EXTRACTA MATHEMATICAE Vol. **19**, Núm. 3, 289 – 311 (2004)

Equivariant Unfoldings of G-Stratified Pseudomanifolds

Dalmagro F.

Escuela de Matemáticas, Facultad de Ciencias, UCV, Caracas, Venezuela
e-mail: dalmagro@euler.ciens.ucv.ve

(Presented by M. de León)

AMS *Subject Class.* (2000): 35S35, 55N33 *Received November* 25, 2003

To Rodolfo Ricabarra, in memoriam.

Introduction

Let X be a Thom-Mather stratified space with depth $d(X) = n$. The De Rham Intersection Cohomology of X with differential forms was defined in [3] by means of an auxiliar construction called *unfolding*, which is a continuous map $\mathcal{L} : \widetilde{X} \to X$ where $\widetilde{X}$ is a smooth manifold obtained trough a finite composition

$$\mathcal{L} : \widetilde{X} = X_n \xrightarrow{\mathcal{L}_n} X_{n-1} \to \cdots \to X_1 \xrightarrow{\mathcal{L}_1} X_0 = X$$

of topological operations $X_i \xrightarrow{\mathcal{L}_i} X_{i-1}$, called *elementary unfoldings*. This iterative construction is possible because the stratification of X is controlled by the existence of a family of conical fiber bundles over the singular strata. Later in [11] we find a more abstract definition of unfoldings, which impose some conditions of transversality over the singular strata. For instance, if the depth of X is 1 then the first elementary unfolding of X is an unfolding in the new sense.

Now let G be a compact Lie group. We introduce the definition of a G-stratified pseudomanifold in the category of stratified pseudomanifolds. Our definition is related to a previous one given in [10]. A G-stratified pseudomanifold is a stratified pseudomanifold in the usual sense together with a continuous action preserving the strata, and whose local model near each singular strata is given by a conical slice. We also give a definition of equivariant unfoldings, which is a suitable adaptation of the usual definition of unfolding to the family of G-stratified pseudomanifolds. We give a sufficient condition for the existence of equivariant unfoldings, which is related to the choice of a good family of tubular neighborhoods and a sequence of equivariant elementary unfoldings of X. Additionally, if G is abelian, then each elementary unfolding induces an elementary unfolding on each orbit space together with a factorization diagram.

The content of this paper is the following. In Sections 1 and 2 we recall the usual definition of stratified pseudomanifolds and give an equivariant version for stratified actions of a compact Lie Group G. In Section 3 we present the G-stratified fiber bundles and show some of their main properties. Section 4 is devoted to the study equivariant tubular neighborhoods, which are equivariant versions of the usual ones. In Section 5 we pass to the orbit space of a stratified action and provide some useful factorization theorems. Finally, in Sections 6 and 7 we develop the construction of equivariant unfoldings trough a finite composition of elementary desingularizations (see, for instance, [1]).

1. Stratified pseudomanifolds

In this section we review the usual definitions of stratified spaces, stratified morphisms and stratified pseudomanifolds. For a more detailed introduction see [7], [9].

1.1. Stratified spaces Let X be a Hausdorff, locally compact and second countable space. A *stratification* of X is a locally finite partition $\mathcal{S}_X$ satisfying:

(i) Each element $S \in \mathcal{S}_X$ is a connected manifold with the induced topology, which a *stratum* of X.

(ii) If $S' \cap \overline{S} \neq \emptyset$ then $S' \subset \overline{S}$ for any two strata S, $S' \in \mathcal{S}_X$. In this case we write $S' \leq S$ and we say that S *incides* on S'.

We say that $(X, \mathcal{S}_X)$ is a *stratified space* whenever $\mathcal{S}_X$ is a stratification of X.

With the above conditions, the incidence relationship is a partial order on $\mathcal{S}_X$. More over, since $\mathcal{S}_X$ is locally finite, any strictly ordered chain

$$S_0 < S_1 < \cdots < S_m$$

in $\mathcal{S}_X$ is finite. The *depth* of X is by definition the supreme (possibly infinite) of the integers m such that there is a strictly ordered chain as above. We write this as $d(X)$.

The maximal (resp. minimal) strata in X are open (resp. closed) in X. A *singular* stratum is a non-maximal stratum in X. The union of the singular strata is the *singular part* of X, denoted by $\Sigma \subset X$, which is closed in X. Its complement $X - \Sigma$ is open and dense in X. The family of minimal strata will often be denoted by $\mathcal{S}_X^{min}$, while the union of minimal strata will be denoted by Σ^{min}, which we call the *minimal part* of X.

1.2. Examples (1) For any manifold M the trivial stratification of M is the family

$$\mathcal{S}_M = \{C : C \text{ is a connected component of } M\}.$$

(2) For any connected manifold M, the space $M \times X$ is a stratified space, with the stratification

$$\mathcal{S}_{M\times X} = \{M \times S : S \in \mathcal{S}_X\}.$$

Notice that $d(M \times X) = d(X)$.
(3) The *cone* of a compact stratified space L is the quotient space

$$c(L) = L \times [0, \infty)/L \times \{0\}.$$

We write $[p, r]$ for the equivalence class of $(p, r) \in L \times [0, \infty)$. The symbol $*$ will be used for the equivalence class of $L \times \{0\}$, this is the *vertex* of the cone. The family

$$\mathcal{S}_{c(L)} = \{*\} \cup \{S \times (0, \infty) : S \in \mathcal{S}_L\}$$

is the canonical stratification of $c(L)$. Notice that $d(c(L)) = d(L) + 1$.

1.3. Stratified subspaces and morphisms Let $(X, \mathcal{S}_X)$ be a stratified space. For each subset $Z \subset X$ the *induced partition* is the family

$$\mathcal{S}_{Z/Y} = \{C : C \text{ is a connected component of } Z \cap S,\ S \in \mathcal{S}_X\}.$$

We will say that Z is a *stratified subspace* of X, whenever the induced partition on Z is a stratification of Z.

Now let $(Y, \mathcal{S}_Y)$ be another stratified space. A *morphism* (resp. *isomorphism*) is a continuous map $f : X \to Y$ (resp. homeomorphism) which smoothly (resp. diffeomorphicaly) sends strata into strata. In particular, f is a *embedding* if $f(X)$ is a stratified subspace of Y and $f : X \to f(X)$ is an isomorphism.

Henceforth, we will write $\mathrm{Iso}(X, \mathcal{S}_X)$ for the group of isomorphisms of a stratified space X. The following statement will be used later, we leave the proof to the reader.

LEMMA 1.4. *Let* $(X, \mathcal{S}_X)$ *be a stratified space, and* $\mathfrak{F} \subset \mathcal{S}_X$ *a subfamily of equidimensional strata. The connected components of* $M = \cup_{S \in \mathfrak{F}} S$ *are the strata in* $\mathfrak{F}$.

Stratified pseudomanifolds were used by Goresky and MacPherson in order to introduce the Intersection Homology and extend the Poincaré duality to the family of stratified spaces [5], [6]; see also [7] for a brief introduction.

1.5. STRATIFIED PSEUDOMANIFOLDS The definition of a stratified pseudomanifold is made by induction on the depth of the space. More precisely:

(1) A stratified pseudomanifold of depth 0 is a manifold with the trivial stratification.

