ has zero mean and its autocovariance function, $K_{v_0}(k,s) = E[v_0(k)v_0^T(s)]$, as well as the autocovariance function of their second-order powers,

$$K_{v_0^{[2]}}(k,s) = E\Big[(v_0^{[2]}(k) - E[v_0^{[2]}(k)])(v_0^{[2]}(s) - E[v_0^{[2]}(s)])^T\Big],$$

are expressed in a semi-degenerate kernel form,

$$K_{v_0}(k,s) = \begin{cases} \alpha(k)\beta^T(s), & 0 \le s \le k, \\ \beta(k)\alpha^T(s), & 0 \le k \le s, \end{cases}$$

$$K_{v_0^{[2]}}(k,s) = \begin{cases} \gamma(k)\delta^T(s), & 0 \le s \le k, \\ \delta(k)\gamma^T(s), & 0 \le k \le s, \end{cases}$$

where the $n \times M''$ matrix functions α, β and the $n^2 \times L''$ matrix functions γ, δ are known.

Also, the covariance function of the coloured noise and its second-order powers, $K_{v_0v_0^{[2]}}(k,s) = E\left[v_0(k)v_0^{[2]T}(s)\right]$, can be expressed in a similar way, namely,

$$K_{v_0v_0^{[2]}}(k,s) = \begin{cases} \epsilon_1(k)\epsilon_2^T(s), & 0 \le s \le k, \\ \rho_1(k)\rho_2^T(s), & 0 \le k \le s, \end{cases}$$

where ϵ_1, ϵ_2, ρ_1 and ρ_2 are $n \times N''$, $n^2 \times N''$, $n \times P''$ and $n^2 \times P''$ known matrix functions, respectively.

(IV) The multiplicative noise $\{U(k);\ k \geq 0\}$, is a sequence of Bernoulli variables with conditional probability matrix $P(k/j)$. As in [6], we assume that the $(2,2)$-element of the conditional probability matrix,

$$[P(k/j)]_{2,2} = P\left[U(k) = 1/U(j) = 1\right] = P_{2,2}(k),$$

is independent of j, for $j < k$.
(V) The signal process, $\{z(k);\ k \geq 0\}$, and the noise processes, $\{U(k);\ k \geq 0\}$, $\{v(k);\ k \geq 0\}$ and $\{v_0(k);\ k \geq 0\}$, are mutually independent.

3. Quadratic estimation problem

Our aim is to obtain the LS quadratic estimator of $z(k)$ based on the observations up to the instant L $(L \geq k)$. By defining the random vectors $y^{[2]}(i) = y(i) \otimes y(i)$, and since $E\left[y^{[2]T}(i)y^{[2]}(i)\right] < \infty$, this estimator is the orthogonal projection of $z(k)$ on the space of n-dimensional linear transformations of $y(1), \ldots, y(L)$ and $y^{[2]}(1), \ldots, y^{[2]}(L)$. In order to treat this problem, let us define the augmented signal and observation vectors by aggregating to the original vectors, $z(k)$ and $y(k)$, their second-order powers $z^{[2]}(k)$ and $y^{[2]}(k)$, that is,

$$\mathcal{Z}(k) = \begin{pmatrix} z(k) \\ z^{[2]}(k) \end{pmatrix}, \quad \mathcal{Y}(k) = \begin{pmatrix} y(k) \\ y^{[2]}(k) \end{pmatrix}.$$

Then, the vector constituted by the first n entries of the linear estimator of $\mathcal{Z}(k)$ based on $\mathcal{Y}(1), \ldots, \mathcal{Y}(L)$ provides the quadratic estimator of $z(k)$.

