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Differential Geometry of Indefinite Complex Submanifolds in Indefinite Complex Space Forms

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Dedicated to Professor Hisao Nakagawa on the occasion of his retirement

Introduction

Classically, a Kaehler structure consists of a Riemann metric and a complex structure, which are related by well known compatibility conditions. The Riemann metric is then called a Kaehler metric. If a Kaehler metric is allowed to be non-degenerate and non-definite, the concept of indefinite Kaehler structure appears naturally. So, we have a geometry which is, at the same time, complex and semi-Riemannian. Besides its purely mathematical interest, indefinite Kaehler geometry (in the case of index 2) could be seen, from the point of view of Physics, as a synthesis of two important geometries: the Lorentzian geometry of space-time and the symplectic geometry of phase space. The indefinite Kaehler metric is a complex version of the Lorentzian metric and the Kaehler form is a covariant version of the classical symplectic form. It is argued in G. Kaiser [22] that indefinite Kaehler geometry could be a unifying geometry to study holomorphic bundles on complexifications of space-time.

M. Barros and the first author [7] systematically introduced indefinite Kaehler manifolds and studied several properties involving their curvature.

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After introducing the standard spaces of constant holomorphic sectional curvature (that is, the simply connected complete indefinite complex space forms) [7], [61], it is natural to think about their complex submanifolds. These are taken to be non-degenerate in the sense that the induced metric becomes non-degenerate. Therefore, a complex submanifold itself has an indefinite Kaehler structure, namely, the inherited one from the ambient space. So, sometimes non-degenerate complex submanifolds are also called as indefinite Kaehler submanifolds.

Our viewpoint to study complex submanifolds will be the differential geometric one, that is, with emphasis on the semi-Riemannian metric as in K. Ogiue [42] and A. Ros [49]. In the definite case, complex submanifolds of relevant Kaehler manifolds were previously well known in Algebraic Geometry, in fact, several complex submanifolds are important algebraic varieties. However, we have to construct good examples of complex submanifolds which we will study later. It will be shown that the behavior of (non-degenerate) complex submanifolds in indefinite complex space forms is quite different to that known in the definite case. We can say that the geometrical properties are richer in the indefinite case, and, sometimes, so rich that some natural geometric assumptions don't lead to expected classifications. We will construct and explain many examples of complex submanifolds in several indefinite complex space forms. Then, we will give a number of results, which will be compared with the definite case. Both for the convenience of the reader and for historical reasons, we will deal first with complex hypersurfaces. Then, we will consider complex submanifolds of higher codimension, although the case of hypersurfaces will be eventually looked at again. Furthermore, some current problems and remarks are explained in the hope of giving an extensive panoramic view of the research on this topic.

To end this section we would like to point out that we have chosen here the so called *complex approach* to complex submanifolds. As it is well known there is another approach (the *real approach*). Our choice treats to unify several results obtained from the two different approaches. In order to keep a reasonable length of the paper we have not used both ones simultaneously. The reader interested can consult [42] where both approaches were simultaneously used in the positive definite case. Using [42] is not difficult to *translate* the content which follows to real notation.

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Chapter 1. Linear preliminaries

1.1. Scalar product vector spaces. Let V be an m-dimensional vector space over $\mathbb{R}$, which is called a real vector space. Given a symmetric bilinear form b on V, two vectors v and w of V are said to be orthogonal with respect to b, written $v \perp w$, provided $b(v,w) = 0$. Thus, b is called a *scalar product* provided that it is non-degenerate, that is, $b(v,w) = 0$ for any vector w in V implies $v = 0$, equivalently when the only vector which is orthogonal to any other is the zero vector. The symmetric bilinear form b is an *inner product* provided that it is positive definite. For a real vector space V equipped with a scalar product b, a vector v in V is said to be *spacelike*, *null* or *timelike*, according as $b(v,v) > 0$ or $v = 0$, $b(v,v) = 0$ and $v \neq 0$ or $b(v,v) < 0$, respectively. It is easily seen that there is a null vector in V if and only if neither b nor $-b$ is an inner product.

In the sequel, V will denote a *scalar product space*, that is, a real vector space furnished with a scalar product b. Let W be a subspace of V. Then the restriction $b|W$ to W of the scalar product is also symmetric and bilinear, but not necessarily non-degenerate. In fact, for a 1-dimensional subspace W spanned by a null vector, $b|W$ is degenerate. A subspace W is said to be *non-degenerate* if $b|W$ is non-degenerate.

The index of the scalar product b on V is defined to be the largest integer $ind\, V$ which is the dimension of a subspace W of V such that $b|W$ is negative definite. Thus the index s satisfies $0 \leq s \leq m$ and $s = 0$ if and only if b is positive definite.

Two subsets A and B of V are said to be *orthogonal*, denoted by $A \perp B$, provided that v and w are orthogonal for all v of A and w of B. For a subspace W of V, the set $W^{\perp}$ consisting of vectors v of V such that $v \perp W$ becomes a subspace, which is called the *W-perpendicular* subspace. The following properties concerning about the perpendicular operation are well known.

Lemma 1.1.1. *If W is a subspace of a scalar product space V, then we have*

(1) $\dim W + \dim W^{\perp} = \dim V$;
(2) $(W^{\perp})^{\perp} = W$;
(3) *W is non-degenerate if and only if V is the direct sum decomposition of W and $W^{\perp}$.*

Observe that, because of this property it follows that W is non-degenerate if and only if $W^{\perp}$ is also so.

A vector v of V is said to be *unit* provided that $b(v, v) = \pm 1$. As usual a set of mutually orthogonal unit vectors is said to be orthonormal. The following result is also well known.

LEMMA 1.1.2. *For a vector space V with a scalar product b we have the following properties:*

(1) *There exists an orthonormal basis $\{e_1, \ldots, e_m\}$ for V, that is, $b(e_i, e_j) = \epsilon_i \delta_{ij}$, where $\epsilon_j = b(e_j, e_j) = \pm 1$;*

(2) *Each vector v of V has a unique expression*

$$v = \sum \epsilon_j b(v, e_j) e_j;$$

(3) *For any orthonormal basis $\{e_1, \ldots, e_m\}$ for V, the number of negative signs in the signature $(\epsilon_1, \ldots, \epsilon_m)$ is equal to the index s of V;*

(4) *If W is non-degenerate, then we have*

$$\mathit{ind}\ V = \mathit{ind}\ W + \mathit{ind}\ W^{\perp}.$$

1.2. COMPLEXIFICATIONS. A *complex structure* on a real vector space V, $\dim V = m \geq 1$, is a linear operator J of V such that $J^2 = -I$, where I stands for the identity transformation of V. A vector space furnished with a complex structure J can be made a vector space over $\mathbb{C}$ by defining a scalar multiplication by complex numbers as follows:

$$(a + ib)v = av + bJv \quad \text{for any } v \in V \text{ and } a, b \in \mathbb{R},$$

where i denotes the imaginary unit. This complex vector space is denoted by V_J. Since $v \neq 0$ and Jv are linearly independent, the dimension m of V must be even and $m/2$ is the complex dimension of V_J.

Conversely, given a complex vector space V of complex dimension n, let J be the linear operator of V defined by $Jv = iv$ for any $v \in V$. Then V can be regarded as a real vector space if real scalars are only used, and it is of dimension $2n$ and J is a complex structure of V.

LEMMA 1.2.1. *For a $2n$-dimensional real vector space V with a complex structure J, there exists a basis $\{v_1, \ldots, v_n, Jv_1, \ldots, Jv_n\}$.*

Now, let V be an m-dimensional real vector space and V^c be the *complexification* of V. The set V^c is by definition the set of vectors $v + iw$ for any vectors v and w of V, and the sum and the scalar multiplication defined naturally on V^c by

$$(v_1 + iw_1) + (v_2 + iw_2) = (v_1 + v_2) + i(w_1 + w_2),$$
$$(a + ib)(v + iw) = (av - bw) + i(aw + bv),$$

for any vectors v, w, v_j, w_j $(j = 1, 2)$ of V and any $a, b \in \mathbb{R}$. It is easy to see that V^c, endowed with these operations, is a complex vector space of complex dimension m. Then V can be regarded as a real subspace of V^c in a natural way. In fact, if we call $v - iw$ the conjugate vector of $v + iw$, noted $\overline{v + iw}$, then a vector of V^c which equals to its conjugate is called a real vector. Thus, V can be naturally regarded as the real vectors of V^c.

For a linear operator f of V, a natural extension f^c from V^c to V^c is defined by $f^c(v + iw) = f(v) + if(w)$. Then it is easily seen that f^c is $\mathbb{C}$-linear and the extension is unique in a natural way. In fact, let f' be another extension of f, that is, $f'(v) = f(v)$ for any vector v of V. Since f' is also $\mathbb{C}$-linear, the relationship

$$f'(v + iw) = f'(v) + if'(w) = f(v) + if(w) = f^c(v + iw)$$

follows for any vectors v and w of V.

Remark 1.2.2. For a vector space V with a complex structure J the complex vector space V_J ought to be distinguished from the complexification of V^c of V.

Concerned with a basis for the complexification V^c of V it is easily seen that the following property holds true.

LEMMA 1.2.3. *A basis $\{v_1, \ldots, v_m\}$ for a real vector space V is also a basis for the complex vector space V^c.*

We assume next that V is a real $2n$-dimensional vector space equipped with a complex structure J. Then J can be uniquely extended to a complex linear operator J^c of V^c and it satisfies $J^{c2} = -I$. The eigenvalues of J^c are therefore i and $-i$. Let $V^{1,0}$ and $V^{0,1}$ be the eigenspaces of J^c corresponding to the eigenvalues i and $-i$, respectively. On the other hand, the dual space V^* of V admits a complex structure J^* defined by $(J^*\omega)(v) = \omega(Jv)$, $\omega \in V^*$, $v \in V$. Then, similarly the complexification V^{*c} of the dual space V^* admits the complex structure J^{*c} which is the natural extension of J^*. Let $V_{1,0}$ and $V_{0,1}$ be the eigenspaces of J^{*c} corresponding to the eigenvalues i and $-i$, respectively. The following result is then easy to prove.

PROPOSITION 1.2.4. *We have the direct sum decompositions as complex vector spaces* $V^c = V^{1,0} \oplus V^{0,1}$, $V^{*c} = V_{1,0} \oplus V_{0,1}$, *where*

$$V^{1,0} = \{v - iJv : v \in V\}, \quad V^{0,1} = \{v + iJv : v \in V\};$$

$$V_{1,0} = \{\omega \in V^{*c} : \omega(v) = 0 \text{ for all } v \in V^{0,1}\},$$

$$V_{0,1} = \{\omega \in V^{*c} : \omega(v) = 0 \text{ for all } v \in V^{1,0}\}.$$

1.3. HERMITIAN SCALAR PRODUCTS. A *Hermitian scalar product* g on a real vector space V with a complex structure J is a scalar product g on V which is J-invariant; that is, g satisfies

$$g(Jv, Jw) = g(v, w) \quad \text{for any } v \text{ and } w \text{ of } V. \tag{1.3.1}$$

It is usually said that the Hermitian scalar product g on V is *compatible* with the complex structure J.

In order to explain this terminology, the concept of a Hermitian scalar product on a complex vector space V' is introduced. It is by definition a $\mathbb{C}$-valued function h on $V' \times V'$ such that

(1) $h(v', w')$ is $\mathbb{C}$-linear in v';
(2) $h(w', v') = \overline{h(v', w')}$;
(3) h is non-degenerate in the sense that $h(v', w') = 0$ for any vector w' of V' implies that $v' = 0$.

Suppose that the complex vector space V_J, induced from a $2n$-dimensional real vector space V with a complex structure J, admits a Hermitian scalar product h. If a real-valued function g on $V \times V$ is defined as the real part of h, then g is a Hermitian scalar product on (V, J).

In fact, for any v and w of V, the function g is given by

$$g(v, w) = \{h(v, w) + h(w, v)\}/2,$$

from which it is easy to see that g is symmetric and bilinear. That $g(v, w) = 0$ for any w of V implies that $h(v, w)$ is pure imaginary, say ia, $a \in \mathbb{R}$. Then $h(v, Jw)$ becomes also so and moreover we have

$$h(v, Jw) = h(v, iw) = -ih(v, w) = a,$$

which yields that

$$a = -ih(v, w) = 0 \text{ for any } w \text{ of } V.$$

It means that g is non-degenerate, since h does so. Furthermore it satisfies

$$g(Jv, Jw) = \text{Re } h(iv, iw) = \text{Re } h(v, w) = g(v, w).$$

In particular, we have

$$h(v, w) = g(v, w) - ig(Jv, w). \tag{1.3.2}$$

Conversely, for a real vector space V with a complex structure J and a Hermitian scalar product g, a complex valued function h on $V_J \times V_J$ is defined by (1.3.2). A simple computation shows that h satisfies conditions (1) and (2) of the previous notion of Hermitian scalar product. Condition (3), that is the non-degeneracy of h, is only here asserted. Suppose that $h(X, Y) = 0$ for any vector Y of V_J. We put $X = (a + ib)v$, $a, b \in \mathbb{R}$, $v \in V$ with $v \neq 0$. For any vector w in $V \subset V_J$ we then have

$$\begin{aligned} h(X, w) &= ah(v, w) + bh(Jv, w) \\ &= \{ag(v, w) + bg(Jv, w)\} + i\{-ag(Jv, w) + bg(v, w)\} \\ &= 0, \end{aligned}$$

which implies that

$$(a^2 + b^2)g(v, w) = (a^2 + b^2)g(Jv, w) = 0,$$

for any w and therefore $a = b = 0$; that is, $X = 0$.

Accordingly, a Hermitian scalar product h on V_J corresponds one to one to that on (V, J), and from this fact it is said that the Hermitian scalar product g is compatible with the complex structure J.

Now, for a real vector space V endowed with a complex structure J and a Hermitian scalar product g, there exists a natural extension of g on V^c which is denoted by the same symbol. The extension g is defined by

$$g(X_1, X_2) = \{g(v_1, v_2) - g(w_1, w_2)\} + i\{g(v_1, w_2) + g(w_1, v_2)\} \tag{1.3.3}$$

for any vectors $X_j = v_j + iw_j$ of V^c, $j = 1, 2$. The proof of the following result is straightforward.

Proposition 1.3.1. *Let g be a Hermitian scalar product on a real vector space V with a complex structure J. Then g can be uniquely extended to a symmetric $\mathbb{C}$-bilinear form on V^c and it satisfies the following properties:*

(1) $g(\bar{X}, \bar{Y}) = \overline{g(X, Y)}$;

(2) g *is non-degenerate;*

(3) $g(X, Y) = 0$ *for any* X *and* Y *in* $V^{1,0}$ *or in* $V^{0,1}$.

Conversely, every symmetric $\mathbb{C}$*-bilinear form* g *on* V^c *satisfying the previous* (1), (2) *and* (3) *is the natural extension of a Hermitian scalar product on* V.

Remark 1.3.2. For the natural extension g to V^c of the Hermitian scalar product g of V, the complex valued function on $V^c \times V^c$ defined by

$$h(X, Y) = g(X, \bar{Y})$$

is a Hermitian scalar product on V^c.

For details in this section, see Kobayashi and Nomizu [25] and O'Neill [44], for instance.

Chapter 2. Indefinite Kaehler manifolds

2.1. Complex manifolds We begin by recalling some basic concepts of complex manifolds. An almost complex structure J of a real manifold M is a tensor field[1] of type (1,1) which satisfies $J_x^2 = -I_x$ at any point x of M, where I_x is the identity transformation of the tangent space T_xM. A manifold furnished with an almost complex structure is called an *almost complex manifold*. As is well known, an almost complex manifold is orientable and of even dimension.

Now, we recall that a complex manifold M, with complex dimension n, carries a natural almost complex structure given as follows:

For a complex coordinate system $\{z^j\}$ with $z^j = x^j + iy^j$ in a coordinate (open) neighborhood around x of M, it is seen that $\{x^1, y^1, \ldots, x^n, y^n\}$ is a real local coordinate system of M and hence

$$\Big(\frac{\partial}{\partial x^1}\Big)_x, \Big(\frac{\partial}{\partial y^1}\Big)_x, \ldots, \Big(\frac{\partial}{\partial x^n}\Big)_x, \Big(\frac{\partial}{\partial y^n}\Big)_x$$

gives a basis for T_xM. An operator J_x of T_xM can be defined by

$$J_x\Big(\frac{\partial}{\partial x^j}\Big)_x = \Big(\frac{\partial}{\partial y^j}\Big)_x, \qquad J_x\Big(\frac{\partial}{\partial y^j}\Big)_x = -\Big(\frac{\partial}{\partial x^j}\Big)_x, \tag{2.1.1}$$

[1] In this lecture note, manifolds and other geometric objects are assumed to be of class C^∞.

for $j = 1, \dots, n$. Then the definition of J_x is independent of the choice of the complex coordinate system around x.

In fact, let $\{w^j\}$ with $w = u^j + iv^j$ be another complex coordinate system. Since $z^j = x^j + iy^j$ is holomorphic, the following Cauchy-Riemann equations hold true on a neighborhood of x,

$$\frac{\partial x^j}{\partial u^k} = \frac{\partial y^j}{\partial v^k}, \qquad \frac{\partial x^j}{\partial v^k} = -\frac{\partial y^j}{\partial u^k},$$

for $j, k = 1, \dots, n$.

On the other hand, we always have for any k

$$\Big(\frac{\partial}{\partial u^k}\Big)_x = \sum_j \Big\{\Big(\frac{\partial x^j}{\partial u^k}\Big)_x \Big(\frac{\partial}{\partial x^j}\Big)_x + \Big(\frac{\partial y^j}{\partial u^k}\Big)_x \Big(\frac{\partial}{\partial y^j}\Big)_x\Big\},$$

from which together with the definition of J_x it follows that

$$\begin{aligned} J\Big(\frac{\partial}{\partial u^k}\Big)_x &= \sum_j \Big\{\Big(\frac{\partial x^j}{\partial u^k}\Big)_x \Big(\frac{\partial}{\partial y^j}\Big)_x - \Big(\frac{\partial y^j}{\partial u^k}\Big)_x \Big(\frac{\partial}{\partial x^j}\Big)_x\Big\} \\ &= \Big(\frac{\partial}{\partial v^k}\Big)_x. \end{aligned}$$

Similarly, we get

$$J_x\Big(\frac{\partial}{\partial v^k}\Big)_x = -\Big(\frac{\partial}{\partial u^k}\Big)_x,$$

which means that the definition (2.1.1) of J_x is independent of the choice of the coordinate neighborhoods. The tensor field J which assigns to each point x of M the operator J_x is smooth, since the components of J relative to the local coordinate system $\{x^1, y^1, \dots, x^n, y^n\}$, induced from the complex coordinate system in a neighborhood, are given by either of 0, 1 and -1. The definition of J yields

$$J_x^2 = -I_x \quad \text{at any point } x,$$

and hence J is an almost complex structure of M. The almost complex structure J on a complex manifold M described here is said to be induced by the complex structure of M, in fact, J is then called a *complex structure*.