(2) An arbitrary stratified space $(X, \mathcal{S}_X)$ is a *stratified pseudomanifold* if, for any singular stratum $S \in \mathcal{S}_X$, there is a compact stratified pseudomanifold L_S depending on S (called a *link* of S) such that each point $x \in S$ has a coordinate neighborhood $U \subset S$ and an embedding onto an open subset of X.

$$\varphi : U \times c(L_S) \to X$$

such that $x \in \mathrm{Im}(\varphi)$. The pair (U, φ) is a *chart* of x *modeled* on L_S.

1.6. EXAMPLES Here are some examples of stratified pseudomanifolds.

(1) If X is a stratified pseudomanifold, then any open subset $A \subset X$ is also a stratified pseudomanifold. Also the product $M \times X$ (with the canonical stratification) is a stratified pseudomanifold, for any manifold M.

(2) If L is a compact stratified pseudomanifold, then $c(L)$ is a stratified pseudomanifold.

2. G-STRATIFIED PSEUDOMANIFOLDS

From now on, we fix an abelian compact Lie group G. We will study the family of actions of G which preserve the strata. Our definition is strongly related to the previous one given in [10]. Also some easy proofs in this section can be seen in [8].

Given a stratified space $(X, \mathcal{S}_X)$ and a effective action $\Phi : G \times X \to X$; we write $\Phi(g, x) = gx$ for any $g \in G$, $x \in X$. We denoted X/K by the K-orbit space for every K closed subgroup of G, and by $\pi : X \to X/K$ the orbit map. The group of G-equivariant isomorphisms of X will be denoted by $\mathrm{Iso}_G(X, \mathcal{S}_X)$.

2.1. G-STRATIFIED SPACES We say that X is *G-stratified* whenever:

(1) For each stratum $S \in \mathcal{S}_X$ the points of S all have the same isotropy group, denoted by G_S.

(2) Each $g \in G$ induces an isomorphism $\Phi_g : X \to X \in \mathrm{Iso}_G(X, \mathcal{S}_X)$.

2.2. EXAMPLES Here are some examples of G-stratified spaces:

(1) Each G-manifold M has a natural structure of G-stratified space, when M is endowed with the stratification given by orbit types.

(2) If X is a G-stratified space, then $M \times X$ is a G-stratified space with the action $g(m, x) = (m, gx)$; for any manifold M.

(3) If L is a compact G-stratified space then $c(L)$, with the action $g[x, r] = [gx, r]$, is a G-stratified space.

Now we introduce the definition of a G-stratified pseudomanifold.

2.3. G-STRATIFIED PSEUDOMANIFOLDS A G-stratified pseudomanifold is a stratified pseudomanifold in the usual sense, endowed with a structure of G-stratified space (i.e. G acts by isomorphisms) and whose local model is described through conical slices. Conical slices were introduced in [10] in order to state a sufficient condition on any continuous action of a compact Lie group (abelian or not) on stratified pseudomanifold so that the corresponding orbit space would remain in the same class of spaces.

Let $(X, \mathcal{S}_X)$ be a G-stratified space. Take a singular stratum $S \in \mathcal{S}_X$ a point $x \in S$. A *conical slice* of x in X is a slice S_x in the usual sense of [2], with a conical part transverse to the stratum S. In other words:

(1) S_x is an invariant G_S-space containing x.

(2) For any $g \in G$, if $gS_x \cap S_x \neq \emptyset$ then $g \in G_S$.

(3) GS_x is open in X. And

(4) There is a G_S-equivalence $\beta : \mathbb{R}^i \times c(L) \to S_x$ where $i \geq 0$ and L is a compact G_S-stratified space. Here the action of G_S on $\mathbb{R}^i$ is trivial (notice that β induces on S_x a structure of G_S-stratified space).

The definition of a *G-stratified pseudomanifold* is made by induction on the depth of the space. A G-stratified pseudomanifold with depth 0 is a manifold with a smooth free action of G. In general, we will say that X is a G-stratified pseudomanifold if, for each singular stratum $S \in \mathcal{S}_X$, there is a compact G_S-stratified pseudomanifold L_S such that each point $x \in S$ has a conical slice

$$\beta : \mathbb{R}^i \times c(L_S) \to S_x$$

and the usual map on the twisted product

$$\alpha : G \times_{G_S} S_x \to X \qquad \alpha([g,y]) = gy$$

is an equivariant (stratified) embedding on an open subset of X. We say that the triple (S_x, β, L_S) is a *distinguished slice* of x.

2.4. Examples Here there are some examples of G-stratified pseudomanifolds.

(1) Take a smooth effective action $\Phi : G \times M \to M$ with fixed points on a manifold M endowed with the stratification by orbit types. By the Equivariant Slice Theorem, M is a G-stratified pseudomanifold.

(2) If X is a G-stratified pseudomanifold then $M \times X$ is a G-stratified pseudomanifold with the obvious action.

(3) If L is a compact G-stratified pseudomanifold, then $c(L)$ is a G-stratified pseudomanifold with the obvious action.

(4) Any invariant open subspace of a G-stratified pseudomanifold is itself a G-stratified pseudomanifold.

2.5. Remark *Each G-stratified pseudomanifold is a stratified pseudomanifold in the previous sense.*

To see this, proceed by induction on the depth. Take a G-stratified pseudomanifold X. For $d(X) = 0$ the statement is trivial. Assume the inductive hypothesis and suppose that $d(X) > 0$. Take a singular stratum $S \in \mathcal{S}_X$, a point $x \in S$ and a distinguished slice (S_x, β, L_S) of x. The isotropy subgroup G_S acts on G by the restriction of the group operation. We fix a slice S_e of the identity element $e \in G$ with respect to this action. Since $G_S S_e$ is open in G, the composition

$$\begin{aligned}(S_e \times \mathbb{R}^i) \times c(L_S) &\to S_e \times (\mathbb{R}^i \times cL_S) \to S_e \times S_x \to \\ &\to S_e \times (G_S \times_{G_S} S_x) \to (G_S S_e) \times_{G_S} S_x \to X\end{aligned}$$

is an embedding. Notice that by induction L_S is a stratified pseudomanifold, since $S_e \times \mathbb{R}^i \simeq S_e G_S (S \cap S_x)$ is open in S. We have obtained a chart of x modeled on L_S.

2.6. Remark *If X is a G-stratified pseudomanifold and K is any closed subgroup of G, then X is also a K-stratified pseudomanifold.*

It is straightforward that X is a K-stratified space. For any singular stratum S and any $x \in S$, in order to choose a distinguished slice in x we proceed as follows: Take a distinguished slice $\beta : \mathbb{R}^i \times c(L_S) \to S_x$ in x with respect to the action of G. Take also a slice V_e of the identity element $e \in G$ with respect to the action of $G_S K$ in G. Then $\imath \times \beta : (V \times \mathbb{R}^i) \times c(L_S) \to V S_x$ is a distinguished slice of x with respect to the action of K.

3. G-stratified fiber bundles

Henceforth we fix a compact, abelian Lie group G. In this section we introduce the notion of a G-stratified fiber bundle. This is a previous step in order to study the family of equivariant tubular neighborhoods. The reader will find in [12] a detailed introduction to the fiber bundles, while [13] provides the usual definition of a tubular neighborhood in the stratified context (see also [2] for the smooth case).