Now, we analyze the properties of the random vectors $\mathcal{Z}(k)$ and $\mathcal{Y}(k)$ which will be utilized to obtain the LS linear estimator of $\mathcal{Z}(k)$. In order to study the properties of $\mathcal{Y}(k)$ we need to obtain an appropriate expression for $y^{[2]}(k)$. By employing the Kronecker product properties [7] and taking into account that $U(k) = U^2(k)$ since $U(k)$ takes the values 0 and 1, the following expression is obtained

$$y^{[2]}(k) = U(k)z^{[2]}(k) + f(k) + f_0(k)$$

with

$$f(k) = (I_{n^2} + K_{n^2})\left[\Big(U(k)z(k) + v_0(k)\Big) \otimes v(k)\right] + v^{[2]}(k) \tag{2}$$

$$f_0(k) = U(k)(I_{n^2} + K_{n^2})(z(k) \otimes v_0(k)) + v_0^{[2]}(k) \tag{3}$$

where I_{n^2} is the $n^2 \times n^2$ identity matrix and K_{n^2} is the $n^2 \times n^2$ commutation matrix, which satisfies $K_{n^2}(z(k) \otimes v(k)) = v(k) \otimes z(k)$.

Then, by denoting

$$Z(k) = \mathcal{Z}(k) - E[\mathcal{Z}(k)], \qquad V(k) = \mathcal{V}(k) - E[\mathcal{V}(k)],$$

$$V_0(k) = [U(k) - p(k)]\, E[\mathcal{Z}(k)] + \mathcal{V}_0(k) - E[\mathcal{V}_0(k)],$$

with

$$\mathcal{V}(k) = \begin{pmatrix} v(k) \\ f(k) \end{pmatrix}, \quad \mathcal{V}_0(k) = \begin{pmatrix} v_0(k) \\ f_0(k) \end{pmatrix}$$

the vectors $Y(k) = \mathcal{Y}(k) - E[\mathcal{Y}(k)]$ satisfy

$$Y(k) = U(k)Z(k) + V(k) + V_0(k). \tag{4}$$

Note that, in view of their own definition, the processes $\{Z(k);\ k \geq 0\}$, $\{V(k);\ k \geq 0\}$ and $\{V_0(k);\ k \geq 0\}$ involved in the above equation (4) all have zero-mean. In the following propositions other statistical properties of these processes are established.

PROPOSITION 1. *Let us suppose that hypotheses* (I)-(V) *are satisfied. Then the autocovariance function of the zero-mean process* $\{Z(k);\ k \geq 0\}$ *is expressed in a semi-degenerate kernel form, specifically*

$$K_Z(k,s) = E[Z(k)Z^T(s)] = \begin{cases} \mathcal{A}(k)\mathcal{B}^T(s), & 0 \leq s \leq k, \\ \mathcal{B}(k)\mathcal{A}^T(s), & 0 \leq k \leq s, \end{cases}$$

being

$$\mathcal{A}(k) = \begin{pmatrix} A(k) & c_1(k) & 0_{n\times P'} & 0_{n\times L'} \\ 0_{n^2\times M'} & 0_{n^2\times N'} & d_2(k) & a(k) \end{pmatrix}$$

$$\mathcal{B}(k) = \begin{pmatrix} B(k) & 0_{n\times N'} & d_1(k) & 0_{n\times L'} \\ 0_{n^2\times M'} & c_2(k) & 0_{n^2\times P'} & b(k) \end{pmatrix}.$$

Moreover, the process $\{Z(k);\ k \geq 0\}$ *is independent of the multiplicative noise* $\{U(k);\ k \geq 0\}$.

Proof. Nakamori et al. [10]. ∎

PROPOSITION 2. *Under hypotheses* (I)-(V), *the zero-mean additive noise* $\{V(k);\ k \geq 0\}$ *of equation* (4) *is a sequence of mutually uncorrelated random vectors with covariance matrices given by*

$$R_V(k) = E[V(k)V^T(k)] = \begin{pmatrix} R_v(k) & R_{vv^{[2]}}(k) \\ R^T_{vv^{[2]}}(k) & R_{22}(k) \end{pmatrix}$$

being

$$R_{22}(k) = (I_{n^2} + K_{n^2})\Big[\Big(p(k)A(k)B^T(k) + \alpha(k)\beta^T(k)\Big) \otimes R_v(k)\Big](I_{n^2} + K_{n^2}) + R_{v^{[2]}}(k). \tag{5}$$

Moreover, $\{V(k);\ k \geq 0\}$ *is uncorrelated with the processes* $\{Z(k);\ k \geq 0\}$ *and* $\{U(k)Z(k);\ k \geq 0\}$.

