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# Almost Contact Metric Submersions and Curvature Tensors

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## 1. Introduction

It is known that the fibres of an almost contact metric submersion of type I (in the sense of Watson [W]) are almost Hermitian manifolds. On such manifolds, $K_i$-curvature identities, $i = 1, 2, 3$, have been studied by A. Gray in [G].

In [W-V1], Watson and Vanhecke have interrelated the $K_i$-curvatures theory with that of almost Hermitian submersions. Let $\alpha$ be a real number, since Janssens and Vanhecke [J-V] have defined the $C(\alpha)$-curvature tensors on almost contact metric manifolds, it seems interesting to examine the analogues interrelations in the field of almost contact metric submersions. That is, in this paper, the main purpose is the following problem:

*Let $f : M \to B$ be an almost contact metric submersion of type* I. *Suppose the total space satisfying a $C(\alpha)$-curvature property, what is the corresponding curvature property on the base space and what kind of $K_i$-identity do have the fibres?*

This paper is organized in the following way. In §2, we recall some basic facts on almost contact metric manifolds and Riemannian submersions.

In order to obtain the desired interrelations, the $\phi$-linearity of the configuration tensors $T$ and $A$ is an important tool. Then, in §3, we determine the defining relations of some almost contact metric structures for which the $\phi$-linearity of $T$ and $A$ can be obtained.

Key words: almost contact metric submersions, almost contact metric manifolds, curvature tensors.

In §4, among the main results, we settle the defining relations of all classes of almost contact metric structures that satisfy the cosymplectic curvature property. Then, we show that for such classes, the cosymplectic curvature property on the total space, which resembles to the Kõhler identity as observed in [J-V], transfers to the base space while the fibres have, as corresponding, the Kõhler identity for a type I submersion. Theorems 4.4 and 4.7 show that the Kenmotsu and Sasakian curvature properties are related to the $K_2$ and $K_3$-curvature identities under supplementaries conditions on the configuration tensors $T$ and $A$.

We follow the terminology of O'Neill [O'N] as follows. Arbitrary vector fields of $\chi(M)$, the Lie algebra of smooth vector fields, will be denoted by $D$, $E$ and $G$. Vector fields of the horizontal distribution $H(M)$, of the tangent bundle of $M$ will be denoted by $X$, $Y$ and $Z$ while those of the vertical distribution $V(M)$ will be $U$, $V$ and $W$.

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## 2. Preliminaries

Let $M$ be a differentiable manifold of odd real dimension $2m+1$. An *almost contact structure* on $M$ is a triple $(\phi,\xi,\eta)$ where:

(1) $\xi$ is a distinguished vector field,

(2) $\eta$ is a 1-form such that $\eta(\xi)=1$, and

(3) $\phi$ is a tensor field of type $(1,1)$ satisfying

$$\phi^2D=-D+\eta(D)\xi \qquad \text{for all } D\in\chi(M).$$

If $M$ is equipped with a Riemannian metric $g$ such that

$$g(\phi D,\phi E)=g(D,E)-\eta(D)\eta(E)\,,$$

then $(g,\phi,\xi,\eta)$ is called an almost contact metric structure. So, the quintuple $(M^{2m+1},g,\phi,\xi,\eta)$ is an almost contact metric manifold. Any almost contact metric manifold admits a *fundamental 2-form*, $\Phi$, defined by

$$\Phi(D,E)=g(D,\phi E)\,.$$

The differential $\mathrm{d}\eta$ of $\eta$ is obtained by

$$2\mathrm{d}\eta(D,E)=D\eta(E)-E\eta(D)-\eta([D,E])\,.$$

We denote by $\nabla$ the Riemannian connection of $M$ and recall the defining relations of almost contact metric structures which will be used in this paper.

An almost contact metric manifold is said to be:

(1) *cosymplectic* if $\nabla\phi = 0$;

(2) *nearly cosymplectic* if $(\nabla_D\phi)D = 0$;

(3) *closely cosymplectic* if $(\nabla_D\phi)D = 0 = \mathrm{d}\eta$;

(4) *nearly Kenmotsu* if $(\nabla_D\phi)D + \eta(D)\phi D = 0 = \mathrm{d}\eta$;

(5) *Kenmotsu* if $(\nabla_D\phi)E = g(\phi D, E)\xi - \eta(E)\phi D$;

(6) *Sasakian* if $(\nabla_D\phi)E = g(D, E)\xi - \eta(E)D$.