3.1. G-stratified fiber bundles Let $\xi = (E, p, B, F)$ be a locally trivial fiber bundle with (maximal) trivializing atlas $\mathcal{A}$. We will say that ξ is a *G-stratified fiber bundle* whenever:

(1) The total space E is a G-stratified space.

(2) The base space B is a manifold, endowed with a smooth action $\Psi : G \times B \to B$ and with constant isotropy $H \subset G$ at all its points.

(3) The fiber F is a H-stratified space.

(4) The projection $p : E \to B$ is G-equivariant.

(5) The group G acts by isomorphisms. In other words, each chart

$$\varphi : U \times F \to p^{-1}(U) \in \mathcal{A}$$

is H-equivariant; and for any two charts $(U, \varphi), (U', \varphi') \in \mathcal{A}$ such that $U' \cap g^{-1}U \neq \emptyset$ for some $g \in G$, there is a map

$$g_{\varphi,\varphi'} : U' \cap g^{-1}U \to \mathrm{Iso}_H(F, \mathcal{S}_F)$$

such that

$$\varphi^{-1} g \varphi'(b, z) = (gb, g_{\varphi,\varphi'}(b) z).$$

Lemma 3.2. *Let $\xi = (E, p, B, F)$ be a G-stratified fiber bundle, H the isotropy of B. If F is an H-stratified pseudomanifold, then E is a G-stratified pseudomanifold.*

Proof. Fix a singular stratum S in E and a point $x \in S$. We must prove the existence of a link L_S depending only on S and, a distinguished slice (S_x, β, L_S) in x. For this purpose, let's take a trivializing chart

$$\varphi : U \times F \to p^{-1}(U) \in \mathcal{A}$$

such that $x \in p^{-1}(U)$. Take $z = p(x)$ and a G-slice V_z in B. Since V_z is contractible, we assume that $V_z \cong \mathbb{R}^k$ and V_z is contained in U.

Write $\varphi^{-1}(x) = (z, y) \in V_z \times F$ and take S' the stratum in F containing y. Since F is an H-stratified pseudomanifold, we can choose a distinguished slice S_y in y; say

$$\beta_0 : S_y \to \mathbb{R}^i \times c(L_{S'}).$$

Consider the following composition

$$\varphi(V_z \times S_y) \stackrel{\varphi^{-1}}{\to} V_z \times S_y \stackrel{\imath \times \beta_0}{\to} V_z \times \mathbb{R}^i \times c(L_{S'}) \cong R^{i+k} \times c(L_{S'}).$$

We will show that

$$(S_x, \beta, L_S) = (\varphi(V_z \times S_y), (\imath \times \beta_0) \circ \varphi^{-1}, L_{S'})$$

is a distinguished slice in x. We proceed in three steps.

• *L_S only depends on S*: If $(U', \psi) \in \mathcal{A}$ is another trivializing chart covering x, $\psi^{-1}(x) = (z, y') \in V_z \times F$ and $\beta'_0 : S_{y'} \to \mathbb{R}^i \times c(L_{S''})$ is a distinguished slice in y'; then the composition $\beta' \beta^{-1}$ induces an H-isomorphism $L_S \stackrel{\cong}{\to} L_{S''}$.

• *S_x is a conical slice*: We verify the conditions (1) to (4) of § 2.3.
(1) Since V_z is a slice of $z \in B$, we have $gp(x) = p(gx) = p(x) \in V_z$ for any $g \in G_S$. So $G_S = H \cap G_S = H_S$, but φ is H-equivariant, hence $G_S = H_S = H_{S'}$. Again, since φ is H-equivariant and S_y is $H_{S'} = G_S$ invariant, we obtain that S_x is G_S-invariant.
(2) Take $g \in G$, $x' \in S_x$ such that $gx' \in S_x$. Then $gp(x') = p(gx') \in V_z$, so $g \in H$ and $gp(x') = p(x')$. Since φ is H-equivariant, if $x' = \varphi(p(x'), y)$ then $g.x' = \varphi(p(x'), gy)$, and $gy \in S_y$; hence $g \in H_{S'} = G_S$.
(3) Take a slice S_e of the identity element $e \in G$ with respect to the action of H. Since S_e is contractible, we can assume that $S_e V_z \subset U$. Notice that $S_e H$

is open in G. Since $GS_x = \bigcup_{g \in G} g(S_eH)S_x$, we only have to show that $(S_eH)S_x$ is open in X. But φ is H-equivariant and the action of H on V_z is trivial, so we get the following equality

$$(S_eH)S_x = S_e\left(H\varphi(V_z \times S_y)\right) = S_e\varphi(V_z \times HS_y).$$

Since HS_y is open in F we deduce that $S_e\varphi(V_zHS_y)$ is open in $S_e\varphi(V_z \times F)$. Finally we show that $S_e\varphi(V_z \times F) = S_ep^{-1}(V_z)$ is open in X: Since p is equivariant and S_eV_z is open in U the set $S_ep^{-1}(V_z) = p^{-1}(S_eV_z) = p^{-1}(S_eHV_z)$ is open in $p^{-1}(U)$ (and so in X).

(4) It is straightforward that the map β is a G_S-equivalence.

- *S_x is a distinguished slice*: We will show that usual the map

$$\alpha : G \times_{G_S} S_x \to X$$

is a (stratified) embedding.

(a) *α preserves the strata*: Take a stratum S^0 in S_x. We will prove that $G'S^0$ is an open subset in some stratum of X, for any connected component $G' \subset G$. It is enough to prove this for the connected component G_0 of the identity element $e \in G$. Let H_0 be the connected component of the identity element $e \in H$. The set S_eH_0 is a connected open subset in S_eH, so is also connected and open in G_0. Since G_0S^0 is connected, we need to prove that $S_eH_0S^0$ is open in some stratum of X. But S_eHS_x is contained in $p^{-1}(S_eV_z)$ and φ is a stratified embedding, and so we only have to show that $\varphi^{-1}(S_eH_0S^0)$ is open in some stratum of $(S_eV_z) \times F$. Consider the map

$$\begin{array}{rccl} f: & S_eH \times V_z \times S_y & \to & (S_eV_z) \times F \\ & (gh, b, l) & \mapsto & \big(ghb, (gh)_{\varphi\varphi}(b)(z)\big) = \big(gb, g_{\varphi\varphi}(b)(hz)\big). \end{array}$$

Let S^1 be the stratum of S_y such that $S^0 = \varphi(V_z \times S^1)$. By hypothesis S_y is a distinguished slice of y in F, and there is a stratum S^2 in F such that H_0S^1 is open in S^2. Notice that

$$\varphi^{-1}(S_eH_0S^0) = f(S_eH_0 \times V_z \times S^1) = f(S_e \times V_z \times H_0S^1).$$

Also, since φ is H-equivariant, we have

$$p\big(\varphi^{-1}(S_eH_0S^0)\big) = S_eV_z.$$

Hence the projection $pr_2 : U \times F \to F$ sends $\varphi^{-1}(S_e H_0 S^0)$ on some open subset of S^2. Notice that $\varphi^{-1}(S_e H_0 S^0)$ is connected, so

$$pr_2\big(\varphi^{-1}(S_e H_0 S^0)\big) = \bigcup_{(g,b)\in S_e\times V_z} g_{\varphi\varphi}(b)(H_0S^1)$$

is a connected subset of F. Each $g_{\varphi\varphi}(b)$ is an H-equivariant stratified isomorphism; hence $g_{\varphi\varphi}(b)(H_0S^1)$ is open is some stratum of F with the same dimension of S^2. Since $e_{\varphi\varphi}(b)(H_0S^1) = H_0S^1 \subset S^2$, by § 1.4 the set

$$\bigcup_{(g,b)\in S_e\times V_z} g_{\varphi\varphi}(b)(H_0S^1)$$

is contained in S^2.