Proof. Using the independence hypothesis (V) on the model and taking into account that $\{v(k);\ k \geq 0\}$ is a white noise, it can be shown that, for $k \neq s$, $E\left[V(k)V^T(s)\right] = 0$. For $k = s$,

$$R_V(k) = \begin{pmatrix} R_v(k) & R_{vv^{[2]}}(k) \\ R^T_{vv^{[2]}}(k) & R_{22}(k) \end{pmatrix}$$

with $R_{22}(k) = E\left[\left(f(k) - E\left[f(k)\right]\right)\left(f(k) - E\left[f(k)\right]\right)^T\right]$. Using the Kronecker product properties, again the independence hypothesis and expression (2), we obtain that

$$R_{22}(k) = (I_{n^2} + K_{n^2})\Big[\Big(p(k)E\big[z(k)z^T(k)\big] + E\big[v_0(k)v_0^T(k)\big]\Big) \otimes E\left[v(k)v^T(k)\right]\Big](I_{n^2} + K_{n^2}) + R_{v^{[2]}}(k).$$

Since, $E\left[z(k)z^T(k)\right] = A(k)B^T(k)$, $E\left[v_0(k)v_0^T(k)\right] = \alpha(k)\beta^T(k)$ and $E[v(k)v^T(k)] = R_v(k)$, expression (5) is obtained.

The uncorrelation between $\{V(k);\ k \geq 0\}$ and the processes $\{Z(k);\ k \geq 0\}$, $\{U(k)Z(k);\ k \geq 0\}$ is derived in a similar way, by using hypotheses (I)-(V) and employing the Kronecker product properties. ∎

PROPOSITION 3. *Let us suppose that hypotheses* (I)-(V) *are verified. Then the zero-mean process* $\{V_0(k);\ \ k \geq 0\}$ *of equation* (4) *is a coloured noise whose covariance function,* $K_{V_0}(k,s) = E[V_0(k)V_0^T(s)]$, *is given by*

(i) $K_{V_0}(k,s) = \begin{cases} \mathcal{C}(k)\mathcal{D}^T(s), & 0 \le s < k, \\ \mathcal{D}(k)\mathcal{C}^T(s), & 0 \le k < s, \end{cases}$

being

$$\mathcal{C}(k) = (\mathcal{C}_1(k), \mathcal{C}_2(k)), \quad \mathcal{D}(k) = (\mathcal{D}_1(k), \mathcal{D}_2(k)),$$

with

$$\mathcal{C}_1(k) = \Big(P_{2,2}(k) - p(k)\Big) E[\mathcal{Z}(k)],$$
$$\mathcal{D}_1(k) = p(k)E[\mathcal{Z}(k)],$$

where

$$E[\mathcal{Z}(k)] = \begin{pmatrix} 0 \\ vec\left(A(k)B^T(k)\right) \end{pmatrix}$$

and

$$\mathcal{C}_2(k) = \begin{pmatrix} \alpha(k) & \epsilon_1(k) & 0_{n\times P''} & 0_{n\times L''} & 0_{n\times M'M''} \\ 0_{n^2\times M''} & 0_{n^2\times N''} & \rho_2(k) & \gamma(k) & \psi(k) \end{pmatrix},$$

$$\mathcal{D}_2(k) = \begin{pmatrix} \beta(k) & 0_{n\times N''} & \rho_1(k) & 0_{n\times L''} & 0_{n\times M'M''} \\ 0_{n^2\times M''} & \epsilon_2(k) & 0_{n^2\times P''} & \delta(k) & \chi(k) \end{pmatrix},$$

where

$$\psi(k) = P_{2,2}(k)\left(I_{n^2} + K_{n^2}\right)\left(A(k) \otimes \alpha(k)\right),$$
$$\chi(k) = p(k)\left(I_{n^2} + K_{n^2}\right)\left(B(k) \otimes \beta(k)\right).$$

