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Cohomology Ring of n-Lie Algebras

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1. Introduction

There is a smart way (see also [9]) to obtain coboundary operators for some types of algebras, e.g. associative algebras or Lie algebras. We will explain it in the following examples.

1.1. Associative algebras. Let V be any vector space and set $M^p = M^p(V) := \operatorname{Hom}(V^{\otimes(p+1)}, V)$ for $p \geq 0$, $M^{-1}(V) := V$, $M^p(V) := 0$ for $p < -1$. Hence $M^p(V)$ is the space of $(p+1)$-linear maps from V to V. Set $M = M(V) := \bigoplus_{p\in\mathbb{Z}} M^p(V)$. If α is in M^p then we write often α^p for α and say that α^p has degree p and dimension $p+1$. Recall the definition of the Gerstenhaber bracket (see [4]):

$$[\alpha^p, \beta^q]^G := (-1)^{pq}\, \alpha^p \circ_G \beta^q - \beta^q \circ_G \alpha^p, \tag{1}$$

where the element $\alpha^p \circ_G \beta^q \in M^{p+q}$ is defined by

$$\begin{aligned} &\alpha^p \circ_G \beta^q\,(v_0, \ldots, v_{p+q}) \\ &\qquad := \sum_{i=1}^{p} (-1)^{iq}\, \alpha(v_0, \ldots, v_{i-1}, \beta(v_i, \ldots, v_{i+q}), \ldots, v_{p+q}). \end{aligned} \tag{2}$$

Let $\pi \in M^1 = \operatorname{Hom}(V \otimes V, V)$. It is easy to check that $[\pi, \pi]^G = 0$ if and only if π is associative, i.e. $\pi(\pi(x,y),z) = \pi(x, \pi(y,z))$. In such a situation we say that associative algebras are canonical structures for the Gerstenhaber bracket. Assume that V is an associative algebra and set $\pi(a,b) = ab$. One can verify that the Hochschild coboundary operator $d^p : M^{p-1} \to M^p$

$$\begin{aligned}(d^p\alpha)(a_1,\dots,a_{p+1}) = {} & a_1\alpha(a_2,\dots,a_{p+1}) \\ & + \sum_{i=1}^{p}(-1)^i\alpha(a_1,\dots,a_{i-1},a_ia_{i+1},\dots,a_{p+1}) \\ & + (-1)^{p+1}\alpha(a_1,\dots,a_p)a_{p+1},\end{aligned}$$

can be written in a very simple form: $d = [\pi,\cdot]^G$.

1.2. Lie algebras. Replace M^p, M and $[\cdot,\cdot]^G$ with $\mathcal{A}^p := \mathrm{Hom}\,(\bigwedge^{(p+1)} V, V)$ (the space of $(p+1)$-linear antysymmetric maps), $\mathcal{A} := \bigoplus_{p\geq -1}\mathcal{A}^p(V)$ and the Nijenhuis-Richardson bracket $[\cdot,\cdot]^{NR}$, respectively. Recall that the Nijenhuis-Richardson bracket is defined by a formula similar to (1) but with $\circ_{NR}$ instead of $\circ_G$, where

$$\alpha^p\circ_{NR}\beta^q\,(v_1,\dots,v_{p+q+1}) := \sum_J(-1)^{(J,I)}\alpha(\beta(v_{j_1},\dots,v_{j_{q+1}}),v_{i_1},\dots,v_{i_p}), \tag{3}$$

the sum over all $(q+1)$-shuffles $J = \{j_1 < \ldots < j_{q+1}\} \subset \{1,\dots,p+q+1\}$. Here $(-1)^{(J,I)}$ stands for the sign of the permutation $(j_1,\dots,j_{q+1},i_1,\dots,i_p)$ of $I\cup J = \{1,\dots,p+q+1\}$, where $I = \{i_1 < \ldots < i_p\}$. Then, for $\pi\in\mathcal{A}^1$, one has: $[\pi,\pi]^{NR} = 0$ if and only if π satisfies the Jacobi identity

$$\pi(\pi(x,y),z) = \pi(x,\pi(y,z)) - \pi(y,\pi(x,z)).$$

The Chevalley-Eilenberg coboundary operator $d^p : \mathcal{A}^{p-1}\to\mathcal{A}^p$,

$$\begin{aligned}(d^p\alpha)(x_1,\dots,x_{p+1}) = {} & \sum_{j<i}(-1)^{j+i}\,\alpha([x_j,x_i],x_1,\dots,\widehat{x_j},\dots,\widehat{x_i},\dots,x_{p+1}) \\ & + \sum_{j=1}^{p+1}(-1)^{j+1}\,[x_j,\alpha(x_1,\dots,\widehat{x_j},\dots,x_{p+1})],\end{aligned}$$

where $\pi(x,y) = [x,y]$ and the hats stand for omissions, has the same simple form $[\pi,\cdot]^{NR}$. We say that Lie algebras are canonical structures for the Nijenhuis-Richardson bracket.