Looking through the first five structures, it appears that they have in common the following relation

$$(\nabla_D\phi)E = \alpha \cdot \{g(\phi D, E)\xi - \eta(E)\phi D\},$$

where $\alpha$, is a real number. Indeed, taking $\alpha = 0$, we get one of the defining relations of cosymplectic, nearly cosymplectic and closely cosymplectic structures. If $\alpha = 1$, we obtain one of the defining relations of a Kenmotsu or a nearly Kenmotsu structure. If $\alpha$ is neither 1 nor zero, then the above equation defines an $\alpha$-Kenmotsu structure as defined in [J-V].

The differential geometry of almost contact metric manifolds is developped in Blair's monograph [Bl1] and its recent expansion [Bl2]. For the basic properties of Riemannian submersions, we refer the reader to the foundational paper of B. O'Neill [O'N]. These primarily concern the orthogonal decomposition $T(M) = V(M) \oplus H(M)$ of the local vector fields of the total space into vertical and horizontal vector fields.

An almost contact metric submersion is a Riemannian submersion whose total space is an almost contact metric manifold. Regarding the structure of the base space, B. Watson has studied in [W] two types of such submersions.

Definition 2.1. Let $(M^{2m+1}, \phi, \xi, \eta, g)$ and $(M'^{2m'+1}, \phi', \xi', \eta', g')$ be almost contact metric manifolds. A Riemannian submersion

$$f : M^{2m+1} \to M'^{2m'+1}$$

that satisfies

$$f_*\phi E = \phi' f_* E \qquad \text{and} \qquad f_*\xi = \xi',$$

is called an *almost contact metric submersion of type* I (see [W]).

When the base space is an almost Hermitian manifold $(B'^{2m'}, g', J')$, then the Riemannian submersion $f : M^{2m+1} \to B'^{2m'}$ is called an *almost contact metric submersion of type* II if $f_*\phi E = J' f_* E$, where $2m'$ is the real dimension of the differentiable manifold $B'$ (see [W]). Hereafter, we will use a prime $'$ to designate objects and tensors on the base space. If a result does not depend upon the type of submersions, we will not specify the dimension of the base space; this is the case with Propositions 3.1 and 3.3.

Now, we recall some of the fundamental properties of such types of submersions (see [T-M1] or [W] for the proof):

PROPOSITION 2.1. *Let $f : M^{2m+1} \to M'^{2m'+1}$ be an almost contact metric submersion of type* I. *We have*

(a) $f^*\eta' = \eta$ ;

(b) *if* $U \in V(M)$, *then* $\phi U \in V(M)$ ;

(c) *if* $X \in H(M)$, *then* $\phi X \in H(M)$ ;

(d) $\xi \in H(M)$ ;

(e) *if* $U \in V(M)$, *then* $\eta(U) = 0$ .

*Proof.* See [T-M1] or [W]. ∎

PROPOSITION 2.2. *Let $f : M^{2m+1} \to B'^{2m'}$ be an almost contact metric submersion of type* II. *We have*

(a) *if* $U \in V(M)$, *then* $\phi U \in V(M)$ ;

(b) *if* $X \in H(M)$, *then* $\phi X \in H(M)$ ;

(c) $\xi \in \ker f_*$ ;

(d) *if* $X \in H(M)$, *then* $\eta(X) = 0$ .

*Proof.* See again [T-M1] or [W]. ∎

By a basic vector field, one understands a horizontal vector field $X$ which is $f$-related to a vector fields $X_*$ of the base space. There is a one-to-one correspondence between basic vector fields and vector fields of the base space.