The complexification T_xM^c of the tangent space at any point x, T_xM, of a real manifold M is called the *complex tangent space* at x. A *complex vector field* Z is uniquely expressed as $Z = X + iY$, where X and Y are real vector fields. If we denote by $D^r M$ the space of r-forms on M, then an element of the complexification $D^r M^c$ of $D^r M$ is called a *complex r-form* on M. Every complex r-form ω may be written as $\omega' + i\omega''$, where ω' and ω'' are real r-forms on M.

Suppose that M is an almost complex manifold with an almost complex structure J. By means of Proposition 1.2.4 we have the direct sum decompositions

$$T_x M^c = T_x M^{1,0} \oplus T_x M^{0,1}, \qquad T_x M^{*c} = D_x^{1,0} \oplus D_x^{0,1},$$

where $T_x M^{1,0}$ and $T_x M^{0,1}$ (resp. $D_x^{1,0}$ and $D_x^{0,1}$) denote the eigenspaces of J (resp. of J^*) corresponding to the eigenvalues i and $-i$. A complex tangent vector at a point x (resp. a complex 1-form) is said to be of type (1,0) or (0,1) if it belongs to $T_x M^{1,0}$ or $T_x M^{0,1}$ (resp. to $D_x^{1,0}$ or $D_x^{0,1}$). In particular, let M be a complex manifold and $\{z^j\}$ with $z^j = x^j + iy^j$ be a complex coordinate system of M. When we set

$$\frac{\partial}{\partial z^j} = \frac{1}{2}\Big(\frac{\partial}{\partial x^j} - i\frac{\partial}{\partial y^j}\Big), \qquad \frac{\partial}{\partial \bar{z}^j} = \frac{1}{2}\Big(\frac{\partial}{\partial x^j} + i\frac{\partial}{\partial y^j}\Big),$$

the complex vector $\frac{\partial}{\partial z^j}$ is the component of type (1,0) of $\frac{\partial}{\partial x^j}$, and $\frac{\partial}{\partial \bar{z}^j}$ is the component of type (0,1) of $\frac{\partial}{\partial x^j}$. Moreover, the complex vector fields

$$\frac{\partial}{\partial z^1}, \cdots, \frac{\partial}{\partial z^n}, \frac{\partial}{\partial \bar{z}^1}, \cdots, \frac{\partial}{\partial \bar{z}^n} \tag{2.1.2}$$

are said to be a *natural complex frame field* which form, at any point x of the corresponding neighborhood, a basis for the complex tangent space $T_x M^c$. From the previous construction of the complex structure J it follows that $\frac{\partial}{\partial z^1}, \cdots, \frac{\partial}{\partial z^n}$ (resp. $\frac{\partial}{\partial \bar{z}^1}, \cdots, \frac{\partial}{\partial \bar{z}^n}$) form a basis for $T_x M^{1,0}$ (resp. for $T_x M^{0,1}$) at each point x of the coordinate neighborhood.

Since $dz^j = dx^j + idy^j$ and $d\bar{z}^j = dx^j - idy^j$, we see that

$$dz^1, \ldots, dz^n, d\bar{z}^1, \ldots, d\bar{z}^n$$

form, at any point x, the dual basis corresponding to the basis (2.1.2) of $T_x M^c$ and $dz^1, \ldots, dz^n$ (resp. $d\bar{z}^1, \ldots, d\bar{z}^n$) form a basis of $D_x^{1,0}$ (resp. of $D_x^{0,1}$).

For the details in this section see Kobayashi and Nomizu [25].

2.2. THE NOTION OF INDEFINITE KAEHLER MANIFOLD. An *indefinite Hermitian metric* on an almost complex manifold M is an indefinite Riemannian metric g invariant under the almost complex structure J, i.e., g is a non-degenerate symmetric tensor field of type (0,2) on M (hence with constant index) and it satisfies

$$g(JX, JY) = g(X, Y)$$

for any vector fields X and Y of M. An indefinite Hermitian metric thus defines a Hermitian scalar product on each tangent space T_xM with respect to the almost complex structure J. A (complex) n-dimensional complex manifold endowed with an indefinite Hermitian metric (with respect to its complex structure J) is called an *indefinite Hermitian manifold*. Accordingly, Jv is spacelike (resp. null or timelike) for any spacelike (resp. null or timelike) vector v of T_xM, and hence the index of g is an even number $2s$, $0 \leq s \leq n$.

Throughout this section the following convention of the ranges on indices is used:

$$\begin{aligned} A, B, \ldots &= 1, \ldots, n, \bar{1}, \ldots, \bar{n}; \\ i, j, \ldots &= 1, \ldots, n; \\ j^* &= j + n. \end{aligned}$$

For a complex coordinate system $\{z^j\}$ of M, we put

$$\{Z_A\} = \{Z_j, Z_{\bar{j}}\}, \qquad Z_j = \frac{\partial}{\partial z^j}, \qquad Z_{\bar{j}} = \bar{Z}_j = \frac{\partial}{\partial \bar{z}^j}. \tag{2.2.1}$$

Given an indefinite Hermitian metric g, the Hermitian scalar product, defined by g_x, on each tangent space T_xM can be extended, making use of Proposition 1.3.1, to a complex symmetric bilinear form g on the complex tangent space T_xM^c. We set

$$g_{AB} = g(Z_A, Z_B). \tag{2.2.2}$$

Since g satisfies (1) and (3) of Proposition 1.3.1 and Z_j (resp. $Z_{\bar{j}}$) is of type (1,0) (resp. of type (0,1)), we have

$$g_{jk} = g_{\bar{j}\bar{k}} = 0 \tag{2.2.3}$$

and $(g_{j\bar{k}})$ is an $n \times n$ Hermitian matrix. It is then customary to express

$$ds^2 = 2 \sum g_{j\bar{k}} dz^j d\bar{z}^k \tag{2.2.4}$$

for the metric g. Let $g_{jk}, g_{jk^*}, g_{j^*k}$ and $g_{j^*k^*}$ be the components of the indefinite Hermitian metric g with respect to the basis $\{\frac{\partial}{\partial x^j}, \frac{\partial}{\partial y^j}\}$, where $z^j = x^j + iy^j$. Since g is J-invariant, we have

$$g_{jk} = g_{j^*k^*}, \qquad g_{j^*k} = -g_{jk^*}, \tag{2.2.5}$$

where the same symbol denotes both the indefinite Hermitian metric g and its natural extension to complex tangent vectors. With respect to $\{\frac{\partial}{\partial x^j}, \frac{\partial}{\partial y^j}\}$ the metric g is written as

$$ds^2 = \sum \big(g_{jk}dx^j dx^k + g_{jk^*}dx^j dy^k + g_{j^*k}dy^j dx^k + g_{j^*k^*}dy^j dy^k\big).$$

Now the fundamental 2-form Φ of M is defined by

$$\Phi(X,Y) = g(X, JY)$$

for any vector fields X and Y on M, which is also called the Kaehler form of g. An indefinite Hermitian manifold M such that its associated fundamental 2-form is closed is said to be an *indefinite Kaehler manifold*.

The fundamental 2-form Φ may be naturally extended to complex vector fields $V = \sum\{dz^j(V)Z_j + d\bar{z}^j(V)\bar{Z}_j\}$ and $W = \sum\{dz^j(W)Z_j + d\bar{z}^j(W)\bar{Z}_j\}$ on M. A simple computation shows that this extension is written

$$\Phi = -2i\sum g_{j\bar{k}}dz^j \wedge d\bar{z}^k. \tag{2.2.6}$$

On the other hand, given complex vector fields U and V on M, say $U = X + iX'$ and $V = Y + iY'$, the bracket $[U,V]$ is defined by

$$[U,V] = \Big\{[X,Y] - [X',Y']\Big\} + i\Big\{[X,Y'] + [X',Y]\Big\}.$$

It is easy to see that

$$\begin{aligned} d\Phi(U,V,W) = {} & d\Phi(X,Y,Z) - d\Phi(X,Y',Z') - d\Phi(X',Y,Z') - d\Phi(X',Y',Z) \\ & + i\Big\{d\Phi(X,Y,Z') + d\Phi(X,Y',Z) + d\Phi(X',Y,Z) - d\Phi(X',Y',Z')\Big\}, \end{aligned}$$

for any complex vector fields, $U = X + iX'$, $V = Y + iY'$ and $W = Z + iZ'$; which implies that the fundamental 2-form Φ is closed if and only if the indefinite metric g satisfies

$$\frac{\partial g_{i\bar{j}}}{\partial z^k} = \frac{\partial g_{k\bar{j}}}{\partial z^i}, \qquad \frac{\partial g_{i\bar{j}}}{\partial \bar{z}^k} = \frac{\partial g_{i\bar{k}}}{\partial \bar{z}^j}.$$

Therefore, by using here exactly the same argument which classically works in the definite case, one finds

Proposition 2.2.1. *Let M be an indefinite Hermitian manifold with an indefinite Hermitian metric g and a complex structure J. The fundamental 2-form Φ is closed if and only if $\nabla J = 0$, where ∇ denotes the Levi-Civita connection corresponding to g.*

We end this section with several remarks relating the existence of an indefinite Kaehler metric on a manifold to some restrictions on its topology.

(a) If (M, g, J) is an indefinite Hermitian manifold, then M admits a J-invariant (i.e. holomorphic) distribution of dimension the index of g.

(b) On the other hand, taking into account the fundamental 2-form of an indefinite Kaehler manifold (M, g, J), M may be contemplated as a symplectic manifold. Thus, if M is assumed to be compact, then, according to the well known Hodge-Lichnerowicz's Theorem, any of its even Betti numbers is not zero (see [45, Theorem 8.8], for instance).

(c) Finally, it is a relevant fact that several examples of compact manifolds which admit indefinite Kaehler metrics but do not admit any positive definite Kaehler metric were found in [6].

2.3. Local formulas for indefinite Kaehler submanifolds. First of all we recall several well known local formulas for the curvature tensor (in complex notation) of an indefinite Kaehler manifold, which are adapted to one of its non-degenerate complex submanifolds. Let $(\widetilde{M}, g', J)$ be an $(n+p)$-dimensional (connected) indefinite Kaehler manifold of index $2(s+t)$, $(n \geq 2,\ 0 \leq s \leq n,\ 0 \leq t \leq p)$, and let M be an n-dimensional non-degenerate complex submanifold of index $2s$ of $\widetilde{M}$. We can choose a local orthonormal frame field $\{E_A\} = \{E_1, ..., E_{n+p}\}$ on an open subset of $\widetilde{M}$ in such a way that, restricted to M, $E_1, ..., E_n$ are tangent to M and the others are normal to M. Here and in the sequel the following convention on the range of indices, unless otherwise stated, is used:

$$\begin{aligned} &A, B, \ldots = 1, \ldots, n, n+1, \ldots, n+p; \\ &i, j, \ldots = 1, \ldots, n; \\ &x, y, \ldots = n+1, \ldots, n+p. \end{aligned}$$

With respect to this frame field, let $\{\omega_A\} = \{\omega_i, \omega_y\}$ be its (local) dual frame field. Namely, it satisfies

$$\omega_A(E_B) = g'(E_A, E_B) = \epsilon_A \delta_{AB},$$

and, therefore we can write $g' = 2\sum_A \epsilon_A\omega_A\otimes\bar\omega_A$ where $\{\epsilon_A\} = \{\epsilon_i,\ \epsilon_x\}$ are given by

$$\epsilon_i = -1 \text{ or } 1 \quad \text{according to} \quad 1\le i\le s \quad \text{or} \quad s+1\le i\le n,$$
$$\epsilon_x = -1 \text{ or } 1 \quad \text{according to} \quad n+1\le x\le n+t \quad \text{or} \quad n+t+1\le x\le n+p.$$

The canonical forms ω_A and the connection forms ω_{AB} of the ambient space $\widetilde M$ satisfy the structure equations:

$$d\omega_A + \sum_B \epsilon_B\omega_{AB}\wedge\omega_B = 0, \qquad \omega_{AB} + \bar\omega_{BA} = 0, \tag{2.3.1}$$

$$\begin{aligned} d\omega_{AB} + \sum_C \epsilon_C\omega_{AC}\wedge\omega_{CB} &= \Omega'_{AB}, \\ \Omega'_{AB} &= \sum_{CD} \epsilon_C\epsilon_D R'_{\bar ABC\bar D}\omega_C\wedge\bar\omega_D, \end{aligned} \tag{2.3.2}$$

where Ω'_{AB} (resp. $R'_{\bar ABC\bar D}$) denotes the components of the Riemannian curvature 2-form Ω' (resp. the components of the Riemannian curvature tensor R') of $\widetilde M$.

The second equation of (2.3.1) means the skew-Hermitian symmetry of Ω'_{AB}, which is equivalent to

$$R'_{\bar ABC\bar D} = \bar R'_{\bar BAD\bar C}.$$

The Bianchi identities $\sum_B \epsilon_B\Omega'_{AB}\wedge\omega_B = 0$ obtained from (2.3.1) and (2.3.2), taking exterior differentiation, give the further symmetric relations

$$R'_{\bar ABC\bar D} = R'_{\bar ACB\bar D} = R'_{\bar DBC\bar A} = R'_{\bar DCB\bar A}. \tag{2.3.3}$$

Now, with respect to the previously chosen frame, the Ricci tensor S' of $\widetilde M$ can be expressed as follows

$$S' = \sum_{CD} \epsilon_C\epsilon_D(S'_{C\bar D}\omega_C\otimes\bar\omega_D + S'_{\bar CD}\bar\omega_C\otimes\omega_D),$$

where $S'_{C\bar D} = \sum_B \epsilon_B R_{\bar BBC\bar D} = S'_{\bar DC} = \bar S'_{\bar CD}$. The scalar curvature K is then given by

$$K = 2\sum_D \epsilon_D S'_{D\bar D}.$$

The indefinite Kaehler manifold $\widetilde{M}$ is said to be *Einstein* if its Ricci tensor S' is proportional to g', that is

$$S'_{C\bar{D}} = \lambda \epsilon_C \delta_{CD}, \qquad \lambda = \frac{K}{2(n+p)},$$

where the constant λ is called the Ricci curvature of the Einstein manifold.

The components $R'_{\bar{A}BC\bar{D};E}$ and $R'_{\bar{A}BC\bar{D};\bar{E}}$ (resp. $S'_{A\bar{B};C}$ and $S'_{A\bar{B};\bar{C}}$) of the covariant derivative of the Riemannian curvature tensor R' (resp. the Ricci tensor S') are respectively defined by

$$\begin{aligned}\sum_E \epsilon_E (R'_{\bar{A}BC\bar{D};E}\omega_E + R'_{\bar{A}BC\bar{D};\bar{E}}\bar{\omega}_E) = dR'_{\bar{A}BC\bar{D}} \\ - \sum_E \epsilon_E (R'_{\bar{E}BC\bar{D}}\bar{\omega}_{EA} + R'_{\bar{A}EC\bar{D}}\omega_{EB} + R'_{\bar{A}BE\bar{D}}\omega_{EC} + R'_{\bar{A}BC\bar{E}}\bar{\omega}_{ED}),\end{aligned}$$

$$\sum_C \epsilon_C (S'_{A\bar{B};C}\omega_C + S'_{A\bar{B};\bar{C}}\bar{\omega}_C) = dS'_{A\bar{B}} - \sum_C \epsilon_C (S'_{C\bar{B}}\omega_{CA} + S'_{A\bar{C}}\bar{\omega}_{CB}).$$

The second Bianchi formula is given by

$$R'_{\bar{A}BC\bar{D};E} = R'_{\bar{A}BE\bar{D};C}, \tag{2.3.4}$$

and hence we have

$$S'_{A\bar{B};C} = S'_{C\bar{B};A} = \sum_D \epsilon_D R'_{\bar{B}AC\bar{D};D}, \qquad K_B = 2\sum_C S_{B\bar{C};C}, \tag{2.3.5}$$

where $dK = \sum_C \epsilon_C (K_C\omega_C + \bar{K}_C\bar{\omega}_C)$.

The components $S'_{A\bar{B};CD}$ and $S'_{A\bar{B};C\bar{D}}$ of the second covariant derivative of S' are expressed by

$$\begin{aligned}\sum_D \epsilon_D (S'_{A\bar{B};CD}\omega_D + S'_{A\bar{B};C\bar{D}}\bar{\omega}_D) = dS'_{A\bar{B};C} \\ - \sum_D \epsilon_D (S'_{D\bar{B};C}\omega_{DA} + S'_{A\bar{D};C}\bar{\omega}_{DB} + S'_{A\bar{B};D}\omega_{DC}).\end{aligned} \tag{2.3.6}$$

Now, taking exterior differentiation of the definition of $S'_{A\bar{B};C}$ and $S'_{A\bar{B};\bar{C}}$, and using (2.3.6), the Ricci formula for the Ricci tensor S' is given as follows

$$S'_{A\bar{B};C\bar{D}} - S'_{A\bar{B};\bar{D}C} = \sum_E \epsilon_E (R'_{\bar{D}CA\bar{E}}S'_{E\bar{B}} - R'_{\bar{D}CE\bar{B}}S'_{A\bar{E}}). \tag{2.3.7}$$

Next we focus our attention on a non-degenerate complex submanifold M of $\widetilde{M}$. Restricting the above canonical forms $\{\omega_A\} = \{\omega_i, \omega_y\}$ to M, we have

$$\omega_x = 0 \tag{2.3.8}$$

and, consequently, the induced indefinite Kaehler metric g of index $2s$ of M is written as $g = 2\sum_j \epsilon_j \omega_j \otimes \bar{\omega}_j$. Therefore, the restriction to M of the frame field $\{E_j\}$ is a local orthonormal frame field with respect to g, and $\{\omega_j\}$ is its dual frame field, which consists of complex 1-forms of type (1,0) on M. Moreover $\omega_1, ..., \omega_n, \bar{\omega}_1, ..., \bar{\omega}_n$ are linearly independent, and they are said to be a set of local canonical 1-forms on M.