(b) *α is smooth on each stratum*: Since $G\times_{G_x} S_x$ has the quotient stratification induced on $G \times S_x$ by the action of H, the stratification of S_x is induced by X and the action of G is smooth on each stratum of $G \times X$. We conclude that the restriction of α to each stratum is smooth. ∎

4. Equivariant tubular neighborhoods

In this section we will study the family of equivariant tubular neighborhoods, which are equivariant version of the usual ones. We fix, as before, a compact Lie Group G and a G-stratified pseudomanifold X. Given a singular stratum S in X, a tubular neighborhood is just a locally trivial fiber bundle over a S whose fiber is $c(L_S)$, the cone of the link of S, and whose structure group is $\mathrm{Iso}_{G_S}(L_S, \mathcal{S}_{L_S})$.

4.1. Equivariant tubular neighborhoods An equivariant tubular neighborhood is a conical locally trivial fiber bundle. For a detailed introduction the reader can see [9], [13]. In [1], the tubular neighborhoods are used in order to show the existence of an unfolding for any manifold endowed with a Thom-Mather structure. We will provide an equivariant version of this fact for any G-stratified pseudomanifold.

Let X be a G-stratified pseudomanifold with $d(X) > 0$. Let's take a singular stratum S in X. An *equivariant tubular neighborhood* of S is a G-stratified fiber bundle $(T_S, \tau_S, S, c(L_S))$ with (maximal) trivializing atlas $\mathcal{A}$, verifying

(1) T_S is an open invariant neighborhood of S and the inclusion $S \to T_S$ is a section of $\tau_S : T_S \to S$.

(2) G preserves the conical radium: For any two charts $(U,\varphi),(U',\varphi^{'})\in\mathcal{A}$ such that $U'\cap g^{-1}U\neq\emptyset$ for some $g\in G$, there is a map

$$g_{\varphi,\varphi'}:U'\cap g^{-1}U\to \mathrm{Iso}_{G_S}(L_S,\mathcal{S}_{L_S})$$

such that

$$\varphi^{-1}g\varphi'\,(b,[l,r])=\Big(gb,[g_{\varphi,\varphi'}(b)l,r]\Big)\,.$$

This allows us to define a (global) *radium* on T_S, as the map $\rho_S:T_S\to[0,\infty)$ satisfying

$$\rho_S(\varphi(z,[l,r]))=r\quad\forall\,(z,[l,r])\in U\times c(L_S);\ (U,\varphi)\in\mathcal{A}.$$

We also define the *radial action* $\delta_S:\mathbb{R}^+\times T_S\to T_S$ as follows

$$\delta_S(r,x)=\varphi(z,[l,rt])$$
$$\forall\,(z,[l,t])\in U\times c(L_S);\ (U,\varphi)\in\mathcal{A}\ \ (\text{for } x=\varphi(z,[l,t])).$$

We will write rx instead of $\delta_S(r,x)$ in the future. These functions satisfy

(a) $\rho_S(rx)=r\rho_S(x)$ and $\rho_S(gx)=\rho_S(x)$ for any $r\in\mathbb{R}^+$, $x\in T_S$, $g\in G$.
(b) $S\cap\rho_S^{-1}(0,\infty)=\emptyset$
(c) The radial action commutes with the action of G on T_S.

4.2. Thom-Mather spaces (see [14], [15]) A *Thom-Mather G-stratified pseudomanifold* is a pair $(X,\mathcal{T})$ where X is a G-stratified pseudomanifold and $\mathcal{T}=\{T_S: S\in\mathcal{S}_X^{sing}\}$ is a family of equivariant tubular neighborhoods satisfying the following condition:

$$T_S\cap T_R\neq\emptyset\ \Leftrightarrow\ R\leq S\ \text{ or }\ S\leq R$$

for any two singular strata R, S in X. We will usually omit the family $\mathcal{T}$ if there is no possible confusion.

4.3. Examples Here are some examples of G-stratified tubular neighborhoods.

(1) Following [2, p. 306], for any manifold M endowed with a smooth action $\Phi: G\times M\to M$ there is a Riemannian metric μ such that G acts by μ-isometries. By the local properties of the exponential map, each singular stratum S of M has a smooth G-equivariant tubular neighborhood which

can be realized as the normal fiber bundle $N_\mu(S)$ over S with respect to μ. The cocycles of this bundle are orthogonal actions. Hence, this tubular neighborhood is actually a G-stratified tubular neighborhood.

(2) If L is a compact G-stratified pseudomanifold, the map $c(L) \to \{\star\}$ is a G-stratified tubular neighborhood of the vertex.

(3) If $\xi = (T_S, \tau_S, S, c(L_S))$ is a G-stratified tubular neighborhood of S in X, then $(M \times T_S, \imath_M \times \tau_S, M \times S, c(L_S))$ is a G-stratified tubular neighborhood of $M \times S$ in $M \times X$; for any connected manifold M.

(4) If $f : Y \to X$ is a G-equivariant isomorphism, then for any G-stratified tubular neighborhood $\xi = (T_S, S, \tau_S, c(L_S))$ of a stratum S in X; the pull-back $f^*(\xi) = (f^{-1}(T_S), f^{-1}\tau_S f, f^{-1}(S), c(L_S))$ is a G-stratified tubular neighborhood of $f^{-1}(S)$ in Y.

5. Orbit Spaces

In this section we will expose some factorization theorems, concerning G-stratified pseudomanifolds and the equivariant tubular neighborhoods. This is made in order to get a consistent theory when passing to the orbit spaces, we do it for any compact *abelian* Lie group G. For similar results in the non-abelian context the reader can see [10].

In the sequel, we fix a G-stratified space X and a closed subgroup $K \subset G$. Write $\pi : X \to X/K$ for the usual orbit map. The orbit space X/K inherits a *canonical stratification* given by the family

$$\mathcal{S}_{X/K} = \{\pi(S) : S \in \mathcal{S}_X\}.$$

Notice also that $d(X) = d(X/K)$.

Lemma 5.1. *The orbit space X/K is a G/K-stratified space.*

Proof. Write $\overline{g} \in G/K$ for the equivalence class of $g \in G$. Consider the quotient action

$$\overline{\Phi} : G/K \times X/K \to X/K, \qquad \overline{g} \cdot \pi(x) = \pi(gx).$$

This action is well defined because G is abelian. So:

• *The isotropy groups are constant over the strata of X/K*:
This is straightforward, since for each stratum $S \in \mathcal{S}_X$ we have

$$(G/K)_{\pi(S)} = KG_S/K.$$

Hence $\pi(S)$ has constant isotropy.