## 3. Properties of the configuration tensors

The O'Neill configuration tensors $T$ and $A$ on the total space of a Riemannian submersion are defined in [O'N] by setting

$$T_D E = \mathcal{H}\nabla_{\mathcal{V}D}\mathcal{V}E + \mathcal{V}\nabla_{\mathcal{V}D}\mathcal{H}E\,,$$
$$A_D E = \mathcal{V}\nabla_{\mathcal{H}D}\mathcal{H}E + \mathcal{H}\nabla_{\mathcal{H}D}\mathcal{V}E\,.$$

Here, $\mathcal{H}$ and $\mathcal{V}$ are respectively horizontal and vertical projections of the fibre bundle, $T(M)$, of the total space on horizontal and vertical distributions.

We recall the fundamental properties of these tensors:

$$T_U V = T_V U\,; \tag{3.1}$$
$$T_E = T_{\mathcal{V}E}\,; \tag{3.2}$$
$$T_X E = 0\,; \tag{3.3}$$
$$g(T_V E, G) = -g(E, T_V G)\,; \tag{3.4}$$
$$\mathcal{H}\nabla_U V = T_U V\,; \tag{3.5}$$
$$A_X Y = -A_Y X\,; \tag{3.6}$$
$$A_E = A_{\mathcal{H}E}\,; \tag{3.7}$$
$$A_V E = 0\,; \tag{3.8}$$
$$g(A_X E, G) = -g(E, A_X G)\,. \tag{3.9}$$

If $X$ is basic, then

$$\mathcal{H}\nabla_U X = A_X U \quad \text{and} \quad [U,X] \text{ is vertical}\,. \tag{3.10}$$

It is known that $T$ is used in the geometry of the fibres and $A$ is the integrability tensor of the horizontal distribution.

Following Watson and Vanhecke [W-V2], the $\phi$-linearity and the $\phi$-symmetry of a smooth tensor field $L$ of type $(1,2)$ on an almost contact metric manifold can be defined by:

(a) $L$ is *$\phi$-linear in the first variable* if $L_{\phi D}E = \phi L_D E\,;$

(b) $L$ is *$\phi$-linear in the second variable* if $L_D \phi E = \phi L_D E\,;$

(c) $L$ is *$\phi$-symmetric* if $L_{\phi D}E = L_D \phi E\,.$

Since $T$ and $A$ are smooth tensors fields of type (1,2), we can examine their $\phi$-linearity properties.

PROPOSITION 3.1. *Let $f : M^{2m+1} \to B$ be an almost contact metric submersion of type* I *or type* II. *If the configuration tensor $T$ (resp. $A$) is $\phi$-linear in one of the variables on the vertical (resp. horizontal) distribution, then it is $\phi$-linear in the other.*

*Proof.* Suppose that $T_{\phi U}V = \phi T_U V$; we have to show that $T_U \phi V = \phi T_U V$. Indeed, since $U$ and $V$ are vertical, it is known that $\phi U$ and $\phi V$ are also vertical by virtue of Proposition 2.1 (b). On the other hand, $T$ is symmetric on the vertical distribution according to (3.1). Thus,

$$T_U \phi V = T_{\phi V} U = \phi T_V U = \phi T_U V \,.$$

In the analogous manner, we can show that if $T_U \phi V = \phi T_U V$, then $T_{\phi U} V = \phi T_U V$.

Consider the configuration tensor $A$. Since $X$ and $Y$ are horizontal vector fields, then $\phi X$ and $\phi Y$ are also horizontal vector fields by virtue of Proposition 2.1 (c); the fact that $A$ is skew-symmetric on horizontal vector fields, according to (3.6), gives rise to

$$A_{\phi X} Y = -A_Y \phi X = -\phi A_Y X = -\phi(-A_X Y) = \phi A_X Y \,.$$

∎

The above proposition is a way to see that, on a given distribution, the $\phi$-linearity implies the $\phi$-symmetry of these tensors.

PROPOSITION 3.2. *Let $f : M^{2m+1} \to M'^{2m'+1}$ be an almost contact metric submersion of type* I. *Suppose the total space satisfying the condition*

$$(\nabla_D \phi) E = \alpha \cdot \{ g(\phi D, E)\xi - \eta(E)\phi D \} \,,$$

*then:*

(a) $T_U \phi X = \phi T_U X - \alpha \cdot \eta(X) \phi U$ ;

(b) $T_U \phi V = \phi T_U V + \alpha \cdot g(\phi U, V)\xi$ ;

(c) $A_X \phi U = \phi A_X U$ ;

(d) $A_X \phi Y = \phi A_X Y$ ;

(e) $A_\xi \xi = 0$ .