It follows from (2.3.8) and the Cartan lemma, by taking exterior differentiation, that

$$\omega_{xi} = \sum_j \epsilon_j h^x_{ij} \omega_j, \quad h^x_{ij} = h^x_{ji}. \tag{2.3.9}$$

The quadratic form h locally defined as

$$\sum_{ijx} \epsilon_i \epsilon_j \epsilon_x h^x_{ij} \omega_i \otimes \omega_j \otimes E_x$$

with values in the normal bundle is called the *second fundamental form* of the submanifold M. The second fundamental form h can be equivalently described as $h(X,Y) = \nabla'_X Y - \nabla_X Y$, for all vector fields X, Y tangent to M, where ∇' and ∇ denote the Levi-Civita connections of the metric of $\widetilde{M}$ and the one of M, respectively. As in the definite case, h satisfies $h(JX, Y) = h(X, JY) = Jh(X, Y)$. This property implies that $h(JX, JY) = -h(X, Y)$, for all X, Y. Consequently, the mean curvature vector field, $(1/2n)\,\mathrm{trace}_g\, h$, of an indefinite complex submanifold vanishes. When $\widetilde{M}$ is a positive definite Kaehler manifold, it is then said that every complex submanifold is *minimal*. However, we are dealing here with indefinite metrics and, therefore, a better sentence for our setting would be to say that every indefinite complex submanifold is *stationary* (i.e. a critical point to the 2n-dimensional area functional). It should be remarked that a stationary positive definite complex submanifold with negative definite normal bundle is usually called *maximal*.

Making use of the structure equations of $\widetilde{M}$ it follows that the structure equations of M are given by

$$d\omega_i + \sum_j \epsilon_j \omega_{ij} \wedge \omega_j = 0, \quad \omega_{ij} + \bar{\omega}_{ji} = 0, \tag{2.3.10}$$

$$\begin{aligned} d\omega_{ij} + \sum_k \epsilon_k \omega_{ik} \wedge \omega_{kj} &= \Omega_{ij}, \\ \Omega_{ij} &= \sum_{kl} \epsilon_k \epsilon_l R_{\bar{i}jk\bar{l}} \omega_k \wedge \bar{\omega}_l, \end{aligned} \tag{2.3.11}$$

where Ω_{ij} (resp. $R_{\bar{i}jk\bar{l}}$) denote the components of the Riemannian curvature form Ω (resp. the components of the Riemannian curvature tensor R) of M. Moreover, the following relationships hold,

$$d\omega_{xy} + \sum_z \epsilon_z \omega_{xz} \wedge \omega_{zy} = \Omega_{xy}, \qquad \Omega_{xy} = \sum_{kl} \epsilon_k \epsilon_l R_{\bar{x}yk\bar{l}} \omega_k \wedge \bar{\omega}_l, \tag{2.3.12}$$

where Ω_{xy} are the components of the normal curvature form on M. For the Riemannian curvature tensors R and R' of M and $\widetilde{M}$, respectively, it follows from (2.3.1), (2.3.2), (2.3.9), (2.3.10) and (2.3.11) that

$$R_{\bar{i}jk\bar{l}} = R'_{\bar{i}jk\bar{l}} - \sum_x \epsilon_x h^x_{jk} \bar{h}^x_{il}, \tag{2.3.13}$$

which is called the Gauss equation. From this equation, the components of the Ricci tensor S and the scalar curvature r of M satisfy

$$S_{i\bar{j}} = \sum_k \epsilon_k R'_{\bar{i}jk\bar{k}} - h^2_{i\bar{j}}, \tag{2.3.14}$$

$$r = 2\sum_j S_{j\bar{j}} = 2\sum_{jk} \epsilon_j \epsilon_k R'_{\bar{j}jk\bar{k}} - 2h_2, \tag{2.3.15}$$

where we have written $h^2_{i\bar{j}} = \sum_{kx} \epsilon_k \epsilon_x h^x_{ik} \bar{h}^x_{kj}$ and $h_2 = \sum_k \epsilon_k h^2_{k\bar{k}}$.

Now the components $h^x_{ij;k}$ and $h^x_{ij;\bar{k}}$ of the covariant derivative of the second fundamental form of M are given by

$$\begin{aligned} \sum_k \epsilon_k (h^x_{ij;k} \omega_k + h^x_{ij;\bar{k}} \bar{\omega}_k) =& dh^x_{ij} - \sum_k \epsilon_k (h^x_{kj} \omega_{ki} + h^x_{ik} \omega_{kj}) \\ &+ \sum_y \epsilon_y h^y_{ij} \omega_{xy}. \end{aligned}$$

Substituting dh^x_{ij} in this definition into the exterior derivative of (2.3.9), using the structure equations (2.3.1), (2.3.2) and (2.3.10), (2.3.11) we get

$$h^x_{ij;k} = h^x_{ji;k} = h^x_{ik;j}, \qquad h^x_{ij;\bar{k}} = -R'_{\bar{x}ij\bar{k}}. \tag{2.3.16}$$

Similarly the components $h^x_{ij;kl}$ and $h^x_{ij;k\bar{l}}$ of the second covariant derivative of the second fundamental form can be defined by

$$\begin{aligned}\sum_l \epsilon_l(h^x_{ij;kl}\omega_l + h^x_{ij;k\bar{l}}\bar{\omega}_l) =& dh^x_{ij;k} - \sum_l \epsilon_l(h^x_{lj;k}\omega_{li} + h^x_{il;k}\omega_{lj} \\ &+ h^x_{ij;l}\omega_{lk}) + \sum_y \epsilon_y h^y_{ij;k}\omega_{xy}.\end{aligned}$$

A straightforward computation give rise to the Ricci formula

$$\begin{aligned} h^x_{ij;kl} =& h^x_{ij;lk}, \\ h^x_{ij;k\bar{l}} - h^x_{ij;\bar{l}k} =& \sum_r \epsilon_r(R_{\bar{l}ki\bar{r}}h^x_{rj} + R_{\bar{l}kj\bar{r}}h^x_{ir}) \\ &- \sum_y \epsilon_y R_{\bar{x}yk\bar{l}}h^y_{ij}. \end{aligned} \tag{2.3.17}$$

Until now, we are dealing with arbitrary indefinite Kaehler manifolds as ambient spaces for complex submanifolds. Next, we will specialize to the important case in which indefinite complex space forms are considered as ambient spaces. But before a few comments about holomorphic sectional curvature is in order. Given an indefinite Kaehler manifold (M, g, J), recall that a plane section π of the tangent space T_xM of M, at any point x, is said to be *non-degenerate* provided that $g_x|\pi$ is non-degenerate. Note that π is non-degenerate if and only if there is a basis $\{u, v\}$ of π such that $g(u, u)g(v, v) - g(u, v)^2 \neq 0$. A holomorphic plane spanned by u and Ju is non-degenerate if and only if it contains some vector v with $g(v, v) \neq 0$. The sectional curvature of a non-degenerate holomorphic plane π spanned by u and Ju is called the *holomorphic sectional curvature*, and it is denoted by $H(\pi) = H(u)$. An indefinite Kaehler manifold M is said to be of *constant holomorphic sectional curvature* if its holomorphic sectional curvature function $H(\pi)$ is constant for all non-degenerate holomorphic plane π and for any point of M. In this case, M is called an *indefinite complex space form*, which is denoted by $M^m_s(c)$, provided that it is of constant holomorphic sectional curvature $c \in \mathbb{R}$, of complex dimension m and of index $2s$. The standard models of indefinite complex space forms are the following three kinds which were given by Barros and the first author [7] and Wolf [61]. Namely, the indefinite complex Euclidean space $\mathbb{C}_s^m$, the indefinite complex projective space $\mathbb{C}P_s^m(c)$ and the indefinite complex hyperbolic space $\mathbb{C}H_s^m(c)$, according as $c = 0, c > 0$ and $c < 0$. For an integer s $(0 < s < m)$ it is seen in [7], extending a well-know result by Hawley and Igusa in the positive definite case (see for instance Kobayashi and Nomizu [25, p. 171]), that $\mathbb{C}_s^m$, $\mathbb{C}P_s^m(c)$ and $\mathbb{C}H_s^m(c)$ are the only geodesically complete, simply connected and connected indefinite complex space forms of dimension m and of index $2s$. Moreover, an indefinite complex space form $M_s^m(c)$ must be locally holomorphically isometric to $\mathbb{C}_s^m$, $\mathbb{C}P_s^m(c)$ or $\mathbb{C}H_s^m(c)$ provided that $c = 0$, $c > 0$ or $c < 0$, respectively.

Let us consider now that the ambient space is an indefinite complex space form $\widetilde{M}_{s+t}^{n+p}(c')$ of constant holomorphic sectional curvature c'. If M_s^n is a (non-degenerate) complex submanifold of $\widetilde{M}_{s+t}^{n+p}(c')$, then (2.3.13), (2.3.14), (2.3.15), (2.3.16) and (2.3.17) specialize to get

$$R_{\bar{i}jk\bar{l}} = \frac{c'}{2}\epsilon_j\epsilon_k(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}) - \sum_x \epsilon_x h_{jk}^x \bar{h}_{il}^x, \tag{2.3.18}$$

$$S_{i\bar{j}} = (n+1)\frac{c'}{2}\epsilon_i\delta_{ij} - h_{i\bar{j}}^2, \tag{2.3.19}$$

$$r = n(n+1)c' - 2h_2, \tag{2.3.20}$$

$$h_{ij;k}^x = h_{ji;k}^x = h_{ik;j}^x, \quad h_{ij;\bar{k}}^x = 0, \tag{2.3.21}$$

$$\begin{aligned} h_{ij;k\bar{l}}^x = {} & \frac{c'}{2}\big(\epsilon_k h_{ij}^x\delta_{kl} + \epsilon_i h_{jk}^x\delta_{il} + \epsilon_j h_{ki}^x\delta_{jl}\big) \\ & - \sum_{ry}\epsilon_r\epsilon_y\big(h_{ri}^x h_{jk}^y + h_{rj}^x h_{ki}^y + h_{rk}^x h_{ij}^y\big)\bar{h}_{rl}^y. \end{aligned} \tag{2.3.22}$$

Besides of the holomorphic sectional curvature, on indefinite Kaehler manifolds it is possible to consider another curvature functions. Precisely, we end this section recalling the notion of *real bisectional curvature* of an indefinite Kaehler manifold, in order to be used later. In [10] R.L. Bishop and S.I. Goldberg introduced the notion of totally real bisectional curvature B on a (positive definite) Kaehler manifold (M, g, J). A totally real plane Span$\{u, v\}$ of T_xM is, by definition, orthogonal to its image by the complex structure Span$\{Ju, Jv\}$. Thus, two orthonormal vectors u and v span a totally real plane if and only if u, v and Jv are orthonormal. The totally real bisectional curvature of a totally real plane Span$\{u, v\}$ is then defined by

$$B(u, v) = \frac{g(R(u, Ju)Jv, v)}{g(u, u)g(v, v)}. \tag{2.3.23}$$

C.S. Houh [18] showed that an $(n{\geq}3)$-dimensional Kaehler manifold has constant totally real bisectional curvature c if and only if it has constant holomorphic sectional curvature $2c$ (see Example 2.3.1 for the sufficient condition).

On the other hand, S.I. Goldberg and S. Kobayashi introduced in [17] the notion of *holomorphic bisectional curvature* $H(u,v)$, which is determined by two holomorphic planes $\mathrm{Span}\{u, Ju\}$ and $\mathrm{Span}\{v, Jv\}$. They asserted that the complex projective space $\mathbb{C}P^n(c)$ is the only compact Kaehler manifold with positive holomorphic bisectional curvature and constant scalar curvature.

If we compare the notion of totally real bisectional curvature $B(u,v)$ with the one of holomorphic bisectional curvature $H(u,v)$ and the one of holomorphic sectional curvature $H(u)$, then $H(u,v)$ turns out to be the totally real bisectional curvature $B(u,v)$ (resp. the holomorphic sectional curvature $H(u)$) when the two holomorphic planes $\mathrm{Span}\{u, Ju\}$ and $\mathrm{Span}\{v, Jv\}$ are orthogonal to each other (resp. coincide with each other). From this assertion it follows that the positiveness of $B(u,v)$ is weaker than the positiveness of $H(u,v)$, because $H(u,v) > 0$ implies that both of $B(u,v)$ and $H(u)$ are positive but we do not know whether $B(u,v) > 0$ implies $H(u,v) > 0$ or not.

Now let us also denote by (M, g, J) an n-dimensional indefinite Kaehler manifold. Of course, the previous formula (2.3.23) may be used to define the totally real bisectional curvature of a non-degenerate totally real plane $\mathrm{Span}\{u, v\}$. Let us remark that the previously mentioned result by Houh was extended to indefinite Kaehler manifolds in [7].

If it is assumed $g(u,u) = g(v,v) = \pm 1$, then we can use the first Bianchi identity in (2.3.23), we get

$$\begin{aligned} B(u,v) &= g(R(u,Jv)Jv,u) + g(R(u,v)v,u) \\ &= K(u,v) + K(u,Jv), \end{aligned} \tag{2.3.24}$$

where $K(u,v)$ and $K(u,Jv)$ mean the sectional curvatures of the planes $\mathrm{Span}\{u,v\}$ and $\mathrm{Span}\{u,Jv\}$, respectively. Hereafter unless otherwise stated, we only consider such a situation; that is, we only consider definite totally real planes.

Now if we put $u' = \frac{1}{\sqrt{2}}(u+v)$ and $v' = \frac{1}{\sqrt{2}}J(u-v)$, then it is easily seen that $g(u',u') = \pm 1$, $g(v',v') = \pm 1$, and $g(u',Jv') = 0$. Thus

$$B(u',v') = \frac{g(R(u',Ju')Jv',v')}{g(u',u')g(v',v')}$$

implies that

$$\begin{aligned} g(u',u')g(v',v')B(u',v') &= g(R'(u',Ju')Jv',v') \\ &= \frac{1}{4}g(u,u)g(v,v)\big\{H(u)+H(v)+2B(u,v)-4K(u,Jv)\big\}, \end{aligned}$$

where $H(u) = K(u, Ju)$, and $H(v) = K(v, Jv)$ are the holomorphic sectional curvatures of the planes $\mathrm{Span}\{u, Ju\}$ and $\mathrm{Span}\{v, Jv\}$, respectively. Since $\mathrm{Span}\{u, v\}$ is definite, it follows that

$$g(u',u')g(v',v') = g(u,u)g(v,v) = 1$$

and therefore

$$4B(u',v') - 2B(u,v) = H(u) + H(v) - 4K(u,Jv). \tag{2.3.25}$$

If we put $u'' = \frac{1}{\sqrt{2}}(u + Jv)$, and $v'' = \frac{1}{\sqrt{2}}(Ju + v)$, then the definiteness of the plane $\mathrm{Span}\{u, v\}$ such that $g(u, u) = g(v, v) = \pm 1$ implies $g(u'', u'') = g(v'', v'') = \pm 1$ and $g(u'', v'') = 0$. Using a similar method to the one considered to get (2.3.25), we have

$$4B(u'',v'') - 2B(u,v) = H(u) + H(v) - 4K(u,v). \tag{2.3.26}$$

Summing up (2.3.25) and (2.3.26), we obtain

$$2B(u',v') + 2B(u'',v'') = H(u) + H(v). \tag{2.3.27}$$

Now we compute the totally real bisectional curvatures of several distinguished examples of indefinite Kaehler manifolds.

EXAMPLE 2.3.1. Let $\widetilde{M}^n_s(c)$ be an indefinite complex space form of constant holomorphic sectional curvature c and of index $2s$, $0 \le s \le n$. Then $\widetilde{M}^n_s(c)$ has constant totally real bisectional curvature $c/2$. In fact, if $\mathrm{Span}\{u, v\}$ is a totally real plane, then $B(u, v) = \frac{g(R(u,Ju)Jv,v)}{g(u,u)g(v,v)} = c/2$ easily follows from the algebraic form of the curvature tensor of $\widetilde{M}^n_s(c)$.

In order to show a Kaehler manifold, with special interest for us, which is not of constant totally real bisectional curvature we will consider the *complex quadric* $\mathbb{C}Q'^n$ in the indefinite complex hyperbolic space $\mathbb{C}H^{n+1}_1(c')$, $c' < 0$ (Example 3.2.3).

EXAMPLE 2.3.2. Let $\mathbb{C}Q_s^n$ be the indefinite complex quadric in the indefinite complex projective space (Example 3.2.3). Then it can be identified to the Hermitian symmetric space $SO^s(n+2)/(SO^s(n)\times SO(2))$. The canonical decomposition of the Lie algebra $\mathfrak{O}^s(n+2)$ of the Lie group $SO^s(n+2)$ is given by

$$\mathfrak{O}^s(n+2) = \mathfrak{H} \oplus \mathfrak{M},$$

where $\mathfrak{H} = \mathfrak{O}(2) \oplus \mathfrak{O}^s(n)$ and

$$\mathfrak{M} = \left\{ \left(\begin{array}{cc} 0 & \begin{pmatrix} \xi_1 & \cdots & \xi_s & -\xi_{s+1} & \cdots & -\xi_n \\ \eta_1 & \cdots & \eta_s & -\eta_{s+1} & \cdots & -\eta_n \end{pmatrix} \\ \begin{pmatrix} \xi_1 & \eta_1 \\ \cdot & \cdot \\ \cdot & \cdot \\ \xi_n & \eta_n \end{pmatrix} & 0 \end{array} \right) \middle| \; \xi_j, \eta_j \in \mathbb{R} \right\}.$$

which can be naturally identified to the tangent space at a point of $\mathbb{C}Q_s^n$. It is classical (see [25, Example XI.10.6]) that several interesting geometric objects on $\mathbb{C}Q_s^n$ can be described as suitable multilinear maps on the subspace $\mathfrak{M}$ of $\mathfrak{O}^s(n+2)$, in particular, the curvature tensor and, therefore, the totally real bisectional curvature [24].

By changing the metric tensor g of $\mathbb{C}Q_s^n$ in $\mathbb{C}P_s^{n+1}(c)$ to its negative, we can also consider $(\mathbb{C}Q_{n-s}^n, -g)$ as a complex hypersurface of the indefinite complex hyperbolic space $\mathbb{C}H_{n+1-s}^{n+1}(c')$, $c' = -c < 0$. Denote by $\mathbb{C}Q'^n_{n-s}$ this complex hypersurface of $\mathbb{C}H_{n+1-s}^{n+1}(c')$. Making use of the *Lagrange multiplier rule* the totally real bisectional curvature $B(u,v)$ of $\mathbb{C}Q_n^n = SO^n(n+2)/(SO^n(n)\times SO(2))$ in $\mathbb{C}P_n^{n+1}(c)$ is computed in [24]. Hence, the totally real bisectional curvature $B'(u,v)$ of $\mathbb{C}Q'^n$ in $\mathbb{C}H_1^{n+1}(c')$, $c' = -c < 0$. In fact, we have

$$\frac{c'}{2} \leq B'(u,v) \leq -\frac{3}{2}c'.$$

On the other hand, from (2.3.27) it follows that

$$2B'(u',v') + c' \leq 2B'(u',v') + 2B'(u'',v'') = H'(u) + H'(v) \leq c'.$$

Thus $B'(u',v')\leq 0$. Consequently we get

$$\frac{c'}{2} \leq B'(u,v) \leq 0.$$

Let us finally note that the totally real bisectional curvature B of the classical complex quadric $\mathbb{C}Q^n$ in $\mathbb{C}P^{n+1}(c)$ satisfies $0 \leq B \leq \frac{c}{2}$.