• *Each $\overline{g}$ induces an isomorphism $\overline{\Phi}_g \in G/K \in \mathrm{Iso}(X/K, \mathcal{S}_{X/K})$*: For each $g \in G$ we have a K-equivariant isomorphism $\Phi_g \in \mathrm{Iso}(X, \mathcal{S}_X)$. Passing to the quotients we obtain an isomorphism $\overline{\Phi}_g \in G/K \in \mathrm{Iso}(X/K, \mathcal{S}_{X/K})$. The differentiability of this map on $\pi(S)$ is immediate from the following commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{\Phi_g} & gS \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ \pi(S) & \xrightarrow{\overline{\Phi}_g} & \pi(gS) \end{array}$$

∎

Now we pass to the conical context.

PROPOSITION 5.2. *Assume that X is a G-stratified pseudomanifold. Then X/K is a G/K-stratified pseudomanifold.*

Proof. Proceed by induction on $l = d(X)$. For $l = 0$ it is straightforward, since $d(X/K) = d(X) = 0$. Assume the inductive hypothesis and suppose that $d(X) > 0$. By § 5.1, X/K is a G/K-stratified space, so we must verify the existence of conical slices.

Take a singular stratum $S \in \mathcal{S}_X$, fix a point $x \in S$ and a distinguished slice (S_x, β, L_S) of a x. The G_S-equivariant isomorphism $\beta : \mathbb{R}^i \times c(L_S) \to S_x$ induces an isomorphism on the orbit spaces

$$\overline{\beta} : \mathbb{R}^i \times c(L_S/G_S \cap K) \to \pi(S_x), \qquad \overline{\beta}(b, [\overline{w}, r]) = \pi(\beta(b, [w, r])).$$

Now we will show that the triple $(\pi(S_x), \overline{\beta}, L_S/G_S \cap K)$ is a distinguished slice of $\pi(x) \in X/K$. We do it in three steps.

• *$\pi(S_x)$ is a slice of $\pi(x)$*: This is straightforward, since $(G/K)_{\pi(x)} = KG_S/K$, the quotient $\pi(S_x)$ is a $(G/K)_{\pi(x)}$-space with the quotient action and the orbit map π is an open map.

• *$\overline{\beta}$ is a KG_S/K-equivalence*: This is immediate, since β is an H-equivalence. Notice that, by induction on the depth, $L_S/G_S \cap K$ is a KG_S/K-stratified pseudomanifold.

• *The induced map $\overline{\alpha} : (G/K) \times_{(G_S/G_S \cap K)} \pi(S_x) \to X/K$ is an embedding*: This $\overline{\alpha}$ is given by the rule $\overline{\alpha}([\overline{g}, \pi(z)]) = \overline{g}.\pi(z)$, and is a homeomorphism.

We consider the following commutative diagram

$$\begin{array}{ccc} G \times_{G_S} S_x & \xrightarrow{\alpha} & X \\ {\scriptstyle \overline{\pi}}\downarrow & & \downarrow{\scriptstyle \pi} \\ (G/K) \times_{(G_S/G_S \cap K)} \pi(S_x) & \xrightarrow{\overline{\alpha}} & X/K \end{array}$$

Since the vertical arrows are submersions, and α is an embedding, we obtain that $\overline{\alpha}$ is an embedding. ∎

The following result provides a factorization theorem for tubular neighborhoods.

PROPOSITION 5.3. *Let S be a singular stratum in X, $\xi = (T_S, \tau_S, S, c(L_S))$ be an equivariant tubular neighborhood of S in X and write*

$$\overline{\tau_S} : \pi(T_S) \to \pi(S)$$

for the induced quotient map. Then $\xi/K = (\pi(T_S), \overline{\tau_S}, \pi(S), c(L_S/G_S \cap K))$ is an equivariant tubular neighborhood of $\pi(S)$ in X/K.

Proof. Since π is an open map, $\pi(T_S)$ is an open neighborhood of $\pi(S)$ in X/K. Also the inclusion $\pi(S) \to \pi(T_S)$ is a section of $\overline{\tau_S} : \pi(T_S) \to \pi(S)$. In order to prove that ξ/K is a G-stratified tubular neighborhood we should first verify that it is a G-stratified fiber bundle, but the conditions § 3.1-(1) to (4) are straightforward.

Now we will prove § 4.1-(2), which implies § 3.1-(5). We will show that the trivializing atlas $\mathcal{A} = \{(U, \varphi)\}$ of ξ induces a trivializing atlas $\mathcal{A}/K = \{(V, \psi)\}$ of ξ/K. Write $\pi' : L_S \to L_S/G_S \cap K$ for the orbit map induced by the action of $G_S \cap K$ in L_S.

• *Trivializing charts*: Take a chart $(U, \varphi) \in \mathcal{A}$ and a point $x \in U$. Take also a K-slice V of x in S, we assume that $V \subset U$. Since G_S acts trivially on V and KV is open in S we deduce that

$$V = V/G_S \cap K = \pi(KV)$$

is open in $\pi(S)$. Since φ is G_S-equivariant, the function

(1) $\quad \psi : V \times c(L_S/G_S \cap K) \to \pi(T_S), \qquad \psi(b, [\pi'(l), r]) = \pi(\varphi(b, [l, r])).$

is well defined. Moreover, ψ is injective because G acts by isomorphisms and V is a K-slice in S. Notice that $W = KV \cap U$ is open in U; since G also preserves the radium in T_S,

$$\mathrm{Im}(\psi) = \pi(\varphi(W \times c(L_S))).$$

Hence $\mathrm{Im}(\psi)$ is open in X/K. It is straightforward that ψ sends smoothly strata onto strata, so actually ψ is an embedding.

• *Atlas and cocycles*: We consider the family $\mathcal{A}/K = \{V, \psi)\}$ of all the pairs (V, ψ) as in (1). We will show that $\mathcal{A}/K$ is a trivializing atlas of ξ/K. Take two charts $(V, \psi); (V', \psi') \in \mathcal{A}/K$ respectively induced by $(U, \varphi); (U', \varphi') \in \mathcal{A}$. Assume that there is some $\overline{g_0} \in G/K$ such that $\overline{g_0}^{-1} V \cap V' \neq \phi$; so $g^{-1}U \cap U' \neq \phi$ for some $g \in g_0 K$. By § 4.1-(2), there is a map

$$g_{\varphi\varphi'} : g^{-1}U \cap U' \to \mathrm{Iso}_{G_S}(L_S, \mathcal{S}_{L_S})$$

satisfying

$$g\varphi^{'}(b, [l, r]) = \varphi(gb, [g_{\varphi\varphi'}(b)(l), r]), \qquad (b, [l, r]) \in (g^{-1}U \cap U') \times c(L_S).$$

Passing to the orbit space $L_S/G_S \cap K$ we obtain the induced map

$$\overline{g_0}_{\psi\psi'} : \overline{g_0}^{-1} V \cap V' \to \mathrm{Iso}_{(G_S/G_S \cap K)}(L_S/G_S \cap K, \mathcal{S}_{L_S/G_S \cap K})$$

satisfying

$$\overline{g_0}\psi'\big(b, [\pi'(l), r]\big) = \psi\big(\overline{g_0}b, [\overline{g_0}_{\psi\psi'}(b)(\pi'(l)), r]\big);$$
$$(b, [\pi'(l), r]) \in (\overline{g_0}^{-1} V \cap V') \times c(L_S/G_S \cap K).$$

Notice that, by definition, G/K preserves the radium of $\pi(T_S)$. ∎

6. Elementary unfoldings

An unfolding of a stratified pseudomanifold is an auxiliar construction which allows us to define the intersection cohomology from the point of view of differential forms [1], [3]. In the rest of this paper we will find conditions for the existence of equivariant unfoldings. For this, we will introduce the elementary unfoldings. The main idea is that, from a finite number of elementary unfoldings one can get an equivariant unfolding in the usual sense.