(ii)

$$K_{V_0}(k,k) = p(k)\left(1 - p(k)\right) E[\mathcal{Z}(k)]\, E\left[\mathcal{Z}^T(k)\right] + \begin{pmatrix} \alpha(k)\beta^T(k) & \epsilon_1(k)\epsilon_2^T(k) \\ \epsilon_2(k)\epsilon_1^T(k) & r_{22}(k) \end{pmatrix}$$

with

$$r_{22}(k) = p(k)(I_{n^2} + K_{n^2})\left[A(k)B^T(k) \otimes \alpha(k)\beta^T(k)\right](I_{n^2} + K_{n^2}) + \gamma(k)\delta^T(k).$$

Moreover, $\{V_0(k);\ k \ge 0\}$ *is uncorrelated with the processes* $\{V(k);\ k \ge 0\}$, $\{Z(k);\ k \ge 0\}$ *and* $\{U(k)Z(k);\ k \ge 0\}$.

Proof. (i) For $s < k$, taking into account that, from hypothesis (IV), $E[U(k)U(s)] = P_{2,2}(k)p(s)$, the hypotheses on the model lead to

$$K_{V_0}(k,s) = (P_{2,2}(k) - p(k))p(s)E\left[\mathcal{Z}(k)\right]E\left[\mathcal{Z}^T(s)\right] + \begin{pmatrix} \alpha(k)\beta^T(s) & \epsilon_1(k)\epsilon_2^T(s) \\ \rho_2(k)\rho_1^T(s) & r_{22}(k,s) \end{pmatrix}$$

with $r_{22}(k,s) = E\Big[(f_0(k) - E\left[f_0(k)\right])\left(f_0(s) - E\left[f_0(s)\right]\right)^T\Big]$. Using the Kronecker product properties, again the independence hypothesis and expression (3), we obtain that

$$r_{22}(k) = P_{2,2}(k)p(s)(I_{n^2} + K_{n^2})\Big(A(k)B^T(s) \otimes \alpha(k)\beta^T(s)\Big)(I_{n^2} + K_{n^2}) + \gamma(k)\delta^T(s)$$

and taking into account that

$$A(k)B^T(s) \otimes \alpha(k)\beta^T(s) = \left(A(k) \otimes \alpha(k)\right)\left(B(s) \otimes \beta(s)\right)^T,$$

we obtain

$$r_{22}(k) = \gamma(k)\delta^T(s) + \psi(k)\chi^T(s).$$

Hence, for $s < k$,

$$K_{V_0}(k,s) = \left(P_{2,2}(k) - p(k)\right)p(s)E\left[\mathcal{Z}(k)\right]E\left[\mathcal{Z}^T(s)\right] + \begin{pmatrix} \alpha(k)\beta^T(s) & \epsilon_1(k)\epsilon_2^T(s) \\ \rho_2(k)\rho_1^T(s) & \gamma(k)\delta^T(s) + \psi(k)\chi^T(s) \end{pmatrix}$$

Analogously, for $k < s$,

$$K_{V_0}(k,s) = \left(P_{2,2}(s) - p(s)\right)p(k)E\left[\mathcal{Z}(k)\right]E\left[\mathcal{Z}^T(s)\right] + \begin{pmatrix} \beta(k)\alpha^T(s) & \rho_1(k)\rho_2^T(s) \\ \epsilon_2(k)\epsilon_1^T(s) & \delta(k)\gamma^T(s) + \chi(k)\psi^T(s) \end{pmatrix}$$

Finally, since $u \otimes v = vec(vu^T)$, using hypothesis (I), it is immediate that

$$E\left[\mathcal{Z}(k)\right] = \begin{pmatrix} 0 \\ vec\left(A(k)B^T(k)\right) \end{pmatrix}$$

and (i) is proved.
(ii) The proof of (ii) is similar to the previous one, taking now into account that $E\left[U(k) - p(k)\right]^2 = p(k)(1-p(k))$ and $E\left[U^2(k)\right] = p(k)$.