Above examples can be generalized (see [1]) to the case of $\mathcal{P}$-algebras, where $\mathcal{P}$ is an arbitrary quadratic operad. In particular, it is the case of

Leibniz algebras, which is a generalization of Lie algebra in which the bracket needs no longer to be antisymmetric.

In this paper we will prove that also n-Lie algebras (a generalization of Lie algebra in which the binary bracket is replaced with an n-linear antisymmetric bracket $[\cdot,\ldots,\cdot]$ satisfying an analog of Jacobi identity), and n-Leibniz algebras (a non-antisymmetric version of n-Lie algebra) are canonical structures for some brackets $[\cdot,\cdot]^L$ and $[\cdot,\cdot]^{nL}$.

This setting allows us to define the corresponding cohomology operators as brackets with the cannonical structure and to introduce a graded Lie algebra structure on the cohomology spaces.

2. Definitions and examples

Here we formulate precise definitions of objects of our interest and give some basic examples.

Definition. Leibniz algebra (see [10, 11]) is a vector space $\mathfrak{g}$ together with a bilinear map $\{\cdot,\cdot\} : \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ satisfying the following version of Jacobi identity:

$$\{\{x,y\},z\} = \{x,\{y,z\}\} - \{y,\{x,z\}\}. \tag{4}$$

Note that the identity (4) says exactly that every operator $\{x,\cdot\} : \mathfrak{g}\to\mathfrak{g}$ is a derivation of $(\mathfrak{g},\{\cdot,\cdot\})$. This notion was introduced by J-L. Loday who called them Leibniz algebras. Leibniz algebras are drawing now more attention, especially in noncommutative geometry.

Definition. n-Leibniz algebra is a vector space A with an n-linear map $[\cdot,\ldots,\cdot] : A\times\ldots\times A\to A$ satisfying so called fundamental identity:

$$\begin{aligned}&[a_1,\ldots,a_{n-1},[b_1,\ldots,b_n]]\\ &\qquad=\sum_{i=1}^{n}[b_1,\ldots,b_{i-1},[a_1,\ldots,a_{n-1},b_i],b_{i+1},\ldots,b_n].\end{aligned} \tag{5}$$

Definition. n-Lie algebra is a vector space A with an n-linear antisymmetric bracket $[\cdot,\ldots,\cdot]$ satisfying (5).

This notion was first introduced by Fillipov [3]. It also appears in the formulation of generalized Hamiltonian mechanics (see [12, 13]).

Example 1. ([3]) Let V be an $(n+1)$-dimensional oriented Euclidian space. Then V with the bracket given by the vector product

$$[v_1, \dots, v_n] := v_1 \times \dots \times v_n$$

is an n-Lie algebra.

Example 2. ([3]) Let A be an associative commutative algebra and D_1, $\dots$, $D_n \in \mathrm{Der}\,(A)$ be commuting derivations of A . Define the Jacobian map $\mathrm{Jac}_n : \bigwedge^n A \to A$,

$$\mathrm{Jac}_n(a_1, \dots, a_n) := det(D_i(a_j))_{i,j=1,\dots,n} \quad .$$

Then (A, Jac_n) is an n-Lie algebra.

Example 3. ([5]) Let A be an n-Leibniz algebra and set $\mathfrak{g} := A^{\otimes(n-1)}$. On the space $\mathfrak{g}$ define a bracket $\{\cdot,\cdot\}$ by

$$\begin{aligned} &\{x_1 \otimes \dots \otimes x_{n-1}, y_1 \otimes \dots \otimes y_{n-1}\} \\ &\quad = \sum_{i=1}^{n-1} y_1 \otimes \dots \otimes y_{i-1} \otimes [x_1, \dots, x_{n-1}, y_i] \otimes y_{i+1} \otimes \dots \otimes y_{n-1}\,. \end{aligned} \tag{6}$$

Then $(\mathfrak{g}, \{\cdot,\cdot\})$ is a Leibniz algebra.