*Proof.* (a) On the total space, the condition under consideration becomes

$$(\nabla_U \phi)X = \alpha \cdot \{g(\phi U, X)\xi - \eta(X)\phi U\}\,.$$

Since $\phi U$ is vertical and $X$ is horizontal, then $g(\phi U, X) = 0$ and this implies that $(\nabla_U \phi)X = -\alpha \cdot \eta(X)\phi U$. The vertical part of this last equation gives (a).

The vanishing of $\eta$ on the vertical vector fields, as shown in Proposition 2.1 (e), leads to

$$(\nabla_U \phi)V = \alpha \cdot g(\phi U, V)\xi\,,$$

which gives the proof of (b) by taking its horizontal projection.

Concerning assertion (c), the condition on the total space becomes

$$(\nabla_X \phi)U = \alpha\,\{g(\phi X, U)\xi - \eta(U)\phi X\}\,.$$

Since, by Proposition 2.1 (e), $\eta(U) = 0$, the condition reduces to $(\nabla_X \phi)U = 0$. Taking the horizontal projection of this equation, we obtain the proof of (c).

To establish (d), we have $(\nabla_X \phi)Y = \alpha \cdot \{g(\phi X, Y)\xi - \eta(Y)\phi X\}$. The vertical projection of this relation gives $\mathcal{V}(\nabla_X \phi)Y = 0$ because $\phi X$ and $\xi$ are horizontal. Therefore, $A_X \phi Y = \phi A_X Y$.

(e) Since $\phi\xi = 0$, we have $\phi A_\xi \xi = 0$ from which $A_\xi \xi = 0$ follows. ∎

PROPOSITION 3.3. *Let $f : M^{2m+1} \to B$ be an almost contact metric submersion of type* I *or type* II. *Suppose the total space satisfying the condition*

$$(\nabla_D \phi)E = 0\,,$$

*then:*

(a) $T_U \phi V = \phi T_U V$ ;

(b) $T_U \xi = 0$ ;

(c) $A_X \phi Y = \phi A_X Y$ ;

(d) $A_X \xi = 0$ .

*Proof.* (a) The condition under consideration on the total space gives

$$\nabla_U \phi V - \phi \nabla_U V = 0\,,$$

from which we deduce $T_U \phi V = \phi T_U V$ by using the horizontal projection.

(b) In the analogous manner, since $\phi\xi = 0$, we have $\phi\nabla_U\xi = 0$ from which the horizontal projection gives $T_U\xi = 0$.

As in (a), $\nabla_X\phi Y - \phi\nabla_X Y = 0$. Thus, using the vertical projection of this equation, we get $A_X\phi Y = \phi A_X Y$ which is the proof of (c).

The last assertion is trivial since $\nabla_X\xi = 0$. ∎

PROPOSITION 3.4. *Suppose that the total space of an almost contact metric submersion of type* II *satisfies the condition*

$$(\nabla_D\phi)E = \alpha \cdot \{g(\phi D, E)\xi - \eta(E)\phi D\}\,,$$

*then:*

(a) $T_U\phi V = \phi T_U V$ ;

(b) $A_X\phi Y = \phi A_X Y + \alpha \cdot g(\phi X, Y)\xi\,.$

*Proof.* (a) On vertical vector fields, the condition on the total space is

$$(\nabla_U\phi)V = \alpha \cdot \{g(\phi U, V)\xi - \eta(V)\phi U\}\,;$$

since, according to Proposition 2.2 (a) and (c), respectively, $\phi U$ and $\xi$ are vertical, $\mathcal{H}(\phi U) = 0 = \mathcal{H}(\xi)$ which imply that $\mathcal{H}(\nabla_U\phi)V = 0$. Therefore $T_U\phi V = \phi T_U V$.