6.1. Elementary unfolding of a G-stratified pseudomanifold The elementary unfolding of a Thom-Mather space is essentially the resolution of singularities given in [4] for the smooth case. This topological operation can be done because the stratification is controlled through a family of tubular neighborhoods. Under certain conditions, after the iterated composition of finitely many elementary unfoldings, one obtains an equivariant unfolding as defined above. We follow the exposition of [1].

Henceforth we fix a Thom-Mather G-stratified pseudomanifold X, a closed (hence minimal) stratum S in X and an equivariant tubular neighborhood $(T_S, \tau_S, S, c(L_S))$ of S. Define the *unitary sub-bundle* as the set $E_S = \rho_S^{-1}(1)$; this is by construction a G-invariant stratified subspace of X. The restriction $\tau_S : E_S \to S$ is a G-stratified fiber bundle with fiber L_S. Consider the map

$$\mathcal{L}_{T_S} : E_S \times \mathbb{R} \to T_S, \qquad \mathcal{L}_{T_S}(x,t) = \begin{cases} |t| * x & \text{if } t \neq 0, \\ \tau_S(x) & \text{if } t = 0. \end{cases} \tag{2}$$

Each chart (U, φ) in the trivializing atlas provides a local description of $\mathcal{L}_{T_S}$ through the following commutative square

$$\begin{array}{ccc} U \times L_S \times \mathbb{R} & \xrightarrow{\widehat{\varphi}} & E_S \times \mathbb{R} \\ {\scriptstyle 1_U \times \mathcal{L}_C}\downarrow & & \downarrow{\scriptstyle \mathcal{L}_{T_S}} \\ U \times cL_S & \xrightarrow{\varphi} & T_S \end{array}$$

where $\widehat{\varphi}(x,l,t) = (\varphi(x,[l,1],t))$ and $\mathcal{L}_C(l,t) = [l,|t|]$. We also obtain the following properties:
(a) The map $\widehat{\varphi}$ is a G_S-equivariant embedding.
(b) The composition $\tau_S \circ \mathcal{L}_{T_S} : E_S \times \mathbb{R} \to S$ is a locally trivial fiber bundle with fiber $L_S \times \mathbb{R}$ and structure group $\mathrm{Iso}_{G_S}(L_S, \mathcal{S}_{L_S})$.
(c) $d(E_S \times \mathbb{R}) = d(E_S) = d(T_S) - 1$.

Now take a disjoint family of equivariant tubular neighborhoods $\{T_S : S \in \mathcal{S}_X^{min}\}$ of the minimal strata. The *elementary unfolding* of X with respect to the family $\{T_S : S \in \mathcal{S}_X^{min}\}$ is the pair $(\widehat{X}, \mathcal{L})$ constructed as follows: First $\widehat{X}$ is the amalgamated sum

$$\widehat{X} = [\bigsqcup_{S \text{ minimal}} E_S \times \mathbb{R}] \bigcup_{\theta} [(X - \Sigma^{min}) \times \{\pm 1\}], \tag{3}$$

where S runs over the family of minimal strata and, for each $S \in \mathcal{S}_X^{min}$, the map θ restricted to E_S is given by

$$\theta : E_S \times \mathbb{R}^* \to [X - \Sigma^{min}] \times \{\pm 1\}, \qquad \theta(x,t) = (|t| * x, |t|^{-1} t). \tag{4}$$

Second, $\mathcal{L}$ is the continuous map given by the rule

$$(5) \qquad \mathcal{L} : \widehat{X} \to X, \quad \mathcal{L}(x) = \begin{cases} \mathcal{L}_{T_S}(x)\,, & x \in E_S \times \mathbb{R}\,, \\ y\,, & x = (y,j) \in (X - \Sigma^{min}) \times \{\pm 1\}\,. \end{cases}$$

Here there are some properties of the elementary unfoldings.

PROPOSITION 6.2. *Let $\mathcal{L} : \widehat{X} \to X$ be the elementary unfolding of a Thom-Mather G-stratified pseudomanifold X. Then*
(1) *$\widehat{X}$ is a G-stratified pseudomanifold, whose stratification is the family $\mathcal{S}_{\widehat{X}}$ consisting of all the following sets*

$$\widehat{R} = [\bigsqcup_{S \text{ minimal}} (E_S \cap R) \times \mathbb{R}] \bigsqcup_{\theta} (R \times \{\pm 1\})\,,$$

where R runs over the non closed strata in X. Moreover, $\widehat{X}$ satisfies the Thom-Mather condition.
(2) *The map $\mathcal{L}$ is a G-equivariant morphism. The restriction*

$$\mathcal{L} : \mathcal{L}^{-1}(X - \Sigma^{min}) \to X - \Sigma^{min}$$

is a (trivial) double covering.
(3) *$d(\widehat{X}) = d(X) - 1$. In particular, if $d(X) = 1$ then $\mathcal{L} : \widehat{X} \to X$ is an equivariant unfolding (see § 7.1).*
(4) *If X is compact, then so is $\widehat{X}$.*
(5) *If G is abelian then, for any closed subgroup $K \subset G$, the induced map*

$$\overline{\mathcal{L}} : \widehat{X}/K \to X/K$$

is an elementary unfolding.

Proof. (1) The stratification of $\widehat{X}$ can be seen in [1]. Since each equivariant tubular neighborhood is a G-stratified pseudomanifold (because they are invariant open subsets of X); so are the unitary sub-bundles (see § 3.2), and hence $\widehat{X}$ is a G-stratified pseudomanifold. Now we verify the Thom-Mather condition: Take a family $\{T_S : S \in \mathcal{S}_X\}$ of equivariant tubular neighborhoods in X. Take also a stratum $\widehat{R}$ in $\widehat{X}$ induced by a non closed stratum R in X. Define

$$T_{\widehat{R}} = \bigsqcup_{S \text{ minimal}} (E_S \cap T_R) \times \mathbb{R} \cup (T_R \times \{\pm 1\}) = \mathcal{L}^{-1}(T_R)$$

where θ is the map given in the equation (4) of § 6.1. This $T_{\widehat{R}}$ is an equivariant tubular neighborhood of $\widehat{R}$ in $\widehat{X}$; we leave the details to the reader.
(2) and (3) are straightforward, see again [1] for more details. The last observation of (3) is a consequence of Definition § 7.1.
(4) Since X is compact, $\mathcal{S}_X^{min}$ is finite. But $\widehat{X}$ is the quotient of the finite family of compact spaces $\bigsqcup_{S \text{ min.}} (E_S \times [-1,1])$ and $[X - \bigsqcup_{S \text{ min.}} \rho_S^{-1}[0,1/2)] \times \{-1,1\}$. Then we get the result.
(5) This is a consequence of § 5.3. ∎

6.3. Remark *With tubular neighborhood of* 4.3-3, $\widehat{M \times X} = M \times \widehat{X}$, *for any manifold* M.

7. Equivariant unfoldings

For a detailed introduction to unfoldings, the reader can see [4], [11]. In this section we introduce equivariant unfoldings, these are a suitable adaptation of the usual unfoldings to the equivariant category. As an example, we show how, for any compact Lie group G and any smooth G-manifold M, there is always an equivariant unfolding. When G is abelian this construction passes well to the orbit space M/K for any closed subgroup $K \subset G$.