Chapter 3. Complex hypersurfaces

3.1. Preliminaries. The Ricci tensor has been a fundamental tool in the study of complex hypersurfaces of (positive definite) complex space forms. Smyth, in his important paper [53], studied complete complex hypersurfaces in a complex space form whose induced Kaehler metric is Einstein. He was motivated by the well known results of Thomas, Cartan and Fialkow on Einstein hypersurfaces of Riemannian manifolds of constant sectional curvature (i.e. of real space forms). He proved that these hypersurfaces are either totally geodesic or a certain hyperquadric of the complex projective space. On the other hand, the quoted results on Einstein hypersurfaces in real space forms, were extended by Magid to the indefinite real case [29], [30] and [31]. Now, we are going to study complex Einstein hypersurfaces in indefinite complex space forms.

Let M_s^n be a (non degenerate) complex hypersurface of (complex) dimension n and index $2s$, $0 \leq s \leq n$, of an $(n+1)$-dimensional indefinite complex space form $\widetilde{M}_{s+t}^{n+1}(c)$, with holomorphic sectional curvature $c \in \mathbb{R}$ and index $2s+2t$, where $t=0$ or $t=1$ according as the normal bundle is positive definite or negative definite, respectively. As we asserted in the last section, we can identify $\widetilde{M}_{s+t}^{n+1}(c)$ with the indefinite complex projective space $\mathbb{C}P_{s+t}^{n+1}(c)$ if $c>0$, with the indefinite complex hyperbolic space $\mathbb{C}H_{s+t}^{n+1}(c)$ if $c<0$, or with the indefinite complex flat space $\mathbb{C}_{s+t}^{n+1}$ if $c=0$. The Ricci tensor of a non-degenerate complex hypersurface M_s^n of $\widetilde{M}_{s+t}^{n+1}(c)$ is given by (2.3.19) or in real notation, [35], [36], by

$$S(X,Y) = \frac{1}{2}(n+1)cg(X,Y) - 2ag(A^2X,Y) \tag{3.1.1}$$

for any vector fields X, Y on M, where $a = g'(\xi,\xi) = +1$ or -1 and A is the shape operator (also called Weingarten operator) associated to the (local) unit normal vector field ξ. Suppose that M_s^n is an Einstein space, that is, $S = \lambda g$, with $\lambda \in \mathbb{R}$. Thus, previous (3.1.1) (or formula (2.3.19)) implies that A satisfies the following polynomial equation

$$x^2 + \beta = 0, \tag{3.1.2}$$

where $\beta = \frac{1}{2}a(\lambda - \frac{1}{2}(n+1)c)$. At this point, Smyth [53], used a simple but important algebraic fact, namely, the shape operator A is (in the definite case) diagonalizable. From this and (2.3.21) it can be proved that a complex hypersurface of a (definite) complex space form is locally symmetric whenever it is assumed to be an Einstein space. Here, we cannot always diagonalize A and, consequently, we must use the well known classification for self-adjoint operators of a scalar product vector space (see for instance Maltsev [32]). So, we have

(i) A is diagonalizable, or

(ii) A is not, but either $A^2 = -b^2 I$, with $b \in \mathbb{R}$, $b \neq 0$, or $A^2 = 0$ and $A \neq 0$.

Except for the last case in (ii) we can also obtain [35] the following result,

THEOREM 3.1.1. *Let M^n_s be a complex Einstein hypersurface of an indefinite complex space form. If the shape operator A associated to a unit normal vector field satisfies $A^2 \neq 0$ or $A = 0$, then M^n_s is locally symmetric.*

Examples 3.3.6 and 3.3.7 below will show that this result is the best possible in that direction.

3.2. PROPER COMPLEX EINSTEIN HYPERSURFACES. In order to do a classification, it is suitable to call a complex Einstein hypersurface *proper* if the shape operator A associated to a unit normal vector field is diagonalizable. Note that if this holds for a concrete unit normal vector field, the same is also true for any normal vector field, so that it is a property of the holomorphic immersion of M^n_s in the indefinite complex space form as a complex hypersurface. Clearly, if A satisfies $A^2 = b^2 I$, $b \in \mathbb{R}$, $b \neq 0$, then M^n_s is proper (as usual, we will refer to the immersion of M^n_s to $\widetilde{M}^{n+1}_{s+t}(c)$ only by saying M^n_s, whenever there is no possible confusion).

We first consider several examples of proper complex Einstein hypersurfaces.

EXAMPLE 3.2.1. The indefinite complex flat space $\mathbb{C}^n_s$ is a totally geodesic (hence Einstein) complex hypersurface of $\mathbb{C}^{n+1}_{s+t}$, $t = 0, 1$, in a natural way.

EXAMPLE 3.2.2. For an indefinite complex projective space $\mathbb{C}P^{n+1}_s(c)$, if $z_1, \ldots, z_s, z_{s+1}, \ldots, z_{n+2}$ is its usual homogeneous coordinate system, then for each j fixed, the equation $z_j = 0$ defines a totally geodesic (hence Einstein) complex hypersurface identifiable with $\mathbb{C}P^n_s(c)$ or $\mathbb{C}P^n_{s-1}(c)$ according as $s + 1 \leq j \leq n+2$ or $1 \leq j \leq s$, respectively. Taking into account that the indefinite complex hyperbolic space $\mathbb{C}H^n_s(-c)$, $c > 0$, is obtained from $\mathbb{C}P^n_{n-s}(c)$ by taking the opposite of its Kaehler metric, the previous discussion shows that

$\mathbb{C}H_s^n(-c)$ is a totally geodesic complex hypersurface of both $\mathbb{C}H_s^{n+1}(-c)$ and $\mathbb{C}H_{s+1}^{n+1}(-c)$.

EXAMPLE 3.2.3. Let $\mathbb{C}Q_s^n$ be the complex hypersurface of $\mathbb{C}P_s^{n+1}(c)$ defined by the equation

$$-\sum_{i=1}^{s} z_i^2 + \sum_{j=s+1}^{n+2} z_j^2 = 0 \tag{3.2.1}$$

in the homogeneous coordinate system of $\mathbb{C}P_s^{n+1}(c)$. Then $\mathbb{C}Q_s^n$ is a non-degenerate complex hypersurface, has index $2s$ and satisfies $A^2 = \frac{c}{4}I$, where I denotes the identity transformation. Therefore, by comparing (2.1.1) and (2.3.19), we have for the second fundamental form

$$h_{i\bar{j}}^2 = \frac{c}{2}\epsilon_i\delta_{ij}.$$

Now, also by using again (2.1.1) or (2.3.19) we achieve that $\mathbb{C}Q_s^n$ is an Einstein space with $S = \frac{nc}{2}g$. $\mathbb{C}Q_s^n$ is called the indefinite complex quadric in $\mathbb{C}P_s^{n+1}(c)$. As in Kobayashi and Nomizu [25, Example XI.10.6], $\mathbb{C}Q_s^n$ is globally holomorphically isometric to the Hermitian symmetric space $SO^s(n+2)/(SO^s(n)\times SO(2))$ endowed with a negative multiple of the Killing form of $SO^s(n+2)$; in particular, [44, Lemma 8.20] gives that $\mathbb{C}Q_s^n$ is geodesically complete. Now consider $\mathbb{C}Q_{n-s}^n$ as a complex hypersurface of $\mathbb{C}P_{n-s}^{n+1}(c)$, $c>0$. If we change the Kaehler metric of $\mathbb{C}P_{n-s}^{n+1}(c)$ by its opposite, we have that $\mathbb{C}Q_{n-s}^n$ endowed with its opposite metric is also an Einstein hypersurface of $\mathbb{C}H_{s+1}^{n+1}(-c)$. Denote this complex hypersurface by $\mathbb{C}Q'^n_s$. As the Ricci tensor is invariant by an homothetical change of the metric we have that $\mathbb{C}Q'^n_s$ and $\mathbb{C}Q_{n-s}^n$ have the same Ricci tensor. Thus $\mathbb{C}Q'^n_s$ is Einstein with $S = \frac{n(-c)}{2}g'$, where g' is the Kaehler metric of $\mathbb{C}Q'^n_s$. A reasoning as above permits us to identify $\mathbb{C}Q'^n_s$ to the Hermitian symmetric space $SO^{n-s}(n+2)/(SO^{n-s}(n)\times SO(2)) = SO^{s+2}(n+2)/(SO^s(n)\times SO(2))$. In particular, the spacelike (i.e. positive definite) complex hypersurface $\mathbb{C}Q'^n$ of $\mathbb{C}H_1^{n+1}(-c)$ is the non-compact Hermitian symmetric space $SO^2(n+2)/(SO(n)\times SO(2))$.

Finally, note that $\mathbb{C}Q_s^n$ can be contemplated as the Grassmannian manifold of all oriented 2-dimensional spacelike subspaces of the indefinite flat Riemannian space $\mathbb{R}_s^{n+2}$. This extends a well-known fact in the positive definite case (i.e. when $s=0$).

These examples are shown to be the only proper complex Einstein hypersurfaces. In fact, following the idea of Smyth [53] and [54], we can investigate the restricted holonomy group of such a hypersurface of an indefinite complex space form. Then, we use the Berger list of symmetric spaces [9] to obtain the following classification result,

THEOREM 3.2.4. (i) *The only proper complete indefinite complex Einstein hypersurfaces of* $\mathbb{C}P^{n+1}_{s+t}(c)$, $c > 0$, $n > 2$ *are* $\mathbb{C}P^n_s(c)$ *with* $t = 0, 1$ *and* $\mathbb{C}Q^n_s$ *with* $t = 0$.

(ii) *The only proper complete indefinite complex Einstein hypersurfaces of* $\mathbb{C}H^{n+1}_{s+t}(-c)$, $c > 0$, $n > 2$ *are* $\mathbb{C}H^n_s(-c)$ *with* $t = 0, 1$ *and* $\mathbb{C}Q'^n_s$ *with* $t = 1$.

(iii) *The only proper complete indefinite complex Einstein hypersurface of* $\mathbb{C}^{n+1}_{s+t}$ $n > 2$ *is* $\mathbb{C}^n_s(c)$ *with* $t = 0, 1$.

This result was obtained by Montiel and the first author in [35] with an extra assumption. In fact, it was assumed that the complex hypersurface is simply connected. However, we can prove that this condition can be dropped. In order to do this, let M be the complex Einstein hypersurface and let φ be the holomorphic immersion of M to $\widetilde{M}(c)$. We denote by $\hat{M}$ the universal semi-Riemannian covering manifold of M and by $\pi : \hat{M} \longrightarrow M$ the corresponding covering map. Clearly, $\hat{M}$ is an indefinite Kaehler manifold and $\pi : \hat{M} \longrightarrow M$ is a holomorphic local isometry. Thus, $\hat{M}$ is a simply connected complete indefinite complex Einstein hypersurface, immersed in $\widetilde{M}(c)$ by $\varphi \circ \pi$. Now, we can use [35, Theorem 4.4] to conclude that if $\widetilde{M}(c) = \mathbb{C}P^{n+1}_{s+t}(c)$ then $\hat{M}$ is holomorphically isometric either to $\mathbb{C}P^n_s(c)$ or to $\mathbb{C}Q^n_s$. By rigidity, the first author [48], Umehara [59] (see also Montiel and the first author [35, pp. 502–503] for a direct proof of the rigidity for our setting) $\hat{M}$ immerses either onto an indefinite complex projective hyperplane or onto an indefinite complex quadric in $\mathbb{C}P^{n+1}_{s+t}(c)$. In either case, $(\varphi \circ \pi)(\hat{M})$ is a simply connected manifold (recall that $\mathbb{C}P^n_s(c)$ and $\mathbb{C}Q^n_s$ are simply connected manifolds) and, therefore, the covering map $\varphi \circ \pi$ must be one-to-one. Thus, π is one-to-one and M is holomorphically isometric to $\mathbb{C}P^n_s(c)$ or to $\mathbb{C}Q^n_s$, according to the case. The same argument can be also applied when $\widetilde{M}(-c) = \mathbb{C}H^{n+1}_{s+t}(-c)$, $c > 0$, or $\mathbb{C}^{n+1}_{s+t}$. Thus, Theorem 3.2.4 is an improved form of [35, Theorem 4.4].

The following local version of Theorem 3.2.4 has been proved by Aiyama, Nakagawa and the second author [3],

THEOREM 3.2.5. *Let* M *be an indefinite complex Einstein hypersurface of an indefinite complex space form* $\widetilde{M}^{n+1}_{s+t}(c)$. *If* M *is proper, then* M *is totally* *geodesic or* $S = \frac{nc}{2}g$, *the latter arising only when* $c > 0$ *and* $t = 0$ *or* $c < 0$ *and* $t = 1$.

3.3. Non-proper complex Einstein hypersurfaces. The non-proper indefinite complex Einstein hypersurfaces present a more irregular geometric behavior. We begin this section with several examples.

Example 3.3.1. Let us consider the complex hypersurface of $\mathbb{C}P^{2n+1}_{n+1}(c)$ defined by the equation

$$\sum_{j=1}^{n+1} z_j z_{n+1+j} = 0 \tag{3.3.1}$$

in the usual homogeneous coordinate system of $\mathbb{C}P^{2n+1}_{n+1}(c)$. This complex hypersurface is non-degenerate, has index $2n$ and satisfies $A^2 = -\frac{c}{4}I$, where I denotes the identity transformation. Therefore, from (3.1.1), we get for the second fundamental form

$$h^2_{i\bar{j}} = -\frac{c}{2}\epsilon_i \delta_{ij}.$$

Now by using (2.3.19) we have that this complex hypersurface is an Einstein space with $S = \frac{(n+2)c}{2}g$. This hypersurface is represented by $\mathbb{C}Q^*_n$ in Montiel and the first author [35] and it is called the *2n-dimensional special quadric*. As in Example 3.2.3, $\mathbb{C}Q^*_n$ is holomorphically isometric to the non-compact Hermitian symmetric space $SO^*(2n+2)/(SO^*(2n) \times SO(2))$ endowed with a negative multiple of the Killing form of the Lie group $SO^*(2n+2)$ (see Berger [9, p. 113], for a description of this Lie group and the corresponding symmetric space). Again, using O'Neill [44, Lemma 8.20], we have that $\mathbb{C}Q^*_n$ is geodesically complete. As in the previous examples, we can also obtain an indefinite complex Einstein hypersurface of $\mathbb{C}H^{2n+1}_n(-c)$, $c > 0$, such that $A^2 = -\frac{c}{4}I$.

Remark 3.3.2. If we write the same equation in Example 3.3.1 as defining a complex hypersurface of the (definite) complex projective space $\mathbb{C}P^{2n+1}(c)$ (recall that $\mathbb{C}P^m_s(c)$ is, as a complex manifold, an open subset of $\mathbb{C}P^m(c)$ for all s) we obtain a complex hypersurface congruent with the usual complex quadric (see Kobayashi and Nomizu [25, Example XI.10.6] or Example 3.2.3 above with $s = 0$).

EXAMPLE 3.3.3. Let M be the complex hypersurface of $\mathbb{C}_n^{2n+1}$, $n \geq 2$, given by

$$\sum_{j=1}^{n}(z_j - z_{n+j})^2 = 2z_{2n+1}. \tag{3.3.2}$$

As it is shown in Romero [47], this non-degenerate complex hypersurface M is holomorphically diffeomorphic to $\mathbb{C}^{2n}$, is geodesically complete, has index $2n$ and its shape operator satisfies $A^2 = 0$ but $A \neq 0$; that is, M is Ricci flat but non flat. Moreover M is locally symmetric [47]. Thus M is a semi-Riemannian symmetric space; in fact, it has been studied from an intrinsic point of view in Cahen and Parker [11, p. 346]. The change of the Kaehler metric of the ambient space $\mathbb{C}_n^{2n+1}$ by its opposite, permits us to have a complex hypersurface of $\mathbb{C}_{n+1}^{2n+1}$, with $A^2 = 0$ and $A \neq 0$. Therefore we get (both cases) for the second fundamental form

$$h_{i\bar{j}}^2 = 0.$$

Note that the rank of A is maximum at any point of M and, thus, we have $\mathrm{Im}(A) = \mathrm{Ker}(A)$ on all M. If we think on the same equation (3.3.2) as defining a complex hypersurface in $\mathbb{C}_s^m$ with $s > n$ and $m > 2n+1$, then we have a complex Einstein hypersurface of $\mathbb{C}_s^m$ satisfying $A^2 = 0$ and $A \neq 0$ and, clearly, with $\mathrm{Im}(A) \neq \mathrm{Ker}(A)$.

Remark 3.3.4. In the definite case, we know by a classical result of Aleeksevski and Kimenfeld (see Berard-Bergery [8, p. 553]) that a Ricci-flat homogeneous Riemannian metric must be flat. Note that the indefinite complex hypersurface M given in Example 3.3.3 is homogeneous and Ricci-flat, but M is not flat.

EXAMPLE 3.3.5. Let M_p', $p \in \mathbb{Z}$, $p \geq 3$, be the non-degenerate complex hypersurface of $\mathbb{C}_n^{2n+1}$ given by

$$\sum_{j=1}^{n}(z_j - z_{n+j})^p = pz_{2n+1}. \tag{3.3.3}$$

Then, [47], each M_p' is holomorphically diffeomorphic to $\mathbb{C}^{2n}$, is geodesically complete, has index $2n$ and satisfies $A^2 = 0$ but $A \neq 0$. Therefore we get for the second fundamental form

$$h_{i\bar{j}}^2 = 0.$$

Now, using again (2.3.19) we have that M'_p is Ricci-flat (but non flat). As in previous example, the rank of A is maximum at any point and, thus, we have $\mathrm{Im}(A) = \mathrm{Ker}(A)$ on all M'_p. As it is shown in Romero [47], the curvature tensor R of M'_p satisfies $\nabla^{2(p-2)+1}R = 0$ and $\nabla^{2(p-2)}R \neq 0$, for any p, where $\nabla^k R$ denotes the k-th covariant derivative of R. Therefore, M'_p and $M'_{p'}$ cannot be (locally) isometric if $p \neq p'$.