(b) It is clear that $(\nabla_X\phi)Y = \alpha \cdot g(\phi X, Y)\xi$ because $\eta(Y) = 0$. Thus, $\mathcal{V}(\nabla_X\phi)Y = \mathcal{V}(\alpha \cdot g(\phi X, Y))\mathcal{V}(\xi)$ which implies that $A_X\phi Y = \phi A_X Y + \alpha \cdot g(\phi X, Y)\xi$. ∎

PROPOSITION 3.5. *Let* $f : M^{2m+1} \to B'^{2m'}$ *be an almost contact metric submersion of type* II. *Suppose the condition*

$$(\nabla_D\phi)E = \alpha \cdot \{g(D, E)\xi - \eta(E)D\}$$

*fulfilled on the total space, then:*

(a) $T_U\phi V = \phi T_U V$ ;

(b) $A_X\phi Y = \phi A_X Y + \alpha \cdot g(X, Y)\xi\,.$

*Proof.* (a) Using vertical vector fields $U$ and $V$, the condition becomes

$$(\nabla_U\phi)V = \alpha \cdot \{g(U, V)\xi - \eta(V)U\}\,.$$

Since $\xi$ and $U$ are vertical we have $\mathcal{H}(\xi) = \mathcal{H}(U) = 0$ so that $\mathcal{H}(\nabla_U\phi)V = 0$ which yields $T_U\phi V = \phi T_U V$.

(b) Since $\eta(Y) = 0$, the condition becomes

$$(\nabla_X\phi)Y = \alpha \cdot g(X,Y)\xi\,,$$

from which the vertical projection gives rise to

$$A_X\phi Y = \phi A_X Y + \alpha \cdot g(X,Y)\xi\,.$$

∎

## 4. Riemannian curvature properties

Recall that the Riemannian curvature tensor $R$ of a Kõhler manifold satisfies the $K_1$-identity (the Kõhler identity) defined by

$$R(D,E,F,G) = R(D,E,JF,JG)\,.$$

Others $K_i$-identities ($i = 1,2,3$) have been studied by A. Gray in [G], but their interrelations with the theory of Riemannian submersions can be found in [W-V1] and [W-V2].

Let $(M^{2m}, g, J)$ be an almost Hermitian manifold. The $K_i$-curvature properties are defined in the following way:

(1) $K_1$: if $R(D,E,F,G) = R(D,E,JF,JG)$;

(2) $K_2$: if $R(D,E,F,G) = R(JD,E,JF,G) + R(JD,JE,F,G) + R(JD,E,F,JG)$;

(3) $K_3$: if $R(D,E,F,G) = R(JD,JE,JF,JG)$.

In their study of curvature tensors of almost contact metric manifolds, D. Janssens and L. Vanhecke [J-V], have obtained the following properties of the Riemannian curvature tensor:

(a) the *cosymplectic curvature property*, defined by

$$R(D,E,F,G) = R(D,E,\phi F,\phi G)\,;$$

(b) the *Kenmotsu curvature property*, defined by

$$\begin{aligned}R(D,E,F,G) = R(D,E,\phi F,\phi G) + g(D,F)g(E,G) - g(D,G)g(E,F)\\ - g(D,\phi F)g(E,\phi G) + g(D,\phi G)g(E,\phi F)\,;\end{aligned}$$

(c) the *Sasakian curvature property*, defined by

$$R(D,E,F,G) = R(D,E,\phi F,\phi G) - g(D,F)g(E,G) + g(D,G)g(E,F) + g(D,\phi F)g(E,\phi G) - g(D,\phi G)g(E,\phi F)\,.$$

The curvature tensors of an almost contact metric manifold are called $C(\alpha)$-curvature tensors where $\alpha$ is a real number. For instance, the cosymplectic curvature tensor is a $C(0)$-curvature tensor, the Kenmotsu curvature tensor is a $C(-1)$-curvature tensor and the Sasakian curvature tensor is a $C(1)$-curvature tensor. For more details, we refer the reader to [J-V]. It is clear that the cosymplectic curvature tensor resembles to the Kõhler identity.

Now, we want to determine the classes of almost contact metric manifolds which satisfy the cosymplectic curvature property.