Finally, as in Proposition 2, the uncorrelation between $\{V_0(k);\ k \geq 0\}$ and the processes $\{V(k);\ k \geq 0\}$, $\{Z(k);\ k \geq 0\}$ and $\{U(k)Z(k);\ k \geq 0\}$ is derived by using hypotheses (I)-(V) and employing the Kronecker product properties. ∎

In view of the properties of the processes involved in equation (4), which have been established in Propositions 1, 2 and 3, the recursive algorithms given in [10] can be applied to obtain the linear filtering and fixed-point smoothing estimators, $\widehat{Z}(k,L)$, $L \geq k$, of the signal $Z(k)$ based on the observations $Y(1),\ldots,Y(L)$. These algorithms, which are presented in the following theorem, allow us to obtain the required quadratic filtering and fixed-point smoothing estimators of the original signal $z(k)$, just by extracting the first n entries of $\widehat{Z}(k,L)$.

THEOREM 1. *The linear fixed-point smoothing estimator,* $\widehat{Z}(k,L)$ *for* $L > k$, *can be recursively obtained from*

$$\begin{aligned}&\widehat{Z}(k,L) = \widehat{Z}(k,L-1) + h(k,L,L)\nu(L)\\ &\nu(L) = Y(L) - P_{2,2}(L)\mathcal{A}(L)O(L-1) - \mathcal{C}(L)Q(L-1)\end{aligned}$$

$$\begin{aligned}O(L) &= O(L-1) + J(L,L)\nu(L), \quad O(0)=0\\ Q(L) &= Q(L-1) + I(L,L)\nu(L), \quad Q(0)=0\end{aligned}$$

being

$$\begin{aligned}J(L,L) &= \left[p(L)\mathcal{B}^T(L) - P_{2,2}(L)r(L-1)\mathcal{A}^T(L) - c(L-1)\mathcal{C}^T(L)\right]\Pi^{-1}(L)\\ I(L,L) &= \left[\mathcal{D}^T(L) - P_{2,2}(L)c^T(L-1)\mathcal{A}^T(L) - d(L-1)\mathcal{C}^T(L)\right]\Pi^{-1}(L)\end{aligned}$$

$$\begin{aligned}\Pi(L) = &\left[p(L)\mathcal{B}(L) - P_{2,2}^2(L)\mathcal{A}(L)r(L-1) - P_{2,2}(L)\mathcal{C}(L)c^T(L-1)\right]\mathcal{A}^T(L)\\ &- \left[P_{2,2}(L)\mathcal{A}(L)c(L-1) + R_V(L) + \mathcal{C}(L)d(L-1)\right]\mathcal{C}^T(L) + K_{V_0}(L,L)\end{aligned}$$

where

$$\begin{aligned}r(L) &= r(L-1) + J(L,L)\Pi(L)J^T(L,L), \quad r(0)=0\\ c(L) &= c(L-1) + J(L,L)\Pi(L)I^T(L,L), \quad c(0)=0\\ d(L) &= d(L-1) + I(L,L)\Pi(L)I^T(L,L), \quad d(0)=0.\end{aligned}$$

The smoothing gain, $h(k,L,L)$, *satisfies*

$$\begin{aligned}h(k,L,L) = &\left[p(L)\mathcal{B}(k)\mathcal{A}^T(L) - P_{2,2}(L)E(k,L-1)\mathcal{A}^T(L)\right.\\ &\left. - F(k,L-1)\mathcal{C}^T(L)\right]\Pi^{-1}(L)\end{aligned}$$

with

$$\begin{aligned}E(k,L) &= E(k,L-1) + h(k,L,L)\Pi(L)J^T(L,L),\\ F(k,L) &= F(k,L-1) + h(k,L,L)\Pi(L)I^T(L,L),\\ F(k,k) &= \mathcal{A}(k)c(k), \quad E(k,k) = \mathcal{A}(k)r(k).\end{aligned}$$

The initial condition is given by the filter, $\widehat{Z}(k,k) = \mathcal{A}(k)O(k)$, *and the filtering error variances,* $P(k,k)$, *satisfy*

$$P(k,k) = \mathcal{A}(k)\mathcal{B}^T(k) - \mathcal{A}(k)r(k)\mathcal{A}^T(k).$$

Proof. Nakamori et al. [10]. ∎

4. Computer simulation results

In order to show the effectiveness of the proposed quadratic estimators, we have performed a program in MATLAB, which simulates the signal value at each iteration, and provides the linear and quadratic estimates, as well as the corresponding error covariance matrices.