On the other hand, if $z \in M'_p$ satisfies $z_j = z_{n+j}$ for all $j = 1, 2, ..., n$, $z_{2n+1} = 0$, then $R = 0$ at this point z. If M'_p would be homogeneous this fact would imply $R \equiv 0$, which is not clearly true. Observe that this example provides us with a complex hypersurface of $\mathbb{C}^{2n+1}_{n+1}$ with the same properties. If we think on the equation (3.3.3) as defining a complex hypersurface in $\mathbb{C}^m_s$ with $s > n$ and $m > 2n+1$, then we have a complex Einstein hypersurface of $\mathbb{C}^m_s$ satisfying $A^2 = 0$ and $A \neq 0$ and, clearly, with $\mathrm{Im}(A) \neq \mathrm{Ker}(A)$.

Example 3.3.6. Let $M^{2n}_s(h_j; c_j)$ be the complex hypersurface of $\mathbb{C}^{2n+1}_s$ given by

$$\sum_{j=1}^{n} h_j\big(z_j + c_j z_{n+j}\big) = z_{2n+1}, \tag{3.3.4}$$

where h_j, $1 \leq j \leq n$, are holomorphic functions on $\mathbb{C}$, and c_j, $1 \leq j \leq n$, complex numbers. Note that $M^{2n}_n(\frac{1}{p}z^p; -1)$, $p \geq 2$, are the complex hypersurfaces in Examples 3.3.3 and 3.3.5. In Aiyama, Ikawa, Kwon and Nakagawa [1] are constructed and studied these hypersurfaces. Among other geometric properties, it is shown that $M^{2n}_s(h_j; c_j)$ is a complete complex hypersurface of index $2s$ in $\mathbb{C}^{2n+1}_s$ if $| c_k | \geq 1$ for any k such that $1 \leq k \leq s$, and it is holomorphically diffeomorphic to $\mathbb{C}^{2n}$. Moreover, if all functions h_q, $s+1 \leq q \leq n$, are linear and $| c_k | = 1$ for any k such that $1 \leq k \leq s$, then $M^{2n}_s(h_j; c_j)$ is Ricci-flat and it is not flat provided there is a function h_k, $1 \leq k \leq s$, which is not linear.

Example 3.3.7. Let M''_p, $p \in \mathbb{Z}$, $p \geq 2$, be the complex hypersurface of $\mathbb{C}P^{2n}_n(c)$, $c > 0$, defined by

$$\sum_{j=1}^{n} (z_j - z_{n+j})^p = z^p_{2n+1}, \quad z_{2n+1} \neq 0. \tag{3.3.5}$$

in the usual homogeneous coordinate system of $\mathbb{C}P^{2n}_n(c)$. As it is shown in Romero [46], each M''_p is non-degenerate, has index $2n$ and satisfies $A^2 = 0$

but $A\neq 0$. Therefore also we get for the second fundamental form

$$h^2_{i\bar{j}} = 0.$$

Now, by using (2.3.19) we have that M''_p is an Einstein space with $S = ncg$. So that, the scalar curvature of M''_p is $2cn^2$, for all p. Moreover, [46], M''_p is not locally symmetric; $\mathrm{Ker}(A)$ is an autoparallel (non parallel) complex differentiable distribution on M''_p; M''_p is not geodesically complete but it is not extendible (as a semi-Riemannian manifold). Finally, if $p\neq p'$ then M''_p and $M''_{p'}$ do not have the same homotopy type. By using once again that the indefinite complex hyperbolic space $\mathbb{C}H^{n+1}_s(-c)$, $c > 0$, can be obtained from $\mathbb{C}P^{n+1}_{n-s+1}(c)$ by replacing the Kaehler metric of $\mathbb{C}P^{n+1}_{n-s+1}(c)$ by its opposite, we can obtain a family M'''_p, $p \in \mathbb{Z}$, $p \geq 2$, of complex hypersurfaces of $\mathbb{C}H^{2n}_n(-c)$, $n > 1$, defined by

$$\sum_{j=1}^{n}(z_j - z_{n+j+1})^p = z^p_{n+1}, \quad z_{n+1}\neq 0. \tag{3.3.6}$$

with analogous properties to previous family of complex hypersurfaces in $\mathbb{C}P^{2n}_n(c)$. These examples have positive definite (resp. negative definite) normal bundle in $\mathbb{C}P^{2n}_n(c)$ (resp. in $\mathbb{C}H^{2n}_n(-c)$). In a similar way, we can obtain another family of complex Einstein hypersurfaces with negative definite (resp. positive definite) normal bundle in $\mathbb{C}P^{2n}_{n+1}(c)$ (resp. in $\mathbb{C}H^{2n}_{n-1}(-c)$).

Remark 3.3.8. The last five examples do not have parallel ones in Smyth's work [53]. That is, they do belong specifically to the area of indefinite hypersurfaces. Examples 3.3.6 and 3.3.7 show us that there exist indefinite complex Einstein hypersurfaces which are not locally symmetric. Recall that in the definite case, Smyth proved in [53] that a complex Einstein hypersurface is always locally symmetric and this is the key of his classification in [53]. On the other hand, it was shown by Smyth in [54] that a complex hypersurface of a (definite) complex space form is homogeneous if and only if it is Einstein. We know from Example 3.3.5 that the same assertion is not true here.

Observe that a lot of complex hypersurfaces in Example 3.3.6, in particular, all the complex hypersurfaces in Example 3.3.5, have the same scalar curvature zero. Also, every complex hypersurface in Example 3.3.7 have the same scalar curvature $2cn^2$. Thus, the scalar curvature does not distinguish among indefinite complex Einstein hypersurfaces of indefinite complex space forms.

On the other hand, if a complex hypersurface of $\mathbb{C}_s^m$ satisfies $A^2 = 0$, then, for its curvature tensor R, we have $R(X,Y)AZ = AR(X,Y)Z = 0$ which implies that these hypersurfaces are *semi-symmetric* in the sense of Szabó [55]; or alternatively, they satisfy *the Nomizu condition* $R.R = 0$, [39]. However, we know that there are a lot of complex hypersurfaces in certain $\mathbb{C}_s^m$ satisfying $A^2 = 0$ and which are not locally symmetric. In the next section, indefinite complex hypersurfaces which satisfy the Nomizu condition will be studied, following Aiyama, Nakagawa and the second author [3].

According to the previous examples, we have that any indefinite complex space form $\widetilde{M}(c)$ always admits some proper complex Einstein hypersurface, independently of the sign of the holomomorphic sectional curvature c. However, for the case of non-proper complex Einstein hypersurfaces we have, Montiel and the first author [35],

Theorem 3.3.9. *Let M be an indefinite complex Einstein hypersurface of $\widetilde{M}(c)$ such that the shape operator A associated to a unit normal vector field ξ satisfies $A^2 = -b^2 I$, $b \neq 0$. Then $c = -4b^2 g(\xi, \xi)$.*

As a consequence, if $\widetilde{M}(c)$ admits a complex hypersurface M under the assumptions in Theorem 3.3.9 then $c \neq 0$ and $c > 0$ (resp. $c < 0$) if and only if the normal bundle is negative definite (resp. positive definite). For the other non-proper case we have (see Romero [46]),

Theorem 3.3.10. *Let M be an indefinite complex hypersurface of a $(n+1)$-dimensional, $n \geq 2$, indefinite complex space form $\widetilde{M}(c)$. Assume M is Einstein with $A^2 = 0$ and not totally geodesic. If M is locally symmetric then $c = 0$.*

Consequently, we get

Corollary 3.3.11. *If $c \neq 0$ then there exists no indefinite complex hypersurface in $\widetilde{M}(c)$ with $A^2 = 0$ and not totally geodesic which is locally symmetric.*

Remark 3.3.12. Observe that the assumption locally symmetric in Theorem 3.3.10 can be replaced by $\mathrm{Ker}(A)$ is a parallel differentiable distribution and the same conclusion holds. Thus, Montiel and Romero [35, Theorem 6.2] is actually a particular case of Corollary 3.3.11 and, therefore, [35, Theorem 6.2] also holds when the index of the hypersurface is equal to 2 (compare with the real case Graves and Nomizu [16] and Magid [31, p. 133]).

Examples 3.3.5, 3.3.6 and 3.3.7 show that Theorem 3.3.10 is the best result in that direction. Moreover, it follows that a full classification is not possible for the non-proper case in which the shape operators are nilpotent and non-zero.

When the ambient space is flat, we have shown complex Einstein hypersurfaces with $A^2 = 0$ and maximal rank at any point, which are locally symmetric (Example 3.3.3), and a lot of such complex Einstein hypersurfaces which are not locally symmetric (Example 3.3.5). It should be pointed out that for such complex Einstein hypersurfaces in $\mathbb{C}_n^{2n+1}$ (or in $\mathbb{C}_{n+1}^{2n+1}$), a local characterization, similar to the obtained one by Magid in the real case [31, Theorem 4.2], may be obtained.

For the non-proper case where $A^2 = -b^2 I$, $b \neq 0$, holds, it is possible to achieve a classification as the following result shows,

THEOREM 3.3.13. *$\mathbb{C}Q_n^*$ is the only indefinite complete complex Einstein hypersurface of $\mathbb{C}P_{n+1}^{2n+1}(c)$ (and of $\mathbb{C}H_n^{2n+1}(-c)$), $c > 0$, which satisfies $A^2 = -b^2 I$, $b \neq 0$. Moreover, if $m \neq 2n$ or $t \neq 1$ (resp. if $m \neq 2n$ or $t \neq 0$) there do not exist complex hypersurfaces satisfying these conditions in $\mathbb{C}P_{n+t}^{m+1}(c)$ (resp. in $\mathbb{C}H_{n+t}^{m+1}(-c)$).*

This result is an improved form of Montiel and the first author [35, Theorem 5.5] where the complex hypersurface was assumed to be simply connected. In order to remove this assumption, let $\hat{M}$ be the semi-Riemannian universal covering of M. Then $\hat{M}$ is a complex Einstein hypersurface which also satisfies $A^2 = -b^2 I$, $b \neq 0$. Taking into account that A is not diagonalizable, we need a different strategy to that previous Theorem 3.2.4. The key fact is that condition $A^2 = -b^2 I$, $b \neq 0$, implies (see Theorem 3.1.1) that M is locally symmetric, and hence $\hat{M}$ is also locally symmetric. Moreover, b is determined from c (Theorem 3.3.9). Thus, $\hat{M}$ is holomorphically isometric to $\mathbb{C}Q_n^*$, where n is the dimension of M. By rigidity, the first author [48], Umehara [59], $\hat{M}$ holomorphically immerses onto $\mathbb{C}Q_n^*$, in $\mathbb{C}P_{n+1}^{2n+1}(c)$ or $\mathbb{C}H_n^{2n+1}(-c)$. Therefore, the covering map must be one-to-one and $M = \hat{M} = \mathbb{C}Q_n^*$.

For the non-proper case where $A^2 = 0$, $A \neq 0$, holds, it is possible to give a characterization in terms of the normal connection. In fact, in Montiel and the first author [35] the following result is proved,

THEOREM 3.3.14. *Let M be an n-dimensional, $n > 2$, complex submanifold of an indefinite complex space form $\widetilde{M}(c)$. Assume M is geodesically* *complete and simply connected and $c \neq 0$. Then, there exist no unit normal vector fields to M which are parallel with respect to the normal connection.*

As a consequence we obtain,

Corollary 3.3.15. *Let M be a complex submanifold as in Theorem 3.3.14. Then the normal curvature tensor vanishes identically if and only if $c = 0$ and $A_\xi^2 = 0$ for any vector field ξ normal to M.*

Remark 3.3.16. In the definite case, Nomizu and Smyth proved [41, Theorem 7] that for a complex hypersurface M in $\mathbb{C}^{n+1}$, the normal curvature tensor is identically zero if and only if M is totally geodesic. Now, from Examples 3.3.5, 3.3.6 and Corollary 3.3.15, we can assert that the situation here is quite different. We refer the reader to [23] for other results on the normal connection.

After the geometric behavior previously exposed for indefinite complex hypersurfaces which satisfy $A^2 = 0$, it is clear that some extra assumption should be imposed to study this family of complex hypersurfaces. The following result uses a natural assumption, we are going to introduce now. Recall that each indefinite complex projective space $\mathbb{C}P_s^m(c)$ is topologically an open subset of the (ordinary) complex projective space $\mathbb{C}P^m(c)$. On the other hand, the complex hypersurfaces M_p'' in Example 3.3.7 are obtained by taking $M_p'' = \mathbb{C}P_n^{2n}(c) \cap \tilde{M}_p$ where $\tilde{M}_p$ is a non-singular and non-closed complex hypersurface of $\mathbb{C}P^{2n}(c)$. Now assume that

$$M = \mathbb{C}P_s^{n+1}(c) \cap \tilde{M},$$

where $\tilde{M}$ is a non-singular closed complex hypersurface of $\mathbb{C}P^{n+1}(c)$. In this case as proved in Romero [46] we have,

Theorem 3.3.17. *Let M be an indefinite complex hypersurface embedded in an indefinite complex projective space $\mathbb{C}P_s^{n+1}(c)$, $n \geq 2$, which is obtained as above. If M is Einstein and satisfies $A^2 = 0$ at any point, then M is totally geodesic and hence an indefinite complex projective space $\mathbb{C}P_s^n(c)$ or $\mathbb{C}P_{s-1}^n(c)$ according the index of M is $2s$ or $2s-2$, respectively.*

The following sketch of proof is inspired from Nomizu [40]. We introduce the complex submanifold of $\mathbb{C}P_s^m(c)$ given by

$$z_1 = z_{s+1}, \ldots, z_t = z_{s+t} \tag{3.3.7}$$

where t is a fixed integer with $1 \leq t \leq \min\{s, m-s+1\}$ and $(z_1, \dots, z_{m+1})$ is the usual complex homogeneous coordinate system of $\mathbb{C}P_s^m(c)$. This complex submanifold has dimension $m-t$ and inherits from $\mathbb{C}P_s^m(c)$ a degenerate metric with constant rank $2(m-2t)$ and index $2(s-t)$. Moreover, it is autoparallel and geodesically complete (with respect to the induced connection from the metric connection of $\mathbb{C}P_s^m(c)$). We call it as the *degenerate complex projective space* $\mathbb{C}P_{s-t;t}^{m-t}(c)$. The key of the proof is to show that, under the assumptions in Theorem 3.3.17, if the rank of A is not zero everywhere and $2r$ denotes the maximum of rank (A) then M contains (after a rigid motion of the ambient space) $\mathbb{C}P_{s-r;r}^{m-r}(c)$. Form this fact, we can conclude that the homogeneous polynomial defining M is of degree 1 and, therefore, M equals $\mathbb{C}P_s^n(c)$ or $\mathbb{C}P_{s-1}^n(c)$, according its index.

Remark 3.3.18. The assumption $A^2 = 0$, at any point, in Theorem 3.3.17 can be changed by rank of M is $\leq n-1$ everywhere, and the same conclusion remains true. In particular, if M is a complex hypersurface of $\mathbb{C}P_s^{n+1}(c)$, $n \geq 3$, satisfying $A^2 = 0$ at any point, its rank cannot be equal to 2 (compare with Nomizu and Smyth [41, Theorem 6]).

On the other hand, note that each complex hypersurface M_p'' in Example 3.3.7 contains the complex submanifold given by

$$z_1 = z_{n+1}, \dots, z_{n-1} = z_{2n-1}, z_n - z_{n+1} = z_{2n+1}, \quad z_{2n+1} \neq 0 \tag{3.3.8}$$

which is an open proper subset of $\mathbb{C}P_{1;n-1}^n(c)$. Thus, the topological assumption in Theorem 3.3.17 cannot be removed.

Finally, note that an analogous result to Theorem 3.3.17 can be stated for the indefinite complex hyperbolic space $\mathbb{C}H_s^{n+1}(-c)$, $c > 0$, $n \geq 2$.

3.4. Complex hypersurfaces with parallel Ricci tensor. We now consider several geometric assumptions on complex hypersurfaces weaker than to be Einstein. In fact, a natural extension to the condition to be Einstein on indefinite Kaehler manifolds is to have parallel Ricci tensor. Last condition, on indefinite Kaehler manifolds, is weaker than to be Einstein. However, if we pay attention only to complex hypersurfaces we have,

Theorem 3.4.1. *Let M be a complex hypersurface of an indefinite complex space form $\widetilde{M}(c)$ with $c \neq 0$. If M satisfies*

$$R(X, Y)S = 0 \tag{3.4.1}$$

where R denotes the curvature tensor, S is the Ricci tensor and $R(X,Y)$ operates on the tensor algebra as a derivation, then M is Einstein.

This result was proved by Aiyama, Ikawa, Kwon and Nakagawa in [1] as an extension of an unpublished previous result by the first author, which stated the same conclusion assuming that the hypersurface has parallel Ricci tensor, clearly a stronger assumption than (3.4.1). Under the assumption of Ricci parallel tensor, we can give an easy proof of this result. In fact, if the Ricci tensor is parallel, then the same fact holds for the operator A^2, being A the shape operator associated to a unit normal vector field on M. Therefore, $R(X,Y)A^2Z = A^2R(X,Y)Z$ holds for all tangent vector fields X, Y, Z. By contracting this formula we obtain $c(A^2 - \frac{\mathrm{trace}(A^2)}{2n}I) = 0$, which gives the desired result when $c\neq 0$. Properly speaking, last argument proves an extension to the indefinite case of a well-known result by Nomizu and Smyth [41] and by Takahashi [56], now using a slightly different method to the Takahashi's one (Nomizu and Smyth use that the shape operator can be diagonalized, but in our case this is not always possible).

On the other hand, Takahashi proves in [57] that a complex hypersurface with parallel Ricci tensor in $\mathbb{C}^{n+1}$ is totally geodesic. From Examples 3.3.5 and 3.3.6, we know that the same assertion is not true now. In the special case of the ambient space is $\mathbb{C}_1^{n+1}$, it is proved by Aiyama, Ikawa, Kwon and Nakagawa [1] that a spacelike (i.e. with positive definite induced metric) complex hypersurface with parallel Ricci tensor must be totally geodesic.

3.5. Semi-symmetric complex hypersurfaces. The section which ends this chapter is concerned with semi-symmetric complex hypersurfaces in the sense of Szabó [55] (or alternatively, which satisfy the Nomizu condition [39]) of an indefinite complex flat space $\mathbb{C}_{s+t}^{n+1}$. Namely, they satisfy

$$R(X,Y)R = 0,$$

for any vector fields X and Y tangent to M. It turns out that the Nomizu condition implies $R.S = 0$ and moreover it has been shown in Theorem 3.4.1 that if an indefinite hypersurface of an indefinite complex space form $\widetilde{M}_{s+t}^{n+1}(c')$, $c'\neq 0$, satisfies $R.S = 0$, then M is Einstein. So, it is natural to think about this condition when the ambient indefinite complex space form is flat.