7.1. Equivariant unfoldings Broadly speaking, an un unfolding of a stratified pseudomanifold X is a manifold $\widetilde{X}$ and a surjective continuous map $\mathcal{L} : \widetilde{X} \to X$ such that $\mathcal{L}^{-1}(X - \Sigma)$ is a union of finitely many disjoint copies of $X - \Sigma$, and which smoothly unfolds the singular part so that the restriction $\mathcal{L} : \mathcal{L}^{-1}(S) \to S$ is a submersion, for any singular stratum S.

As for the usual unfoldings, the definition of an equivariant unfolding is made by induction on the depth. Fix a compact abelian Lie group G. Let X be a G-stratified pseudomanifold. An *equivariant unfolding* of X is a manifold $\widetilde{X}$ together with a smooth free action $\widetilde{\Phi} : G \times \widetilde{X} \to \widetilde{X}$; a surjective continuous equivariant map

$$\mathcal{L} : \widetilde{X} \to X$$

and a family of equivariant unfoldings $\{\mathcal{L}_{L_S} : \widetilde{L_S} \to L_S\}_S$ where S runs on the singular strata of X; satisfying:

(1) The restriction $\mathcal{L} : \mathcal{L}^{-1}(X - \Sigma) \to X - \Sigma$ is a smooth finite trivial covering.

(2) For each singular stratum S and each $x \in S$, there is a *liftable modeled*

chart, i.e.; a commutative square

$$\begin{array}{ccc} U \times \widetilde{L_S} \times \mathbb{R} & \xrightarrow{\widetilde{\varphi}} & \widetilde{X} \\ {\scriptstyle \mathcal{L}_c}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{L}} \\ U \times c(L_S) & \xrightarrow{\varphi} & X \end{array}$$

such that

(a) (U, φ) is a G_S-equivariant chart of x modeled on L_S.
(b) $\widetilde{\varphi}$ is a G_S-equivariant smooth embedding on an open subset of $\widetilde{X}$.
(c) The map $\mathcal{L}_c$ is given by the rule $\mathcal{L}_c(u, z, t) = (u, [\mathcal{L}_{L_S}(z), |t|])$.

A G-stratified pseudomanifold X is said to be *unfoldable* whenever it has an equivariant unfolding.

7.2. Examples Here are some examples of equivariant unfoldings.
(1) For any free smooth action $\Phi : G \times M \to M$ the identity $\imath : M \to M$ is an equivariant unfolding.
(2) If $\mathcal{L} : \widetilde{X} \to X$ is an equivariant unfolding, then for any manifold M the product $\imath : M \times \widetilde{X} \to M \times X$ is also an equivariant unfolding.
(3) For any equivariant unfolding $\mathcal{L} : \widetilde{L} \to L$ over a compact G-stratified pseudomanifold L, the map $\mathcal{L}_c : \widetilde{L} \times \mathbb{R} \to c(L)$ defined above is also an equivariant unfolding.

7.3. Iteration of elementary unfoldings From now on, we fix a Thom-Mather G-stratified pseudomanifold X. Our main goal is to prove that, from a finite composition of elementary unfoldings, one gets an equivariant unfolding. This is not surprising since, as we have already seen, for any elementary unfolding $\mathcal{L} : \widehat{X} \to X$, the space $\widehat{X}$ is again a Thom-Mather G-stratified pseudomanifold and satisfies $d(\widehat{X}) = d(X) - 1$. This allows us to ask for the behavior of a chain

$$X_l \xrightarrow{\mathcal{L}_l} X_{l-1} \xrightarrow{\mathcal{L}_{l-1}} \dots \xrightarrow{\mathcal{L}_2} X_1 \xrightarrow{\mathcal{L}_1} X \tag{6}$$

of elementary unfoldings, where $l = d(X)$. As we shall see, under certain conditions on the tubular neighborhoods, this iterative process leads us to an equivariant unfolding

$$\mathcal{L} : \widetilde{X} \to X \,,$$

where $\widetilde{X} = X_l$ and $\mathcal{L} = \mathcal{L}_1 \dots \mathcal{L}_l$.

Recall the definition of a *saturated subspace* [1]. Let $Y \subset X$ be a stratified subspace of X. We say that Y is *saturated* whenever

$$Y \cap T_S = \tau_S^{-1}(Y \cap S)\,, \qquad \forall S \in \mathcal{S}_X.$$

For instance, if S is a singular stratum and $U \subset S$ is open, then $Y = \tau_S^{-1}(U)$ is a saturated. Also the unitary sub-bundle $Y = E_S$ is saturated.

7.4. Transverse morphisms Now we introduce the family of transverse morphisms, whose main feature is the preservation of the tubular neighborhoods. Let $H \subset G$ be a closed subgroup, Y a Thom-Mather H-stratified pseudomanifold and M be a connected manifold. A morphism

$$\psi : M \times Y \to X$$

is *transverse* whenever:

(1) $\mathrm{Im}(\psi)$ is a saturated open subspace of X.

(2) If $\psi(M \times S) \subset R$ then $\psi^{-1}(T_R) = M \times T_S$, for any $R \in \mathcal{S}_X, S \in \mathcal{S}_Y$.

Now let $\psi : M \times Y \to X$ be a transverse morphism. The *lifting* of ψ is, by definition, the map

$$\widehat{\psi} : M \times \widehat{Y} \to \widehat{X}\,, \quad \widehat{\psi}(m,z,t) = \begin{cases} (\psi(m,z),t), & (m,z,t) \in M \times E_S \times \mathbb{R}\,, \\ (\psi(m,z),t), & (m,z,t) \in M \times (Y - \Sigma^{min}) \times \{\pm 1\}\,. \end{cases}$$

This is the unique morphism such that the diagram

$$\begin{array}{ccc} M \times \widehat{Y} & \xrightarrow{\widehat{\psi}} & \widehat{X} \\ {\scriptstyle \imath_M \times \mathcal{L}_Y}\downarrow & & \downarrow{\scriptstyle \mathcal{L}_X} \\ M \times Y & \xrightarrow{\psi} & X \end{array}$$

commutes.

7.5. Examples For any smooth effective action of G, the charts of the tubular neighborhoods are transverse morphisms: Take a smooth action $\Phi : G \times M \to M$ and an invariant metric μ in M. Then M has a structure of Thom-Mather G-stratified pseudomanifold, where $\mathcal{S}_M$ is the stratification induced by the orbit types of the action. For any singular stratum S with codimension $\mathrm{codim}(S) = q + 1 > 0$, the equivariant tubular neighborhood

$T_S = N_\mu(S)$ is the normal fiber bundle over S induced by μ (see § 4.3). Then any a trivializing chart

$$\varphi : U \times c(\mathbb{S}^q) \to \tau_S^{-1}(U)$$

is transverse: Notice that $\operatorname{Im}(\varphi)$ is a saturated open subspace in M, so we only have to verify § 7.4-(2). Let S' be a stratum in $c(\mathbb{S}^q)$, R a stratum in M. Suppose that $\varphi(U \times S') \subset R$. There are two cases:

• $S' = \{\star\}$ *is the vertex*: It is straightforward, since $R = S$ and $T_{S'} = c(\mathbb{S}^q)$.