This program has been applied to a scalar signal generated by the following first-order autoregressive model

$$z(k+1) = 0.95z(k) + w(k)$$

where $\{w(k);\ k \geq 0\}$ is a zero-mean white Gaussian noise with $Var\,[w(k)] = 0.1$, for all k.

The autocovariance and crosscovariance functions of this signal and its second-order powers are

$$\begin{array}{l} K_z(k,s) = 1.025641 \times 0.95^{k-s}, \quad 0 \leq s \leq k \\ K_{z^{[2]}}(k,s) = 2.1038795 \times 0.95^{2(k-s)}, \quad 0 \leq s \leq k \\ K_{zz^{[2]}}(k,s) = 0, \quad \forall\ s,k. \end{array}$$

So, according to hypothesis (I),

$$\begin{array}{ll} A(k) = 1.025641 \times 0.95^{k}, & B(k) = 0.95^{-k}, \\ a(k) = 2.1038795 \times 0.95^{2k}, & b(k) = 0.95^{-2k}, \\ c_1(k) = c_2(k) = d_1(k) = d_2(k) = 0. & \end{array}$$

As in [6], we consider that the signal can be transmitted through one of two channels, with observation equations:

$$\begin{array}{ll} \text{Channel I:} & y(k) = z(k) + v(k) + v_0(k) \\ \text{Channel II:} & y(k) = \beta(k)z(k) + v(k) + v_0(k) \end{array}$$

where $\{\lambda(k);\ k \geq 0\}$ are independent Bernoulli random variables with $P[\lambda(k) = 1] = \check{p}$, for all k. The noise $\{v(k);\ k \geq 0\}$ is a white sequence with

$$E[v(k)] = 0, \quad R_v(k) = 9.142857,$$
$$R_{vv^2}(k) = -62.693878, \quad R_{v^2}(k) = 429.900875$$

and $\{v_0(k);\ k \geq 0\}$ is a coloured noise generated by

$$v_0(k+1) = 0.5v_0(k) + v_1(k)$$

where $\{v_1(k);\ k \geq 0\}$ is a zero-mean white Gaussian noise with $Var\,[v_1(k)] = 0.075$, for all k. For this model, we have

$$\begin{aligned} &K_{v_0}(k,s) = 0.1 \times 0.5^{k-s}, \quad 0 \leq s \leq k \\ &K_{v_0^2}(k,s) = 0.02 \times 0.5^{2(k-s)}, \quad 0 \leq s \leq k \\ &K_{v_0 v_0^2}(k,s) = 0, \quad \forall\ s,k \end{aligned}$$

and, according to hypothesis (III),

$$\begin{aligned} &\alpha(k) = 0.1 \times 0.5^{k}, \quad \beta(k) = 0.5^{-k}, \\ &\gamma(k) = 0.02 \times 0.5^{2k}, \quad \delta(k) = 0.5^{-2k} \\ &\epsilon_1(k) = \epsilon_2(k) = \rho_1(k) = \rho_2(k) = 0. \end{aligned}$$

Assuming that Channel II is randomly picked with probability q, the observations can be expressed as

$$y(k) = U(k)z(k) + v(k) + v_0(k)$$

being $U(k) = (1-\theta) + \theta\lambda(k)$ and θ a Bernoulli variable with $P\,[\theta = 1] = q$. Then, $U(k)$ are Bernoulli variables with

$$\begin{aligned} p(k) &= P\,[U(k) = 1] = \check{p}q + (1-q) \\ P_{2,2}(k) &= \left[1 - q(1-\check{p}^2)\right] / \left[1 - q(1-\check{p})\right]. \end{aligned}$$

In order to compare the linear and quadratic estimates of this signal, we have performed two hundred iterations of the respective algorithms for several values of the parameters $\check{p}$ and q, which lead to different situations for the sequence $\{U(k);\ k \geq 0\}$.