THEOREM 4.1. *Let $(M^{2m+1}, g, \phi, \xi, \eta)$ be an almost contact metric manifold. If $M$ satisfies the condition*

$$(\nabla_D \phi)E = 0\,,$$

*then it has the cosymplectic curvature property.*

*Proof.* For an almost contact metric manifold, the Ricci identity is given by

$$R(D,E)\phi - \phi R(D,E) = [\nabla_D, \nabla_E]\,\phi - \nabla_{[D,E]}\phi\,. \tag{4.1}$$

The condition on $M$ being equivalent to $\nabla\phi = 0$, the right hand side of (4.1) vanishes. We get $R(D,E)\phi F - \phi R(D,E)F = 0$ which gives

$$g(R(D,E)\phi F, \phi G) = g(\phi R(D,E)F, \phi G) = -g(R(D,E)F, \phi^2 G)$$

from which we get

$$g(R(D,E)\phi F, \phi G) = -g(R(D,E)F, -G) - g(R(D,E)F, \eta(G)\xi)\,. \tag{4.2}$$

It remains to show that $g(R(D,E)F, \eta(G)\xi) = 0$. Indeed,

$$g(R(D,E)F, \eta(G)\xi) = g(R(D,E)F, \xi)\eta(G)\,;$$

but

$$g(R(D,E)F, \xi) = R(D,E,F,\xi) = -R(D,E,\xi,F) = -g(R(D,E)\xi, F)\,.$$

Since, in such a situation, $\nabla_D \xi = 0$, we get $R(D,E)\xi = 0$ from which we deduce $g(R(D,E)F,\xi) = 0$ so that (4.2) becomes

$$g(R(D,E)\phi F, \phi G) = g(R(D,E)F,G)\,;$$

hence $R(D,E,\phi F,\phi G) = R(D,E,F,G)$ follows immediately. ∎

THEOREM 4.2. *Let $f : M^{2m+1} \to M'^{2m'+1}$ be an almost contact metric submersion of type* I. *Suppose that the total space satisfies the condition*

$$(\nabla_D \phi)E = 0\,,$$

*then the base space has the cosymplectic curvature property and, on the fibres, this property corresponds to the Kõhler identity.*

*Proof.* We begin by establishing that the structure of the total space transfers to the base space [T-M2]. Let $X$ and $Y$ be basic vector fields, that is, they are $f$-related to the vector fields $X_*$ and $Y_*$ respectively of the base space. In [C2, Proposition 2.3], it is shown that $\mathcal{H}(\nabla_X \phi)Y$ is basic associated to $(\nabla'_{X_*}\phi')Y_*$. Thus, since $(\nabla_X\phi)Y = 0$, we deduce that $(\nabla'_{X_*}\phi')Y_* = 0$. Therefore, according to the preceding Theorem 4.1, the base space has the cosymplectic curvature property.

Now, consider the vector fields $U$, $V$, $W$ and $S$ tangent to the fibres. For a Riemannian submersion, the Gauss equation is given by

$$R(U,V,W,S) = \hat{R}(U,V,W,S) - g(T_U W, T_V S) + g(T_V W, T_U S)\,. \tag{4.3}$$

This equation can be transformed in

$$\begin{aligned} R(U,V,\phi W,\phi S) = \hat{R}(U,V,\hat{\phi} W,\hat{\phi} S) &- g(T_U \phi W, T_V \phi S) \\ &+ g(T_V \phi W, T_U \phi S)\,. \end{aligned} \tag{4.4}$$

Since $T$ is $\phi$-linear in the second variable, as shown in Proposition 3.3 (a), we have

$$\begin{aligned} g(T_U\phi W, T_V\phi S) &= g(\phi T_U W, \phi T_V S) = -g(T_U W, \phi^2 T_V S) \\ &= g(T_U W, T_V S) - g(T_U W, \eta(T_V S)\xi)\,; \end{aligned}$$

but $\eta(T_V S) = g(\xi, T_V S) = -g(S, T_V \xi) = 0$ because $T_V\xi = 0$. Therefore,

$$\begin{aligned} g(T_U\phi W, T_V\phi S) &= g(T_U W, T_V S)\,, \\ g(T_V\phi W, T_U\phi S) &= g(T_V W, T_U S)\,. \end{aligned}$$

In such a case, (4.4) leads to

$$R(U,V,\phi W,\phi S) = \hat{R}(U,V,\hat{\phi}W,\hat{\phi}S) - g(T_U W, T_V S) + g(T_V W, T_U S)\,. \quad (4.5)$$