• $S' = S'' \times \mathbb{R}^+$ *for some stratum* S'' *in* $\mathbb{S}^q$: Then $S < R$. We consider in T_S the following decomposition of the metric:

$$\mu \,|_{T_S} = \mu_H + \mu_V$$

corresponding to the orthogonal decomposition of the tangent $T(T_S)$ in the horizontal and vertical sub-bundles. Hence

$$\varphi^{-1}(T_R) = \varphi^*(N_\mu(R)) = N_{\varphi^*(\mu)}(U \times S') = U \times N_{\mu_V}(S') = U \times T_{S'}\,.$$

Now we show two easy properties of the transverse morphisms.

PROPOSITION 7.6. *Let K, H a closed subgroups of G, L a Thom-Mather H-stratified pseudomanifold, $\psi : M \times L \to X$ a transverse morphism. Then*

(1) The lifting $\widehat{\psi} : M \times \widehat{L} \to \widehat{X}$ is transverse.

(2) If additionaly G is abelian, then the induced quotient map $\overline{\psi} : M \times (L/H \cap K) \to X/K$ is transverse.

Proof. (1) is straightforward from Definition § 7.4. (2) is a consequence of § 5.3. ∎

Finally, we give a sufficient condition for the existence of equivariant unfoldings.

THEOREM 7.7. *Let X be a Thom-Mather G-stratified pseudomanifold. Suppose that for any singular stratum S, each trivializing chart*

$$\varphi : U \times c(L_S) \to T_S$$

is transverse. Then

(1) The composition of the l elementary unfoldings of starting at X induces an equivariant unfolding $\mathcal{L} : \widetilde{X} \to X$ where $\widetilde{X}$ is the last (non trivial) elementary unfolding and $\mathcal{L} = \mathcal{L}_1\mathcal{L}_2 \ldots \mathcal{L}_l$ (see eq. (6) at the beginning of this section).

(2) If G is abelian then, for any closed subgroup $K \subset G$, the induced map $\overline{\mathcal{L}} : \widetilde{X}/K \to X/K$ is an unfolding.

Proof. (1) Take a family of equivariant tubular neighborhoods in X with transverse trivializing charts. Let

$$X_l \xrightarrow{\mathcal{L}_l} X_{l-1} \xrightarrow{\mathcal{L}_{l-1}} \dots \xrightarrow{\mathcal{L}_2} X_1 \xrightarrow{\mathcal{L}_1} X$$

be the chain of elementary unfoldings induced by the tubular neighborhoods. Proceed by induction on $l = d(X)$; for $l = 1$ it is straightforward. For $l > 1$ we assume the inductive hypothesis, so $\mathcal{L}' : \widetilde{X} \to X_1$ is an equivariant unfolding, for $\widetilde{X} = X_l$ and $\mathcal{L}' = \mathcal{L}_2 \dots \mathcal{L}_l$. Take a closed stratum S and a transverse trivializing chart

$$\varphi : U \times c(L_S) \to \tau_S^{-1}(U) \subset T_S\,.$$

Apply the chain of elementary unfoldings and use § 7.6; you will get the following commutative diagram:

$$\begin{array}{ccc}
U \times \widetilde{L_S} \times \mathbb{R} & \xrightarrow{\widetilde{\psi_1}} & \widetilde{X} \\
{\scriptstyle \imath_U \times \mathcal{L}_{L_S} \times \imath_{\mathbb{R}}}\downarrow & & \downarrow{\scriptstyle \mathcal{L}'} \\
U \times L_S \times \mathbb{R} & \xrightarrow{\psi_1} & X_1 \\
{\scriptstyle \imath_U \times \mathcal{L}_S}\downarrow & & \downarrow{\scriptstyle \mathcal{L}_1} \\
U \times c(L_S) & \xrightarrow{\psi} & X
\end{array}$$

We conclude that $\mathcal{L} = \mathcal{L}_1 \mathcal{L}' : \widetilde{X} \to X$ is an equivariant unfolding.
(2) This is a consequence of § 6.2-(5). ∎

Corollary 7.8. (Unfolding of a G-manifold) *Let M be a manifold, $\Phi : G \times M \to M$ a smooth effective action, possibly with fixed points. Endow M with the stratification induced by the orbit types and the usual structure of a Thom-Mather G-stratified pseudomanifold. Then there is an equivariant unfolding $\mathcal{L} : \widetilde{M} \to M$.*

Proof. Apply the above theorem to the transverse charts obtained in § 7.5. ∎

Acknowledgements

We thank M. Saralegi and R. Popper for some useful conversations, so as the referee of the journal for some helpful comments. This article was supported by the Université D'Artois; the ECOS-Nord project and the CDCH-U.C.V. While writing this work, the author received the hospitality of the Math Departments of the Université D'Artois and the Euskal Herriko Unibertsitatea.

References

[1] Brasselet, J.P., Hector, G., Saralegi, M., Théoréme de de Rham pour les variétés stratifiées, *Ann. Global Anal. Geom.,* **9** (3) (1991), 211 – 243.

[2] Bredon, G., "Introduction to Compact Transformation Groups", Pure and Appl. Math., Vol. 46, Academic Press, New York-London, 1972.

[3] Brylinski, J.L., Equivariant intersection cohomology, *Contemp. Math.,* **132** (1992), 5 – 32.

[4] Davis, M., Smooth G-manifolds as collections of fiber bundles, *Pacific J. Math.,* **77** (2) (1978), 315 – 363.

[5] Goresky, M., MacPherson, R. , Intersection homology theory, *Topology,* **19** (2) (1980), 135 – 162.

[6] Goresky, M., MacPherson, R., Intersection homology II, *Invent. Math.,* **71** (1) (1983), 77 – 129.

[7] Haefliger, A., Introduction to piecewise linear intersection homology, in "Intersection Cohomology-Swiss Seminars (Bern, 1983)", *Progress in Mathematics,* Vol. 50, 1 – 22, Birkhäuser Boston, Boston, 1984.

[8] Padilla, G., Intersection cohomology of $\mathbb{S}^1$-actions on pseudomanifolds, preprint, 2003.

[9] Pflaum, M.J., "Analytic and Geometric Study of Stratified Spaces", Lecture Notes in Mathematics, Vol. 1768, Springer-Verlag, Berlin, 2001.

[10] Popper, R., Compact Lie groups acting on pseudomanifolds, *Illinois J. Math.,* **44** (1) (2000), 1 – 19.

[11] Saralegi, M., Homological properties of stratified spaces, *Illinois J. Math.,* **38** (1) (1994), 47 – 70.

[12] Steenrod, N., "The Topology of Fibre Bundles", Princeton Mathematical Series, Vol. 14, Princeton University Press, Princeton, N. J., 1951.

[13] Thom, R., Ensembles et morphismes stratifiés, *Bull. Amer. Math. Soc.,* **75** (1969), 240 – 284.

[14] Verona, A., Le théoreme de de Rham pour les préstratifications abstraites, *C.R. Acad. Sci. Paris Ser A-B,* **273** (1971), A886 – A889.

[15] Verona, A., "Stratified Mappings, Structure and Triangulability", Lecture Notes in Mathematics, Vol. 1102, Springer-Verlag, Berlin, 1984.