The linear and quadratic filtering error variances for $p(k) = 0.72$ and different values of $P_{2,2}(k)$, namely $P_{2,2}(k) = 0.72,\ 0.8833$ and 1, are displayed in Figure 1, which shows that the quadratic filtering error variances are always less than the linear ones. This figure also shows that the error variances increase with $P_{2,2}(k)$; so, as $P_{2,2}(k)$ increases, the estimations of the signal are worse.

Figure 2 displays the simulated signal and the quadratic filtering estimates for the same probability, $p(k) = 0.72$, and the above different values $P_{2,2}(k) = 0.72, 0.8833, 1$.

Figure 3 and Figure 4 display a simulated signal together with the linear and quadratic filtering estimates for $p(k) = 1$ (the signal is always present in the observations and, hence, $P_{2,2}(k) = 1$) and $p(k) = 0.72$, $P_{2,2}(k) = 0.8833$, respectively. In both cases, the figures show that the quadratic filtering estimate follows the signal evolution better than the linear one, agreeing with the comments about Figure 1.

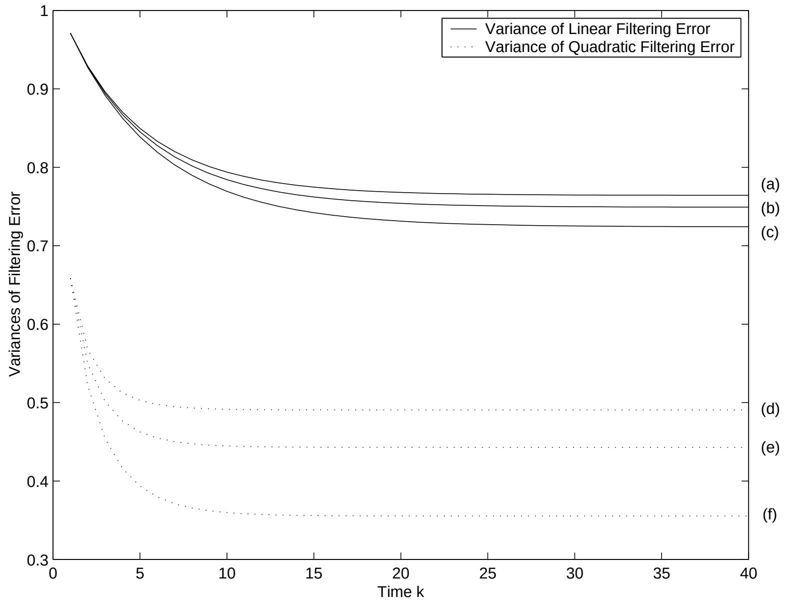

Figure 1: Linear and quadratic filtering error variances when $p(k) = 0.72$ and (a), (d): $P_{2,2}(k) = 1$, (b), (e): $P_{2,2}(k) = 0.8833$, (c), (f): $P_{2,2}(k) = 0.72$.