Now, suppose that M is an indefinite complex hypersurface satisfying the condition $R.S = 0$ of $\widetilde{M}_{s+t}^{n+p}(c')$. It is seen that the condition $R.S = 0$ is equivalent to

$$S_{i\bar{j};k\bar{l}} - S_{i\bar{j};\bar{l}k} = 0,$$

and, combining (2.3.7) together with (2.3.21) and (2.3.22), we have for an indefinite complex hypersurface M of $\mathbb{C}^{n+1}_{s+t}$,

$$h_{ik}\bar{h}^3_{jl} - \bar{h}_{jl}h^3_{ik} = 0, \tag{3.5.1}$$

where we put $h^3_{jk} = \epsilon \sum_{rs} \epsilon_r \epsilon_s h_{jr}\bar{h}_{rs}h_{sk}$, where $\epsilon = -1$ or $+1$, which implies that

$$h^3_{jk} = fh_{jk} \text{ for a function } f \text{ on } M,$$

whenever the set consisting of points of M at which the function h_2 $(= \sum_k \epsilon_k h^2_{k\bar{k}})$ vanishes is of zero measure.

Now, let us recall the notion of cylindrical hypersurface. A complex hypersurface M of index $2s$ of $\mathbb{C}^{n+1}_{s+t}$ is said to be *cylindrical*, if M is a (semi-Riemannian) product manifold of $\mathbb{C}^{n-1}_a$ and a complex curve in $\mathbb{C}^2_b$ orthogonal to $\mathbb{C}^{n-1}_a$ in $\mathbb{C}^{n+1}_{s+t}$ $(a + b = s + t)$. It is clear that a cylindrical complex hypersurface M, with index $2s$, of $\mathbb{C}^{n+1}_{s+t}$ satisfies (3.4.1), but it is not necessary Einstein. In the definite case, Takahashi [57] proved that cylindrical complex hypersurfaces are the only complete complex hypersurfaces of $\mathbb{C}^{n+1}$ satisfying the condition (3.4.1), except for $\mathbb{C}^n$. However, this property cannot be extended to an indefinite complex flat space. In fact, we can find counter-examples among the complex hypersurfaces in Examples 3.3.6.

In connection with cylindrical complex hypersurfaces, it is natural to state the following question:

Do there exist indefinite complex hypersurfaces satisfying $R.S = 0$ of $\mathbb{C}^{n+1}_{s+t}$ which are not Einstein and not cylindrical?

In order to settle this problem affirmatively, an indefinite complex hypersurface M of $\mathbb{C}^{n+1}_{s+t}$ satisfying the Nomizu condition is considered. By the twice exterior differentiation of the Riemannian curvature tensor R of M (see section 2.3) the Ricci formula for R is as follows:

$$\begin{aligned} R_{\bar{i}jk\bar{l};m\bar{n}} - R_{\bar{i}jkl;\bar{n}m} = \sum_r \epsilon_r \Big(& - R_{\bar{n}mr\bar{i}}\, R_{\bar{r}jk\bar{l}} + R_{\bar{n}mj\bar{r}}\, R_{\bar{i}rk\bar{l}} \\ & + R_{\bar{n}mk\bar{r}}\, R_{\bar{i}jr\bar{l}} - R_{\bar{n}mr\bar{l}}\, R_{\bar{i}jk\bar{r}} \Big). \end{aligned}$$

Taking into account the Gauss equation (2.3.18), we have

$$\big(h^2_{m\bar{l}}\bar{h}_{ni} + h^2_{m\bar{i}}\bar{h}_{nl}\big)h_{jk} - \big(h^2_{k\bar{n}}h_{mj} + h^2_{j\bar{n}}h_{mk}\big)\bar{h}_{il} = 0, \tag{3.5.2}$$

which implies the following two equations

$$h_{ik}\bar{h}^3_{jl} = h^3_{ik}\bar{h}_{jl}, \tag{3.5.3}$$

$$h_2 h^3_{ij} = h_4 h_{ij}. \tag{3.5.4}$$

where we set $h_4 = \sum_{ij} \epsilon_i \epsilon_j h^2_{i\bar{j}} h^2_{\bar{j}\bar{i}}$. Making use of these results, we give the following,

Proposition 3.5.1. *Let M be an indefinite complex hypersurface of $\mathbb{C}^{n+1}_{s+t}$. Then M satisfies $R(X,Y)R = 0$, for any vector fields X and Y, if and only if*

$$h_{il}h_{kl} = h_{ik}h_{jl} \quad \textit{or} \quad h^2_{i\bar{j}} = 0. \tag{3.5.5}$$

Now we come back to the family of complex hypersurfaces $M^{2n}_s(h_j; c_j)$ of $\mathbb{C}^{2n+1}_s$ given in Example 3.3.6. As it is shown by Aiyama, Ikawa, Kwon and Nakagawa in [1], at any point z of $M^{2n}_s(h_j; c_j)$ the vector ξ_z defined by

$$\xi_z = (\bar{h}'_a, -\bar{h}'_x, -\bar{c}_a\bar{h}'_{a^*}, -\bar{c}_x\bar{h}'_{x^*}, 1),$$

where $1 \le a \le s$, $s+1 \le x \le n$, $a^* = a+n$, $x^* = x+n$ and $h'_j = \partial h_j/\partial z_j$ is spacelike and normal. By setting $\xi' = \xi/\mid \xi \mid$, we have a unit spacelike normal vector field ξ' on $M^{2n}_s(h_j; c_j)$.

The components of the second fundamental form derived from ξ' relative to the tangent frame $\{f_A\}$ defined by

$$f_A = (0, \dots, 1, \dots, 0, 0, \dots, 0, h'_A),$$

where $1 \le A \le 2n$, are given by

$$h_{ij} = \delta_{ij}h''_i/|\xi|, \quad h_{ij^*} = c_i\delta_{ij}h''_i/|\xi|,$$
$$h_{i^*j} = c_i\delta_{ij}h''_i/|\xi|, \quad h_{i^*j^*} = c_i^2\delta_{ij}h''_i/|\xi|,$$

where $h''_i = \partial h'_i/\partial z_i$. This means that it turns out that the first equation of (3.5.5) holds true, which implies that $M^{2n}_s(h_j; c_j)$ satisfies the Nomizu condition $R.R = 0$. Thus one finds,

Theorem 3.5.2. *There exist many indefinite complex hypersurfaces of $\mathbb{C}^{2n+1}_s$, for any s as above, with $R.R = 0$ which are not Einstein and not cylindrical.*

CHAPTER 4. COMPLEX SUBMANIFOLDS

4.1. EXAMPLES OF INDEFINITE COMPLEX SUBMANIFOLDS. Before going to give several classification results, we will explain several relevant examples of complex submanifolds of indefinite complex space forms.

EXAMPLE 4.1.1. Let $f : M \to \widetilde{M}_S^N(c')$ be a holomorphic isometric immersion of an indefinite Kaehler manifold M into a complete and simply connected indefinite complex space form $\widetilde{M}_S^N(c')$. Then f is said to be *full* if $f(M)$ is not contained in a proper totally geodesic (degenerate or not) complex submanifold of $\widetilde{M}_S^N(c')$. It is seen in [48] that

$\mathbb{C}P_s^n(c)$ admits a full holomorphic isometric immersion into $\mathbb{C}P_S^N(c')$ if and only if

$$c' = kc \quad \text{for some positive integer} \quad k,$$
$$N = \binom{n+k}{k} - 1,$$

and

$$S = \sum_{j=0}^{[(k+1)/2]-1} \binom{s+2j}{2j+1}\binom{n-s+k-2j-1}{k-2j-1}$$

if $s > 0$, $[(k+1)/2]$ denoting the greatest integer less than or equal to $(k+1)/2$, and

$$S = 0 \qquad \text{if} \qquad s = 0.$$

Changing the (negative definite) Kaehler metric of $\mathbb{C}P_n^n(c)$ by its opposite, we conclude that:

There is a full holomorphic isometric immersion of the (classical) complex hyperbolic space $\mathbb{C}H^n(-c)$ into the indefinite complex hyperbolic space $\mathbb{C}H_{S'(n,k)}^{N(n,k)}(-kc)$, where $S'(n,k) = N(n,k) - S(n,n,k)$ and

$$S(n,n,k) = \sum_{j=0}^{[(k+1)/2]-1} \binom{n+2j}{2j+1}.$$

It is seen that $N(n,2) - n = S'(n,2) = n(n+1)/2$ and $N(n,k) - n > S'(n,k)$ if $k > 2$.

EXAMPLE 4.1.2. For the homogeneous coordinate systems $z_1, \ldots, z_s, z_{s+1}, \ldots, z_{n+1}$ of $\mathbb{C}P_s^n(c)$ and $w_1, \ldots, w_t, w_{t+1}, \ldots, w_{m+1}$ of $\mathbb{C}P_t^m(c)$, a mapping Ψ of

$$\mathbb{C}P_s^n(c) \times \mathbb{C}P_t^m(c) \quad \text{into} \quad \mathbb{C}P_{R(n,m,s,t)}^{N(n,m)}(c),$$

with $N(n,m) = n + m + nm$ and $R(n,m,s,t) = s(m-t) + t(n-s) + s + t$, is defined by

$$\Psi[(z,w)] = [(z_a w_u, z_r w_x, z_b w_y, z_s w_v)],$$

where

$$a, b, \ldots = 1, \ldots, s; \qquad r, s, \ldots = s+1, \ldots, n+1,$$
$$x, y, \ldots = 1, \ldots, t; \qquad u, v, \ldots = t+1, \ldots, m+1.$$

Then Ψ is a well defined holomorphic mapping and it is seen by Ikawa, Nakagawa and the first author [20] that Ψ is also a full isometric embedding, which is called the *indefinite Segre embedding.* In particular, if $s = t = 0$, then Ψ is the classical Segre embedding (Nakagawa and Takagi [37]). Recall that, in the definite case, $\mathbb{C}P^1(c) \times \mathbb{C}P^1(c)$ is the usual complex quadric $\mathbb{C}Q^2$ in $\mathbb{C}P^3(c)$. However, $\mathbb{C}Q^2_2$ and $\mathbb{C}Q^*_1$ are two different quadrics in $\mathbb{C}P^3_2(c)$, namely, the indefinite complex quadrics of Examples 3.2.3 and 3.3.1, respectively.

Using once again how $\mathbb{C}H^n_s(-c)$ is obtained from the indefinite complex projective space $\mathbb{C}P^n_{n-s}(c)$, another indefinite Segre embedding

$$\Psi : \mathbb{C}H^n_s(-c) \times \mathbb{C}H^m_t(-c) \rightarrow \mathbb{C}H^{N(n,m)}_{S(n,m,s,t)}(-c)$$

is given, where $S(n,m,s,t) = (n-s)(m-t) + st + s + t$. In particular, for $s = t = 0$ we have a holomorphic isometric embedding

$$\Psi : \mathbb{C}H^n(-c) \times \mathbb{C}H^m(-c) \rightarrow \mathbb{C}H^{N(n,m)}_{nm}(-c)$$

which permits to see a Riemannian product of two complex hyperbolic spaces as a spacelike complex submanifold of certain indefinite complex hyperbolic space. Moreover, this submanifold has negative definite normal bundle and parallel second fundamental form [20]. Recall that, Nakagawa and Takagi [37], an analogue to the Segre embedding for complex hyperbolic spaces cannot be stated. In fact, they proved that if there exists a holomorphic isometric immersion of a product of two Kaehler manifolds into a complex space form $\widetilde{M}(c)$, then its holomorphic sectional curvature satisfies $c \geq 0$. The fact of the metric on each normal space is, of course, positive definite is a crucial point for the proof of this result. Thus, last indefinite Segre embedding gives an alternative answer to the negative Nakagawa and Takagi's result in the area of indefinite Riemannian geometry.

4.2. Rigidity of holomorphic isometric immersions. Extending classical Calabi's rigidity theorem [12] for full holomorphic isometric immersions into complete and simply connected complex space forms, the first author [48] and Umehara [59] independently proved the following result,

Theorem 4.2.1. *Let $f : M \to \widetilde{M}_S^N(c)$ and $f' : M \to \widetilde{M}_{S'}^{N'}(c)$ be two full holomorphic isometric immersions of the same indefinite Kaehler manifold M into simply connected complete indefinite complex space forms of holomorphic sectional curvature $c \in \mathbb{R}$, $\widetilde{M}_S^N(c)$ and $\widetilde{M}_{S'}^{N'}(c)$, where N, N' denote the complex dimensions and $2S$, $2S'$ are the respective index. Then*

$$N = N', \qquad S = S'$$

and there exists a unique holomorphic rigid motion Φ of $\widetilde{M}_S^N(c)$ such that

$$\Phi \circ f = f'.$$

Recall that the assumption of being full is an affine notion; i.e. only depends on the Levi-Civita connection of the ambient space. It should be pointed out that the assumption: $f(M)$ is not contained in a non-degenerate proper totally geodesic complex submanifold of $\widetilde{M}_S^N(c)$, which is weaker to the previously given one, does not get to the rigidity. For a counter example let us consider $f, f' : \mathbb{C} \to \mathbb{C}_1^3$ given by $f(z) = (z^2+1, z^2+2, z)$ and $f'(z) = (z^3+1, z^3+2, z)$. Both are holomorphic isometric immersions. Clearly, there exists no holomorphic rigid motion Φ of $\mathbb{C}_1^3$ such that $\Phi \circ f = f'$. Note that $f(\mathbb{C})$ and $f'(\mathbb{C})$ are contained in the degenerate complex hyperplane $z_1 - z_2 + 1 = 0$ of $\mathbb{C}_1^3$, but $f(\mathbb{C})$ and $f'(\mathbb{C})$ are not contained in a proper non-degenerate complex affine subspace of $\mathbb{C}_1^3$.

On the other hand, Aiyama, Nakagawa and the second author proved in [3] the following two results,

Theorem 4.2.2. *Let $M_s^n(c)$ be an n-dimensional indefinite complex space form which admits a holomorphic isometric immersion in another one $\widetilde{M}_{s+t}^{n+p}(c')$.*

(1) *If $c' \neq 0$, then $c' = kc$ and $n + p \geq \binom{n+k}{k} - 1$ for some positive integer k.*
(2) *$c' = 0$ if and only if $c = 0$.*

Proposition 4.2.3. *Let $M_s^n(c)$ be an n-dimensional indefinite complex space form which admits a holomorphic isometric immersion in $\widetilde{M}_{s+t}^{n+p}(c'), c' \neq 0$ and $t = p$.*

(1) *If $c' > 0$, then $c' = c$ (and M_s^n is totally geodesic).*

(2) *If $c' < 0$, then $c' = c$ or $2c$, the first case arising only when M_s^n is totally geodesic and the other one arising only when $s = 0$.*

The indefinite Segre embedding described in Example 4.1.2 can be characterized as follows [20],

THEOREM 4.2.4. *Let M_s^n and M'^m_t be complete indefinite Kaehler manifolds with complex dimensions n and m and index $2s$ and $2t$, respectively. Assume there exists a holomorphic isometric immersion $\varphi : M_s^n \times M'^m_t \to \mathbb{C}P_R^N(c)$. Then*

(1) *$N \geq N(n,m)$ and $R \geq R(n,m,s,t)$.*

(2) *If $N = N(n,m)$ then $R = R(n,m,s,t)$, M_s^n is holomorphically isometric to $\mathbb{C}P_s^n(c)$, M'^m_t is holomorphically isometric to $\mathbb{C}P_t^m(c)$ and, by identifying $M_s^n \times M'^m_t$ with $\mathbb{C}P_s^n(c) \times \mathbb{C}P_t^m(c)$, the immersion φ is an embedding obtained by the composition of the indefinite Segre embedding Ψ given in Example 4.1.2 and a rigid motion of $\mathbb{C}P^{N(n,m)}_{R(n,m,s,t)}(c)$.*

This result can be considered as an answer to the "converse" problem for the statement of the indefinite Segre embedding. Moreover, it extends to the indefinite case a well known theorem by Chen in [13]. As a consequence, we have [20],

COROLLARY 4.2.5. *Let M^n and M'^m be complete Kaehler manifolds. Assume there exists a holomorphic isometric immersion $\varphi : M^n \times M'^m \to \mathbb{C}H_S^N(-c),\ c > 0$. Then*

(1) *$N \geq n + m + nm$ and $S \geq nm$,*

(2) *If $N = n + m + nm$, then $S = nm$, M^n is holomorphically isometric to $\mathbb{C}H^n(-c)$, M'^m is holomorphically isometric to $\mathbb{C}H^n(-c)$ and, by identifying $M^n \times M'^m$ with $\mathbb{C}H^n(-c) \times \mathbb{C}H^m(-c)$, the immersion φ is an embedding obtained by the composition of the indefinite Segre embedding $\Psi : \mathbb{C}H^n(-c) \times \mathbb{C}H^m(-c) \to \mathbb{C}H_{nm}^{n+m+nm}(-c)$ in Example 4.1.2 and a rigid motion of $\mathbb{C}H_{nm}^{n+m+nm}(-c)$.*

Now let us note that the second fundamental form h of the indefinite Segre embedding Ψ is parallel; i.e. it is invariant by parallel translation with respect to the normal connection, or equivalently, it satisfies

$$\nabla h = 0.$$

In the definite case, this property gives a well known characterization of the classical) Segre embedding (see Nakagawa and Takagi [37], for instance). Now, this fact also characterizes Ψ and gives, as a consequence, the following result of particular interest [20],

THEOREM 4.2.6. *Let M^n and M'^m be complete Kaehler manifolds. Assume there exists a holomorphic isometric immersion $\varphi : M^n \times M'^m \to \mathbb{C}H^N_{nm}(-c)$, $c > 0$. If the second fundamental form of φ is parallel, then φ is obtained from the Segre embedding*

$$\Psi : \mathbb{C}H^n(-c) \times \mathbb{C}H^m(-c) \to \mathbb{C}H^{n+m+nm}_{nm}(-c) \subset \mathbb{C}H^N_{nm}(-c)$$

and a rigid motion of $\mathbb{C}H^{n+m+nm}_{nm}(-c)$.

Finally, following the same idea as in [13] for the definite case, it is given in [20] another characterization of the indefinite Segre embedding

$$\Psi : \mathbb{C}H^n(-c) \times \mathbb{C}H^m(-c) \to \mathbb{C}H^{n+m+nm}_{nm}(-c),$$

now in terms of the square of the length of its second fundamental form, which is given by $h_2 = -cnm < 0$.

