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An Example of a Banach Space V such that all the Degree $d \geq 2$ Hypersurfaces of $\mathbf{P}(V)$ are Singular

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1. The example

For any complex Banach space let $\mathbf{P}(V)$ denote the projective space of all one-dimensional linear subspaces of V. For any integer $d \geq 1$ let $P^d(V)$ be the set of all continuous degree d complex valued homogeneous polynomials on V. By definition a degree d hypersurface of $\mathbf{P}(V)$ is the zero-locus of some $f \in P^d(V)$, $f \neq 0$. Hence a degree one hypersurface is just a closed hyperplane. Here we want to show the existence of a Banach space V (not separable) such that for every integer $d \geq 2$ there is no smooth degree d hypersurface of $\mathbf{P}(V)$. We do not have any example of a separable Banach space with the same property. Smooth hypersurfaces are important, because one hopes to use analytic tools on them ([3]). This example shows that the situation for Banach projective spaces is dramatically different from the situation for finite-dimensional projective spaces, in which Bertini's theorem states that a general choice of a finite number of polynomial equations define a nonsingular set. The same example gives the non-existence of non-linear smooth complete intersection with finite codimension. This example is not new. We just extracted it from the literature ([2] and [1, Prop. 8]) and proved that it is interesting also from this new point of view.

Example. Fix an uncountable discrete set A and let $C_0(A)$ be the Banach space of all complex valued functions on A which vanish at infinity, with the supremum norm. For any open subset U of $C_0(A)$ every holomorphic function

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on U depends only from a countable number of variables ([2] or [1, Prop. 8]). Hence every continuous homogeneous degree d polynomial f on $C_0(A)$, $f \neq 0$, depends only from countably many variables. Hence the degree d hypersurface $\{f = 0\} \subset \mathbf{P}(C_0(A))$ is a cone with as vertex W a projective space $\mathbf{P}(B)$ with uncountable algebraic dimension. If $d \geq 2$ every point of W is a singular point of the hypersurface $\{f = 0\}$. Since B has a supplement isomorphic to $C_0(A')$ with A' countable, it is easy to see that for every integer $s \geq 1$ and any continuous homogeneous polynomials f_i on V, $1 \leq i \leq s$, $f_i \neq 0$, $\deg(f_i) \geq 2$ for all i, the closed analytic subset $\{f_1 = \cdots = f_s = 0\}$ of $\mathbf{P}(C_0(A))$ is a cone with infinite-dimensional vertex and in particular it is not smooth.

References

[1] Dineen, S., Growth properties of pseudo-convex domains and domains of holomorphy in locally convex topological vector spaces, *Math. Ann.* **226** (3) (1977), 229 – 236.

[2] Josefson, B., A counterexample in the Levi problem, in "Proc. on Infinite Dimensional Holomorphy", Lect. Notes in Math. 364, Springer, Berlin, 1974, 168 – 177.

[3] Kotzev, B., "Vanishing of the First Dolbeaut Cohomology Group of Line Bundles on Complete-Intersections in Infinite-Dimensional Projective Space", Ph. D. thesis, Purdue University, December 2001.