Subtracting (4.5) from (4.3) we get

$$R(U,V,W,S) - R(U,V,\phi W,\phi S) = \hat{R}(U,V,W,S) - \hat{R}(U,V,\hat{\phi}W,\hat{\phi}S)\,.$$

Since $R(U,V,W,S) = R(U,V,\phi W,\phi S)$, then $\hat{R}(U,V,W,S) = \hat{R}(U,V,\hat{\phi}W,\hat{\phi}S)$ which shows that the fibres have the $K_1$-curvature identity. ∎

The above theorem can be viewed as a way to establish the following

THEOREM 4.3. *Let $f : M^{2m+1} \to M'^{2m'+1}$ be an almost contact metric submersion of type I. Suppose the following conditions satisfied:*

(a) *the total space satisfies the cosymplectic curvature property;*

(b) *the configuration tensor $T$ is $\phi$-linear on the vertical distribution;*

(c) $T_U\xi = 0$ *for all vertical vector fields $U$.*

*Then the fibres have the Kõhler identity.*

THEOREM 4.4. *Let $f : M^{2m+1} \to M'^{2m'+1}$ be an almost contact metric submersion of type* I *satisfying the following conditions:*

(a) *the total space satisfies the Kenmotsu curvature property;*

(b) *the configuration tensor $T$ is $\phi$-linear on the vertical distribution;*

(c) $T_U\xi = 0$ *for all vertical vector fields $U$.*

*Then the fibres verify the $K_2$-curvature identity.*

*Proof.* Since $T$ is $\phi$-linear and $T_U\xi = 0$, by calculation we get $g(T_U\phi W, T_V\phi S) = g(T_U W, T_V S)$ and $g(T_{\phi U}\phi W, T_V S) = -g(T_U W, T_V S)$. By virtue of the Kenmotsu curvature property, we have

$$\begin{aligned} R(U,V,W,S) &= R(U,V,\phi W,\phi S) + R(\phi U,V,\phi W,S) \\ &\quad + R(\phi U,V,W,\phi S)\,. \end{aligned} \quad (4.6)$$

So the Gauss equation gives

(1) $R(U,V,\phi W,\phi S) = \hat{R}(U,V,\hat{\phi}W,\hat{\phi}S) - g(T_U W, T_V S) + g(T_V W, T_U S)\,,$

(2) $R(\phi U, V, \phi W, S) = \hat{R}(\hat{\phi} U, V, \hat{\phi} W, S) + g(T_U W, T_V S) + g(T_V W, T_U S)$,

(3) $R(\phi U, V, W, \phi S) = \hat{R}(\hat{\phi} U, V, W, \hat{\phi} S) - g(T_U W, T_V S) - g(T_V W, T_U S)$.

Therefore, summing 1, 2 and 3, we obtain

$$R(U, V, \phi W, \phi S) + R(\phi U, V, \phi W, S) + R(\phi U, V, W, \phi S) = R(U, V, W, S)$$
$$-\hat{R}(U, V, W, S) + \hat{R}(U, V, \hat{\phi} W, \hat{\phi} S) + \hat{R}(\hat{\phi} U, V, \hat{\phi} W, S) + \hat{R}(\hat{\phi} U, V, W, \hat{\phi} S);$$

with (4.6) in mind, we get

$$\hat{R}(U, V, W, S) = \hat{R}(U, V, \hat{\phi} W, \hat{\phi} S) + \hat{R}(\hat{\phi} U, V, \hat{\phi} W, S) + \hat{R}(\hat{\phi} U, V, W, \hat{\phi} S),$$

which shows that the fibres verify the $K_2$-curvature identity. ∎

Now we are going to examine the analogous properties in the case of almost contact metric submersions of type II.

THEOREM 4.5. *Let $f : M^{2m+1} \to B'^{2m'}$ be an almost contact metric submersion of type* II. *Suppose the condition*

$$(\nabla_D \phi) E = 0$$

*fulfilled on the total space, then the fibres verify the cosymplectic curvature property which corresponds to the $K_1$-curvature identity on the base space.*

*Proof.* By Theorem 4.1, the total space satisfies the cosymplectic curvature property. As in Theorem 4.2, setting $D = U$ and $E = V$ in the given condition on the total space, we obtain $(\hat{\nabla}_U \hat{\phi}) V = 0$, hence the fibres verify the cosymplectic curvature property.