4.3. SPACELIKE COMPLEX SUBMANIFOLDS. We begin this section with the statement of a uniqueness result due to Aiyama, Kwon and Nakagawa [2]. In fact, they proved the following Bernstein-type result,

THEOREM 4.3.1. *Let M^n be a complete spacelike complex submanifold of an indefinite complex space form $\widetilde{M}^{n+s}_s(c)$. If $c \geq 0$ then M^n must be totally geodesic.*

Remark 4.3.2. (1) Note that, in this result, the index of the ambient space agrees with the real codimension of the submanifold; i.e., the normal bundle is assumed to be negative definite. Clearly, this assumption cannot be removed (consider, for instance, the classical complex quadric $\mathbb{C}Q^n$ as a spacelike complex submanifold of $\mathbb{C}P^{n+s+1}_s(c)$, $s > 0$).

(2) In particular, Theorem 4.3.1 classifies complete spacelike complex hypersurfaces of $\mathbb{C}P^{n+1}_1(c)$ and of $\mathbb{C}^{n+1}_1$, without the assumption of being Einstein.

(3) Theorem 4.3.1 cannot be extended to the case $c < 0$. In fact, the complex quadric $\mathbb{C}Q'^n$ of Example 3.3.3 is a complete spacelike complex hypersurface of $\mathbb{C}H^{n+1}_1(-k)$, $k > 0$, which is not totally geodesic. Even more, the

indefinite Segre embedding, which we have given in Example 4.1.2, provides us with $(n+m)-$dimensional, $n \neq m$, complete spacelike complex submanifolds of $\mathbb{C}H^{n+m+nm}_{nm}(-k)$, $k>0$, which are not Einstein. Finally, we refer the reader to [23] for another related Bernstein-type result obtained using the same technique as in [2].

Next, we will focus our attention in the proof of Theorem 4.3.1. Let us consider the function h_2 on M; i.e., the square length of the second fundamental form. Clearly, it satisfies $h_2 \leq 0$ from the assumption on the normal bundle, and $h_2 = 0$ if and only if M is totally geodesic. In [2] an adapted version for indefinite metrics of classical Simon's formula, [42, Proposition 3.1], is used. In fact, for a complex submanifold M^n_s of an indefinite complex space form $M^{n+p}_{s+t}(c)$, by means of (2.3.22), the Laplacian of the function h_2 can be computed as follows

$$\triangle h_2 = (n+2)\frac{c}{2}h_2 - (2h_4 + A_2) + \sum_{xijk} \epsilon_x\epsilon_i\epsilon_j\epsilon_k h^x_{ijk}\bar{h}^x_{ijk} \tag{4.3.1}$$

where $h_4 = \sum_{ij}\epsilon_i\epsilon_j h^2_{i\bar{j}}h^2_{j\bar{i}}$, and $A_2 = \sum_{xy}\epsilon_x\epsilon_y A^x_y A^y_x$, with $A^x_y = \sum_{ij}\epsilon_i\epsilon_j h^x_{ij}\bar{h}^y_{ij}$.

Now let M be an n-dimensional spacelike complex submanifold of an indefinite complex space form $\widetilde{M}^{n+p}_p(c)$. From the fact that M is spacelike we know that the Hermitian matrix $(h^2_{j\bar{k}})$ is, at any point, negative semi-definite, and hence its eigenvalues λ_j are non-positive real valued functions. On the other hand, the Hermitian matrix (A^x_y) is by definition, at any point, positive semi-definite, whose eigenvalues are denoted by λ_x. Then λ_x are non-negative and, making use of the Cauchy-Schwarz inequality, we have

$$\begin{aligned} (-h_2)^2 \geq h_4 = \sum_x \lambda_x^2 \geq (-h_2)^2/n, \\ h_2^2 \geq A_2 = \sum_x \lambda_x^2 \geq (\sum_x \lambda_x)^2/p = h_2^2/p. \end{aligned} \tag{4.3.2}$$

From (4.3.1), we get

$$\triangle h_2 \leq \{np(n+2)ch_2 - 2(n+2p)h_2^2\}/2np,$$

where the equality holds true if and only if

$$\lambda_j = \lambda,\ \lambda_x = \mu \quad \text{for any indices } j \text{ and } x,$$

and the second fundamental form h is parallel.

This means for the non-negative function $f = -h_2$ the following inequality

$$\triangle f \geq \{np(n+2)cf + 2(n+2p)f^2\}/2np. \tag{4.3.3}$$

And therefore, using $c \geq 0$ we get

$$\triangle f \geq \frac{n+2p}{np} f^2 \tag{4.3.4}$$

Next, we can use the following result proved by Nishikawa in [38],

THEOREM 4.3.3. *Let M be a complete Riemannian manifold whose Ricci curvature is bounded from below, and let f be a non-negative smooth function on M. If f satisfies*

$$\triangle f \geq k f^2$$

where k is a positive constant, then $f = 0$.

In order to end this sketch for the proof of Theorem 4.3.1, we only have to note that the Ricci curvature of a spacelike complex submanifold M^n of $\widetilde{M}_p^{n+p}(c)$ is bounded from below by $(n+1)c/2$, as a consequence of (2.3.19).

It should be pointed out that Theorem 4.3.3 is in fact a nice consequence of the well known maximum principle by Omori [43] and Yau [62]. Following [3], we will state here this result in a slightly different form to the original one.

THEOREM 4.3.4. *Let M be a complete Riemannian manifold whose Ricci curvature is bounded from below and let F be a function of class C^2 on M. If F is bounded from below, then for any point p and any $\epsilon > 0$ there exists a point $q \in M$ such that*

$$|\nabla F(q)| < \epsilon, \quad \triangle F(q) > -\epsilon \quad \textit{and} \quad F(q) \leq F(p).$$

Now, we will explain a new proof of Theorem 4.3.1 for the case $c > 0$. It is inspired on the arguments in the paper by A. Ros [49]. First note that M^n must be compact when $c > 0$. In fact, in this case the Ricci curvature is bounded from below by the positive constant $(n+1)c/2$. This fact and completeness imply compactness, because of the classical Myers theorem.

As in A. Ros [49], take $c = 1$ and let UM be the unitary tangent bundle to M; i.e. the hypersurface of the tangent bundle TM consisting of all tangent vectors u which satisfy $\langle u, u\rangle = 1$, where we are denoting by $\langle\ ,\ \rangle$ both the metric of $\widetilde{M}_p^{n+p}(c)$ and M. Now consider the smooth function $f : UM \to \mathbb{R}$, given by

$$f(u) = \langle h(u,u), h(u,u)\rangle,$$

where h is the second fundamental form of the submanifold. Clearly, this function satisfies $f(u) \leq 0$ and $f = 0$ if and only if M is totally geodesic.

Note that UM is compact because M is assumed to be compact and the fiber of the unitary tangent bundle UM is always compact. Therefore f attains its minimum at some point $v \in UM$. If we prove that $f(v) = 0$ then $f = 0$, as desired.

The formula (2.8) in Ros's paper [49] (which is obtained from the Hessian of f at v) remains true here if v is the minimum (Ros arguments in his proof on the unitary tangent vector where f attains its maximum). Now, because the normal bundle is negative definite, the right hand side of the quoted formula (2.8) is ≤ 0. But from our assumption (v is minimum) the same right member is ≥ 0. Hence

$$f(v)\big(1 - 4f(v)\big) = 0,$$

which implies $f(v) = 0$.

We would like point out now that Theorem 4.3.1 can be seen as the complex version of another Bernstein-type result, but now in the real case, which was obtained by Ishihara in [21]. The main tool used by Ishihara was also the classical approach using Simon's formula for the square of the length of the second fundamental form. In the case $c > 0$, a new proof to Ishihara's theorem can be performed with a slight modification of the previously explained method. On the other hand, by means of certain integral formulas introduced by Alias and the first author [4], see also [5], it is proved the following result,

THEOREM 4.3.5. *Let M be an n-dimensional complete spacelike submanifold with zero mean curvature in the indefinite sphere $\mathbb{S}_q^{n+p}(1)$, $1 \leq q \leq p$. If the Ricci tensor S of M satisfies*

$$S(u,u) \geq (n-1)g(u,u)$$

for all tangent vector u, then M must be totally geodesic.

Note that in the case $p = q$ the inequality in Theorem 4.3.5 is automatically satisfied. Thus, this result generalizes Ishihara's theorem [21] when the ambient space is $\mathbb{S}_p^{n+p}(1)$. From previous results, it arises in a natural way to decide if the following assertion is true, CONJECTURE. Let M^n be a complete spacelike complex submanifold of an indefinite complex space form $\widetilde{M}_q^{n+p}(c)$, $1 \leq q \leq p$, $c > 0$. If the Ricci tensor S of M satisfies

$$S(u,u) \geq \frac{(n+1)c}{2} g(u,u)$$

for all tangent vector u, then M must be totally geodesic.

Of course, without the restriction on the Ricci tensor, the conclusion M *is totally geodesic*, in previous conjecture, cannot be achieved (consider, for instance, the classical complex quadric $\mathbb{C}Q^n$ as a spacelike complex submanifold of $\mathbb{C}P_s^{n+s+1}(c)$, $s > 0$).

Now we come back to comment Theorem 4.3.1. From Remark 4.3.2 we know that it cannot be extended to the case $\widetilde{M}_p^{n+p}(c)$, $c < 0$. Thus, a new assumption is necessary to impose on some curvature of the complex submanifold. Therefore, consider next, following Aiyama, Nakagawa and the second author [3], spacelike complex submanifolds with constant scalar curvature of an indefinite complex space form $\widetilde{M}_p^{n+p}(c)$, $c < 0$.

The square of the norm of the tensor field (locally) defined by

$$\sum \{\epsilon_x h^x_{jk} h^x_{il} - h_2(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl})/n(n+1)\}$$

gives rise to the inequality

$$A_2 \geq 2h_2^2/n(n+1),$$

where the equality holds true if and only if M is of constant holomorphic sectional curvature. From this it follows that

$$\triangle f \geq (n+2)\{n(n+1)cf + 4f^2\}/2n(n+1). \tag{4.3.5}$$

As a direct consequence of these estimates, one finds without any assumption on completeness the following (see [3])

PROPOSITION 4.3.6. *Let M be a spacelike complex submanifold of $\widetilde{M}_p^{n+p}(c)$ with constant scalar curvature r.*

(1) *If $c \geq 0$, then M is totally geodesic.*
(2) *If $c < 0$ and $r \geq n^2(n+p+1)c/(n+2p)$, then M is Einstein, $r = n^2(n+p+1)c/(n+2p)$ and the second fundamental form is parallel.*

(3) *If* $c < 0$ *and* $r \geq n(n+1)c/2$, *then* M *is a complex space form* $\widetilde{M}^n(c/2)$ *and* $p \geq n(n+1)/2$.

As we pointed out in previous Remark 4.3.2(2), in the case where $c < 0$, there are many complete spacelike complex submanifolds which are not Einstein. However, it is proved in [3] the following,

THEOREM 4.3.7. *Let* M *be an* $n(\geq 3)$*-dimensional complete spacelike complex submanifold of* $\mathbb{C}H_p^{n+p}(c)$, $p > 0$, $c < 0$.

(1) *If* $r \geq n^2(n+p+1)/(n+2p)$, *then* M *is Einstein,* $r = n^2(n+p+1)c/(n+2p)$ *and the second fundamental form is parallel.*

(2) *If* $r \geq n(n+1)c/2$, *then* $M = \mathbb{C}H^n(c/2)$, $p \geq n(n+1)/2$.

COROLLARY 4.3.8. *Let* M *be a complete spacelike complete complex submanifold of* $\widetilde{M}_p^{n+p}(c)$, $c < 0$.

(1) *If every Ricci curvature of* M *is greater than or equal to* $n(n+p+1)c/2(n+2p)$, *then* M *is Einstein.*

(2) *If every Ricci curvature of* M *is greater than or equal to* $(n+1)c/4$, *then* M *is a complex space form* $M^n(c/2)$.

4.4. COMPLEX SUBMANIFOLDS OF DEFINITE COMPLEX SPACE FORMS. In the last section of this chapter, motivate from previous results, let us study complete complex submanifolds with constant scalar curvature of a (definite) complex space form $\widetilde{M}^{n+p}(c)$. Let M be an n-dimensional complex submanifold of $\widetilde{M}^{n+p}(c)$. First, recall that the components $S_{i\bar{j};kl}$ and $S_{i\bar{j};k\bar{l}}$ of the second covariant derivative of the Ricci tensor S of M are expressed by (2.3.6). On the other hand, the Ricci formula for S is given by (2.3.7), (primes in both formulas should be forgotten here). Assume now that the scalar curvature r of M is constant. Since we have $\sum_j S_{j\bar{j};k} = 0$, for all k, it follows that

$$\begin{aligned}\triangle S_{k\bar{j}} &= \sum_l (S_{k\bar{l}}S_{l\bar{j}} - R_{\bar{i}kl\bar{j}}S_{i\bar{l}})\\ &= c(2nS_{k\bar{j}} - r\delta_{kj})/4\\ &\quad - \sum_{ls}\{h^2_{k\bar{l}}S_{l\bar{j}} - h^x_{kl}\bar{h}^x_{js}S_{s\bar{l}}\}.\end{aligned}$$

On the other hand, by combining the relation

$$\sum_{jk} S_{j\bar{k}}S_{k\bar{j}} - r^2/4n = h_4 - h_2^2/n$$

together with the above equation, the following one

$$\begin{aligned}\triangle(h_4 - h_2^2/n) = & \, nc(h_4 - h_2^2/n) + 2\sum h^x_{ls;i}\bar{h}^y_{ls;j}h^y_{jt}\bar{h}^x_{ti} \\ & - \sum_{iljx}(h^x_{il}S_{\bar{l}j} - S_{i\bar{l}}h^x_{lj})\overline{(h^x_{is}S_{\bar{s}j} - S_{i\bar{s}}h^x_{sj})}\end{aligned} \tag{4.4.1}$$

is derived. Then by using Theorem 4.3.7 next result is proved (see [3]),

THEOREM 4.4.1. *Let M be a complete complex submanifold with constant scalar curvature of a complex space form $\widetilde{M}^{n+p}(c)$, $c > 0$. If the Ricci tensor of M and any shape operator are commutative, then M is Einstein.*

Recall that in the case where M is compact, this result was proved by Kon in [26]. A complex hypersurface M of $\widetilde{M}^{n+1}(c)$ are next considered. Assume that the scalar curvature r of M is constant. Then (4.4.1) is simplified to give

$$\triangle h_4 = nc(h_4 - h_2^2/n) + 2\sum_{ijls} h^2_{i\bar{j}}h_{jl;s}\bar{h}_{il;s}.$$

On the other hand, since we have

$$\sum_{ls} h_{ls;i}\bar{h}_{ls;j} = -c\{h_2\delta_{ij} + 2h^2_{i\bar{j}}\}/2 + \{2h^4_{i\bar{j}} + h_2h^2_{i\bar{j}}\},$$

because of the constant scalar curvature and (2.3.22), the above equation is reduced to

$$\triangle h_4 = c\{(n-2)h_4 - 2h_2^2\} + 2(2h_6 + h_2h_4),$$

where $h_6 = \sum_{jrs} h^2_{j\bar{r}}h^2_{r\bar{s}}h^2_{s\bar{j}}$. For the eigenvalues λ_j of the Hermitian matrix $(h^2_{j\bar{k}})$, the function h_6 equals to $\sum_j \lambda_j^3$ and hence we have

$$h_6 \geq h_2h_4/n.$$

Thus we achieve

$$\triangle h_4 \geq c(n-4)h_4 + 2(n+2)h_2h_4/n, \tag{4.4.2}$$

provided that $c \leq 0$. By means of this inequality, the following theorem for complex hypersurfaces is proved,

THEOREM 4.4.2. *Let M be a complete complex hypersurface of $\widetilde{M}^{n+1}(c)$. If the scalar curvature of M is constant, then the following statements hold true:*

(1) *If* $c \geq 0$, *then* M *is totally geodesic or* $S = \frac{nc}{2}g$, *the later case arising only when* $c > 0$.

(2) *If* $c < 0$ *and* $n \leq 4$, *then* M *is totally geodesic.*

Remark 4.4.3. As recalled previously, B. Smyth, [53], [54], classified complete Einstein Kaehler hypersurfaces in a simply connected complete complex space form $\widetilde{M}^{n+1}(c)$, and asserted that they are totally geodesic for $c \leq 0$, and they are either totally geodesic or the complex quadric $\mathbb{C}Q^n$ for $c > 0$. For higher codimension the situation seems to be more complicated. Let us remark that Smyth's theorem was generalized by Y. Matsuyama [34] in case of complete Einstein complex surfaces M^2 in a complex space form $\widetilde{M}^{n+2}(c)$, and by K. Tsukada [58] for codimension 2, i.e., in the case of complete Einstein Kaehler manifolds M^n in a complex space form $\widetilde{M}^{n+2}(c)$. More generally, M. Umehara [59] proved that every complete Einstein Kaehler n-submanifold of $\mathbb{C}^{n+p}$ or $\mathbb{C}H^{n+p}$, $n \geq 1$, (i.e. with arbitrary codimension) must be always totally geodesic. It should noted that his proof is based on several properties of the diastasis of Einstein Kaehler submanifolds (the diastasis of a Kaehler manifold was introduced by E. Calabi [12] and used by the first author [48] to prove rigidity, in the indefinite setting, of full holomorphic isometric immersions, Theorem 4.2.1).

Chapter 5. Totally real bisectional curvature

5.1. Motivations. In Chapter 2, we recalled the notion of totally real bisectional curvature defined on a Kaehler manifold $\widetilde{M}$. Concerning several results with such a kind of curvature, in section 5.2 we will show that a complete Kaehler manifold $\widetilde{M}$ with positively lower bounded totally real bisectional curvature $B(u,v) \geq b > 0$ and constant scalar curvature is holomorphically isometric to a complex projective space $\mathbb{C}P^n(c)$. Before obtaining this result we should verify that a complete Kaehler manifold $\widetilde{M}$ with $B(u,v) \geq b > 0$ must be Einstein. Moreover we also show that the positive constant b in the above estimation is the best possible. This means that the condition of a positive lower bound for the totally real bisectional curvature cannot be replaced by the non-negativity of this curvature; in fact, it is not difficult to show an example of complete Kaehler manifold with non-negative totally real bisectional curvature $B(u,v) \geq 0$ but not Einstein (see Remark 5.2.3).

S.I. Goldberg and S. Kobayashi [17] showed that a complete Kaehler manifold $\widetilde{M}$ with positive holomorphic bisectional curvature $H(u,v) > 0$ must be Einstein. In order to get this result they should verified that, under their assumptions, the Ricci tensor of $\widetilde{M}$ is positive definite. In the proof they used the fact that the holomorphic sectional curvature $H(u)$ is positive, which necessarily follows from the condition $H(u,v) > 0$. But the condition of $B(u,v) > 0$ carries less information than the condition of $H(u,v) > 0$, and it gives us no meaning to use the S.I. Goldberg and S. Kobayashi method to derive the fact that M is Einstein. That is, we cannot use the condition of $H(u) > 0$. However, in spite of this weaker condition $B(u,v) \geq b > 0$ by making use of the maximum principle due to Omori [43] and Yau [62], Theorem 4.3.4, we can also obtain the desired result.