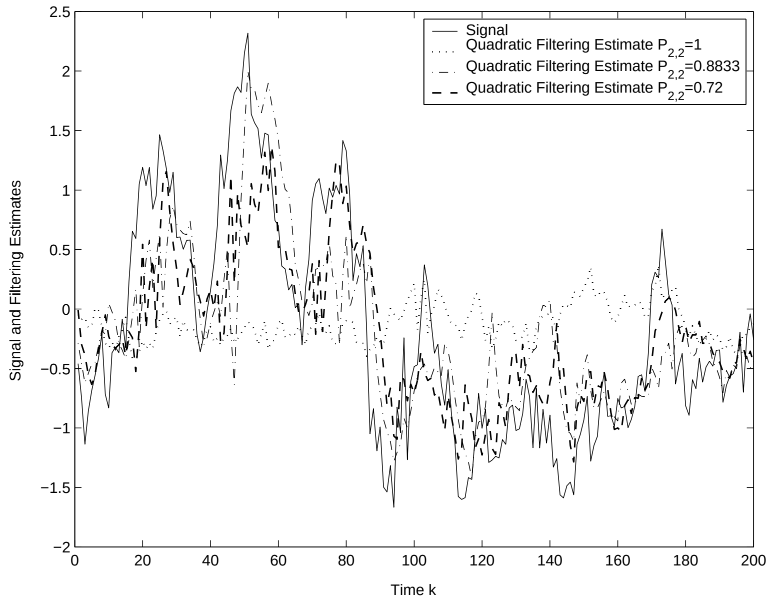

Figure 2: Signal and the quadratic filtering estimates for $p(k) = 0.72$ and $P_{2,2}(k) = 0.72,\ 0.8833,\ 1$.

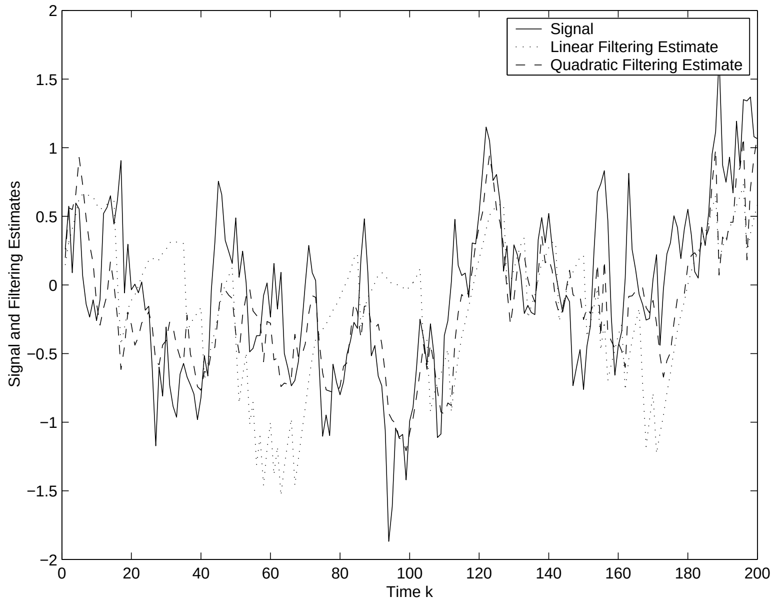

Figure 3: Signal, linear and quadratic filtering estimates for $p(k) = 1$ and $P_{2,2}(k) = 1$.

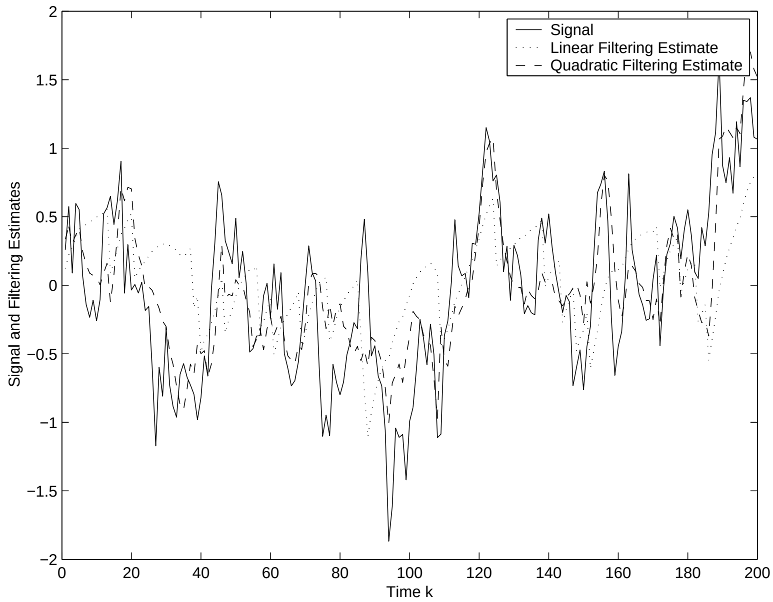

Figure 4: Signal, linear and quadratic filtering estimates for $p(k) = 0.72$ and $P_{2,2}(k) = 0.8833$.

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