To see that the base space verifies the Kõhler identity, let $X$, $Y$, $Z$ and $P$ be basic vector fields. As in Theorem 4.2, the Gauss equation is

$$\begin{aligned} R(X, Y, Z, P) = R'(X_*, Y_*, Z_*, P_*) - 2g(A_X Y, A_Z P, ) \\ + g(A_Y Z, A_X P) + g(A_X Z, A_Y P), \end{aligned} \tag{4.7}$$

which gives

$$\begin{aligned} R(X, Y, \phi Z, \phi P) = R'(X_*, Y_*, \phi' Z_*, \phi' P_*) - 2g(A_X Y, A_{\phi Z} \phi P) \\ + g(A_Y \phi Z, A_X \phi P) + g(A_X \phi Z, A_Y \phi P). \end{aligned} \tag{4.8}$$

Taking account the fact that, in the context of Proposition 3.3, the configuration tensor $A$ is $\phi$-linear on the horizontal distribution and $A_X\xi = 0$, then (4.8) can be rewritten in the following way

$$\begin{aligned} R(X,Y,\phi Z,\phi P) &= R'(X_*,Y_*,\phi' Z_*,\phi' P_*) - 2g(A_X Y, A_Z P) \\ &\quad + g(A_Y Z, A_X P) + g(A_X Z, A_Y P)\,. \end{aligned} \tag{4.9}$$

Thus, subtracting (4.9) from (4.7), we get

$$\begin{aligned} &R(X,Y,Z,P) - R(X,Y,\phi Z,\phi P) = \\ &\qquad R'(X_*,Y_*,Z_*,P_*) - R'(X_*,Y_*,\phi' Z_*,\phi' P_*)\,. \end{aligned} \tag{4.10}$$

Since the total space satisfies the cosymplectic curvature property, then $R(X,Y,Z,P) = R(X,Y,\phi Z,\phi P)$, which implies that $R'(X_*,Y_*,Z_*,P_*) = R'(X_*,Y_*,\phi' Z_*,\phi' P_*)$.

The base space being an almost Hermitian manifold, it follows that it verifies the $K_1$-curvature identity. ∎

THEOREM 4.6. *Let* $f : M^{2m+1} \to B'^{2m'}$ *be an almost contact metric submersion of type* II *that satisfies the following conditions:*

(a) *the total space satisfies the cosymplectic curvature property,*

(b) *the configuration tensor* $A$ *is* $\phi$*-linear on the horizontal distribution, and*

(c) $A_X\xi = 0$ *for all horizontal vector fields* $X$.

*Then, the fibres have the cosymplectic curvature property and, on the base space, this property corresponds to the Kõhler identity.*

*Proof.* Similar to the preceding. ∎

THEOREM 4.7. *Let* $f : M^{2m+1} \to B'^{2m'}$ *be an almost contact metric submersion of type* II *such that*

(a) *the total space* $M$ *has the Sasakian curvature property,*

(b) *the configuration tensor* $A$ *is* $\phi$*-linear on the horizontal distribution, and*

(c) $A_X\xi = 0$ *for all horizontal vector fields* $X$.

*Then the fibres have the Sasakian curvature property which corresponds to the* $K_3$*-curvature identity on the base space.*

*Proof.* Similar to that of Theorem 4.4. ∎

An almost contact metric structure is said to be *nearly-K-cosymplectic* if

$$(\nabla_D \phi)D = 0\,,$$
$$\nabla_D \xi = 0\,.$$

In [C1, Theorem 2.3], D. Chinea has proved that if the total space of an almost contact metric submersion of type I is nearly-K-cosymplectic, then the configuration tensor $T$ satisfies

$$T_U \phi V = T_{\phi U} V = \phi T_U V$$
$$T_U \xi = 0\,.$$

On the other hand, it is well known that for a submersion of this class, the fibres are nearly Kõhler [C2, Theorem 2.1]. Since the total space has the cosymplectic curvature property, the fibres have the Kõhler identity and therefore are Kõhler.

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