As mentioned at the end of section 2.3, the totally real bisectional curvature $B(u,v)$ can be also considered for non degenerate totally real planes $\mathrm{Span}\{u,v\}$ in any indefinite Kaehler manifold. In [7], M. Barros and the first author asserted that the above mentioned Houh's result [18] can be extended to indefinite Kaehler manifolds.

We show in section 4.3, Aiyama, Nakagawa and the second author [3], the classification problem of spacelike complex submanifolds, with bounded scalar curvature, of the indefinite complex hyperbolic space $\mathbb{C}H_p^{n+p}(c)$, $c < 0$. Being motivated by this result, in section 5.3 we also study those classification problems with bounded totally real bisectional curvature. Finally in section 5.4, we will deal with the classification of complex submanifolds M^n, with bounded totally real bisectional curvature, of the complex projective space $\mathbb{C}P^{n+p}(c)$, $c > 0$.

5.2. Complete Kaehler manifolds with positive totally real bisectional curvature

Let $(\widetilde{M}, g, J)$ be an n-dimensional Kaehler manifold. It is well known that its Ricci 2-form is harmonic if and only the scalar curvature of $\widetilde{M}$ is constant. In order to prove that the second Betti number of a compact connected Kaehler manifold $\widetilde{M}$ with positive holomorphic bisectional curvature $H(u,v) > 0$ is one, S.I. Golberg and S. Kobayashi [17] used the fact that $H(u) > 0$. Thus the Ricci 2-form is proportional to the Kaehler 2-form, so that $\widetilde{M}$ becomes an Einstein manifold.

But, as mentioned, the assumption $B(u,v) > 0$ is weaker than $H(u,v) > 0$. Thus in order to get the previous result it is impossible for us to use $H(u) > 0$ with the condition $B(u,v) > 0$. From this point of view, the maximum principle due to Omori [43] and Yau [62], Theorem 4.3.4, has been used by Ki and the second author in [24] to obtain the following result.

THEOREM 5.2.1. *Let $\widetilde{M}$ be an n-dimensional complete Kaehler manifold with constant scalar curvature. Assume that the totally real bisectional curvature is lower bounded for some positive constant b. Then $\widetilde{M}$ is Einstein.*

In order to prove this theorem, we need the following result [24],

LEMMA 5.2.2. *Under the same assumption as stated in Theorem 5.2.1, the Ricci curvature of $\widetilde{M}$ is bounded from below.*

Next by using Lemma 5.2.2, we will complete the proof of Theorem 5.2.1. In order to do that, we fix a constant $a > 0$, and consider the smooth positive function

$$F = (f + a)^{-\frac{1}{2}},$$

where we put $f = S_2 - \frac{r^2}{4n}$, with $S_2 = \sum_{BC} S_{B\bar{C}} S_{C\bar{B}}$ (see section 2.3). Note that $f \geq 0$ and $f = 0$ if and only if the manifold is Einstein.

We may assume that $S_{B\bar{C}} = \lambda_B \delta_{BC}$ locally, where λ_B is a real valued function. Since r is constant, from (2.3.5) it follows that

$$\sum_B S_{B\bar{B};C} = \sum_C S_{C\bar{B};B} = 0.$$

This fact, together (2.3.5) and (2.3.7), and taking into account (2.3.3), permit us to compute the Laplacian of the function S_2 to get

$$\frac{1}{2} \triangle S_2 = \frac{1}{2} \mid \nabla S \mid^2 \; + \sum_{BC} S_{\bar{C}B} \Big(\lambda_B S_{B\bar{C}} - \sum_A \lambda_A R_{\bar{A}AB\bar{C}} \Big),$$

where $\mid \nabla S \mid^2 = 2 \sum S_{A\bar{B};C} \bar{S}_{A\bar{B};C}$. Therefore

$$\triangle S_2 \geq \sum_{AB} (\lambda_A - \lambda_B)^2 R_{\bar{A}AB\bar{B}}, \tag{5.2.1}$$

and the equality holds if and only if $\nabla S = 0$.

According to our assumptions and the above definition of f, from (5.2.1) we get the following differential inequality

$$\triangle f \; \geq \; 2nbf, \tag{5.2.2}$$

and the equality holds if and only if $\nabla S = 0$.

Now, Lemma 5.3.2 is claimed in order to apply Theorem 4.3.4 (Omori [43] and Yau [62]) to the function $F = (f+a)^{-\frac{1}{2}}$ which has been previously constructed. Thus, for each positive number $\epsilon > 0$, there exists a point p_ϵ such that

$$|\nabla F|(p_\epsilon) < \epsilon, \quad \triangle F(p_\epsilon) > -\epsilon, \quad F(p_\epsilon) < \operatorname{Inf} F + \epsilon. \tag{5.2.3}$$

It follows from these inequalities that

$$\epsilon\,(3\epsilon + 2F(p_\epsilon)) > F(p_\epsilon)^4 \triangle f(p_\epsilon) \geq 0. \tag{5.2.4}$$

Thus for a convergent sequence $\{\epsilon_m\}$, $\epsilon_m \in \mathbb{R}$, such that $\epsilon_m > 0$, and $\epsilon_m \to 0$ as $m \to \infty$, there is a point sequence $\{p_m\}$ in M so that the sequence $\{F(p_m)\}$ satisfies (5.2.3) and converges to F_0, by taking a subsequence if necessary, because the sequence $\{F(p_m)\}$ is bounded. Making use of (5.2.3) we have $F_0 = \operatorname{Inf} F$ and hence $f(p_m) \to f_0 = \operatorname{Sup} f$. It follows from (5.2.4) that we have

$$\epsilon_m\{3\epsilon_m + 2F(p_m)\} > F(p_m)^4 \triangle f(p_m) \;\geq\; 0, \tag{5.2.5}$$

and the left hand side converges to 0 because the function F is bounded. Thus we get

$$F(p_m)^4 \triangle f(p_m) \to 0$$

as $m \to \infty$. As is already seen, the Ricci curvature is bounded from below i.e., so is any λ_B. Since $r = 2\sum_B \lambda_B$ is constant, λ_B is bounded from above. Hence $F = (f+a)^{-\frac{1}{2}}$ is bounded from below by a positive constant. From (5.2.5) it follows that

$$\triangle f(p_m) \to 0$$

as $m \to \infty$. But, from (5.2.2) we have that

$$\triangle f(p_m) \geq 2nb\, f(p_m) \geq 0.$$

Thus we get $f(p_m) \to 0 = \operatorname{Inf} f$. Since $f(p_m) \to \operatorname{Sup} f$, $\operatorname{Sup} f = \operatorname{Inf} f = 0$. Hence $f = 0$ on M. That is, M is Einstein. This ends the sketch of proof for Theorem 5.2.1.

Remark 5.2.3. The assumption of the constant bound $b > 0$ in Theorem 5.2.1 is the best possible. This means that the condition of the existence of a positive lower bound for the totally real bisectional curvature cannot be replaced by the non-negativity of this curvature. In fact, there is a complete Kaehler manifold with non-negative totally real bisectional curvature $B(u, v) \geq 0$ but not Einstein constructed as follows.

Consider the Riemannian product manifold $\mathbb{C}P^{n_1}(c_1)\times\mathbb{C}P^{n_2}(c_2)$. It is not difficult to see that the totally real bisectional curvature of this Kaehler manifold is given by

$$R_{\bar{A}AB\bar{B}} = \begin{cases} R_{\bar{a}ab\bar{b}} = \frac{c_1}{2} & \text{if } A=a,\ B=b, \\ 0 & \text{if } A=a,\ B=s, \\ R_{\bar{r}rs\bar{s}} = \frac{c_2}{2} & \text{if } A=r,\ B=s, \end{cases}$$

where $A, B\,(A\neq B), ... = 1, ..., n_1, n_1+1, ..., n_1+n_2;\quad a, b, ... = 1, ..., n_1,$ and $r, s, ... = n_1+1, ..., n_1+n_2$.

Taking now into account the form of the curvature tensor of a Riemannian product, its Ricci tensor satisfies

$$\begin{aligned} S_{A\bar{B}} = \sum_C R_{\bar{B}AC\bar{C}} &= \sum_a R_{\bar{B}Aa\bar{a}} + \sum_r R_{\bar{B}Ar\bar{r}} \\ &= \begin{cases} \frac{n_1+1}{2}c_1\delta_{bc} & \text{if } B=c,\ A=b, \\ 0 & \text{if } B=s,\ A=b, \\ \frac{n_2+1}{2}c_2\delta_{ts} & \text{if } B=s,\ A=t. \end{cases} \end{aligned}$$

Thus for the case where $(n_1+1)c_1 \neq (n_2+1)c_2$, $\mathbb{C}P^{n_1}(c_1)\times\mathbb{C}P^{n_2}(c_2)$ is not Einstein.

Now we come back to Theorem 5.2.1. Since a complete Kaehler manifold $\widetilde{M}$ under the assumptions of that result, is known to be Einstein and its scalar curvature r is a positive constant, its Ricci tensor is clearly positive definite. Thus by using the classical Myers theorem we can assert that $\widetilde{M}$ must be compact. Next let us recall a result of S.I. Goldberg and S. Kobayashi [17], stated in a slight different form to the original one.

THEOREM 5.2.4. *An n-dimensional compact connected Kaehler manifold with an Einstein metric and of positive totally real bisectional curvature must be globally isometric to the complex projective space $\mathbb{C}P^n$ with its Fubini-Study metric.*

Though in the original result in [17] was assumed positive holomorphic bisectional curvature, it can be easily checked that the conclusion in Theorem 5.2.4 also holds if we assume positive totally real bisectional curvature. Thus combining Theorems 5.2.1 and 5.2.4 we can state the following result [24],

THEOREM 5.2.5. *Let $\widetilde{M}$ be an $n(\geq 3)$-dimensional complete Kaehler manifold with constant scalar curvature. Assume that its totally real bisectional* *curvature is lower bounded for some positive constant b. Then $\widetilde{M}$ is globally isometric to $\mathbb{C}P^n$ with its Fubini-Study metric.*

5.3. Spacelike complex submanifolds with lower bounded totally real bisectional curvature. Let $\mathbb{C}H_p^{n+p}(c)$, $c<0$, be the $(n+p)$-dimensional indefinite complex hyperbolic space of index $2p(>0)$, and let M be an $n(\geq 3)$-dimensional spacelike complex submanifold of $\mathbb{C}H_p^{n+p}(c)$. Then, from the Gauss equation (2.3.23) we get

$$R_{\bar{i}ij\bar{j}} = \frac{c}{2} - \sum_x \epsilon_x h_{ij}^x \bar{h}_{ij}^x \geq \frac{c}{2}, \tag{5.3.1}$$

where we have used the fact that $\epsilon_x = -1$, for all x, because the normal space of M is negative definite.

Now we will give here some remarks on the totally real bisectional curvature for complex submanifolds of indefinite complex space forms.

Remark 5.3.1. (a) For a complex submanifold M of a (positive definite) complex space form $\widetilde{M}^{n+p}(c)$, from (2.3.23) we have

$$R_{\bar{i}ij\bar{j}} = \frac{c}{2} - \sum_x h_{ij}^x \bar{h}_{ij}^x \leq \frac{c}{2}.$$

As it was shown in Example 2.3.3, the totally real bisectional curvature B of the complex quadric $\mathbb{C}Q^n$ in $\mathbb{C}P^{n+1}(c)$ satisfies $0 \leq B \leq \frac{c}{2}$. Moreover, the holomorphic sectional curvature H of $\mathbb{C}Q^n$ is holomorphically pinched as $\frac{c}{2} \leq H \leq c$, [42].

(b) From Theorem 4.3.1 we know that if M is a complete spacelike complex submanifold of an indefinite complex space form $\widetilde{M}_p^{n+p}(c)$ with $c\geq 0$, then, M is totally geodesic. Thus, its totally real bisectional curvature satisfies $B = \frac{c}{2}$.

(c) Suppose that an n-dimensional indefinite complex space form $M_s^n(c)$ admits a holomorphic isometric immersion into an indefinite complex space form $\widetilde{M}_{s+p}^{n+p}(c')$, $c' \neq 0$. Making use of Proposition 4.2.3, we can assert that if $c' > 0$, then $c' = c$. Thus M is totally geodesic and $B = \frac{c'}{2}$. On the other hand, if $c' < 0$, then $c' = c$ or $2c$, the first case arising only when M is totally geodesic, and the other arising only when $s = 0$ and thus $B = \frac{c'}{4}$.

(d) Let $\mathbb{C}Q'^n$ be the spacelike complex quadric of the complex hyperbolic space $\mathbb{C}H_1^{n+1}(c)$, $c<0$. Recall that $\mathbb{C}Q'^n$ is Einstein, and that (see Example 2.3.32) its totally real bisectional curvature satisfies $\frac{c}{2} \leq B \leq 0$.

Therefore, there is no meaning to consider complete spacelike submanifolds of $\widetilde{M}_p^{n+p}(c)$, $c\geq 0$, with lower bounded totally real bisectional curvature. Thus, we are focusing now on

The classification problem of the complete spacelike submanifolds of $\mathbb{C}H_p^{n+p}(c)$, $c<0$, *with lower bounded totally real bisectional curvature.*

So, suppose that there exists $b\in\mathbb{R}$ such that

$$R_{\bar{i}ij\bar{j}} \geq b \quad \text{for any} \quad i,j \quad (i\neq j). \tag{5.3.2}$$

From this assumption and (5.3.1) it follows that

$$2\sum_x \epsilon_x h_{ij}^x \bar{h}_{ij}^x \leq c-2b \quad \text{for any} \quad i,j \quad (i\neq j), \tag{5.3.3}$$

where recall that now and in the following reasoning $\epsilon_x=-1$, for all x.

Making use of (2.3.20), the above formulas and Lemma 5.2.2, we deduce

$$2nb \leq \sum_j R_{\bar{j}jj\bar{j}} \leq n(n+1)\frac{c}{2} - h_2 - n(n-1)b.$$

Thus we have

$$2h_2 \leq n(n+1)(c-2b), \tag{5.3.4}$$

where the above equality holds if and only if $R_{\bar{j}jj\bar{j}}=2b$ for any j. That is, M is of constant holomorphic sectional curvature $2b$.

On the other hand, by using Lemma 5.3.2 and (2.3.20), we have that

$$(n-2)R_{\bar{j}jj\bar{j}} \geq (n-1)(n+4)b - n(n+1)\frac{c}{2} + h_2. \tag{5.3.5}$$

Using (2.3.19), the holomorphic sectional curvature $R_{\bar{j}jj\bar{j}}$ equals to $c+\sum_x h_{jj}^x\bar{h}_{jj}^x$, from which it follows that

$$-\sum_x h_{jj}^x\bar{h}_{jj}^x = c - R_{\bar{j}jj\bar{j}} \leq \frac{(n-1)(n+4)(c-2b)-2h_2}{2(n-2)}\,. \tag{5.3.6}$$

With these estimates previously given, it is shown in [24] the following,

THEOREM 5.3.2. *Let M be an $n(\geq 3)$-dimensional complete complex submanifold of $\mathbb{C}H_p^{n+p}(c)$, $p>0$, with totally real bisectional curvature $\geq b$. Then the following assertions hold:*

(1) *b is smaller than or equal to $c/4$.*

(2) *If $b = c/4$, then M is a complex space form of holomorphic sectional curvature $c/2$, $p \geq n(n+1)/2$.*

(3) *If $b = n(n+p+1)c/2(n+2p)(n+1)$, then M is a complex space form of holomorphic sectional curvature $c/2$, $p = n(n+1)/2$.*

5.4. Complex submanifolds with lower bounded totally real bisectional curvature of definite complex space forms. In this last section, as an application to the previous development, we will study n-dimensional complex submanifolds M of an $(n+p)$-dimensional complex projective space $\mathbb{C}P^{n+p}(c)$, $c > 0$, with bounded totally real bisectional curvature. Of course, in this case all the signs ϵ_i and ϵ_x in formulas (2.3.18) and (2.3.19) are just $+1$.

Let us recall that, as in Chapter 2, h_2 denotes the function $h_2 = \sum_{i,j,x} h^x_{ij}\bar{h}^x_{ij}$. Thus by using (2.3.21) and (2.3.22) we have

$$\begin{aligned}(h_2)_{k\bar{l}} = &\sum h^x_{ij;k}\bar{h}^x_{ij;l} + \sum \Big\{ \frac{c}{2}\big(h^x_{ij}\delta_{kl} + h^x_{jk}\delta_{il} + h^x_{ki}\delta_{jl}\big)\bar{h}^x_{ij} \\ &- \big(h^x_{ri}h^y_{jk} + h^x_{rj}h^y_{ki} + h^x_{rk}h^y_{ij}\big)\bar{h}^y_{rl}\bar{h}^x_{ij}\Big\}.\end{aligned} \tag{5.4.1}$$

As in section 3.5, we consider now the function $h_4 = \sum h^2_{i\bar{j}}h^2_{j\bar{i}} = \sum h^x_{ij}\bar{h}^x_{jk}h^y_{kl}\bar{h}^y_{il}$. Now, from (5.4.1) and using again (2.3.22), we can compute the Laplacian of h_4 (see [24]) as follows

$$\begin{aligned}\triangle h_4 = &2\sum \big\{ \frac{n+2}{2}ch^x_{ij} - \big(h^x_{ir}h^2_{j\bar{r}} + h^x_{jr}h^2_{i\bar{r}} + {A_z}^x h^z_{ij}\big)\big\}\bar{h}^x_{jk}h^y_{kl}\bar{h}^y_{li} \\ &+ \sum \big(h^x_{ij;m}\bar{h}^x_{jk;m}h^2_{k\bar{i}} + h^x_{ij;m}\bar{h}^x_{jk}h^y_{kl}\bar{h}^y_{li;m}\big).\end{aligned}$$

By using these formulas we have the following result (see Ki and the second author [24]).

Theorem 5.4.1. *Let M be an $n(\geq 3)$-dimensional complete complex submanifold of a complex projective space $\mathbb{C}P^{n+p}(c)$. If there exists a positive constant b such that $b > \frac{n^3+2n^2+2n-2}{2n(n^2+2n+3)}c$ and the totally real bisectional curvature of M is greater than or equal to b, then M is totally geodesic, hence a complex projective space $\mathbb{C}P^n(c)$.*

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