Example 4. ([7, 8]) Let M be a smooth manifold and $\{\cdot, \dots, \cdot\} : \bigwedge^n C^\infty(M) \to C^\infty(M)$ be a skew-symmetric n-bracket defined on the algebra of smooth functions on M which satisfies the fundamental identity and also the following Leibniz rule

$$\{f_1 f_2, g_2, \dots, g_n\} = f_1\{f_2, g_2, \dots, g_n\} + f_2\{f_1, g_2, \dots, g_n\}.$$

Then the pair $(M, \{\cdot, \dots, \cdot\})$ is called Nambu-Poisson manifold. Thanks to Leibniz rule, every such a bracket is given by an n-vector field π on M so that

$$\{\pi, df_1 \wedge \dots \wedge df_n\} = \{f_1, \dots, f_n\}.$$

It is shown in [7] that if $n \geq 3$ then n-Poisson manifolds give rise to a Leibniz bracket $[[\cdot,\cdot]]$ on the space of $(n-1)$-forms on M so that the following formula holds

$$\begin{aligned} &[[df_1 \wedge \dots \wedge df_{n-1}, dg_1 \wedge \dots \wedge dg_{n-1}]] \\ &\quad = \sum_{i=1}^{n-1} dg_1 \wedge \dots \wedge dg_{i-1} \wedge d\{f_1, \dots, f_{n-1}, g_i\} \wedge dg_{i+1} \wedge \dots \wedge dg_{n-1}. \end{aligned}$$

DEFINITION. ([4]) An algebra $(A, [\cdot,\cdot])$ is a $\mathbb{Z}$-graded Lie algebra if

1. It is a graded algebra, i.e. is a direct sum of vector subspaces, $A = \bigoplus_{p\in\mathbb{Z}} A_p$, such that $[A_p, A_q] \subset A_{p+q}$.
2. The bracket $[\cdot,\cdot]$ in A is graded skew-symmetric, i.e.

$$[x, y] = -(-1)^{pq}[y, x],$$

for $x \in A_p$, $y \in A_q$, and

3. satisfies the so called the graded Jacobi identity:

$$[[x, y], z] = [x, [y, z]] - (-1)^{pq}[y, [x, z]]. \tag{7}$$

for $x \in A_p$, $y \in A_q$ and $z \in A_r$.

It is easy to check that if $\pi \in A_1$ is such that $[\pi, \pi] = 0$ then the map $\delta^p_\pi : A_p \to A_{p+1}$, $\delta_\pi(x) = [\pi, x]$ is a coboundary map, i.e. $\delta^{p+1}_\pi \circ \delta^p_\pi = 0$. In fact, from (7) we have

$$0 = [[\pi, \pi], z] = [\pi, [\pi, z]] - (-1)[\pi, [\pi, z]] = 2\,\delta_\pi(\delta_\pi(z)).$$

It is known and easy to prove the following

THEOREM 1. *Let $A = \bigoplus_{p\in\mathbb{Z}} A_p$ be a graded Lie algebra, π be an element of A_1 such that $[\pi, \pi] = 0$, $\delta^p = \delta^p_\pi$ be a homomorphism $[\pi, \cdot] : A_p \to A_{p+1}$ and $\delta : A \to A$ be such that $\delta_{|A_p} = \delta^p_\pi$. Set $H^p(A) = \ker \delta^p / \operatorname{im} \delta^{p-1}$. Then*

1. $\delta[x^p, y^q] = [\delta x^p, y^q] + (-1)^p[x, \delta y^q]$.
2. *The bracket on A factors to the map*

$$[\cdot,\cdot] : H^p(A) \times H^q(A) \to H^{p+q}(A).$$

Remark. The element $\pi \in A_1$ satisfying the assumptions of the above theorem will be called a canonical structure for the graded Lie algebra A.

Remark. Notice that the natural gradation by dimension in the spaces of cochains of associative algebras, Lie algebras and also, as we will see, Leibniz and n-Leibniz algebras is shifted by 1 with the comparison to the natural gradation by degree of the associated graded Lie algebras.

3. Canonical structures

Here we build up some brackets and show that Leibniz algebras as well as n-Leibniz algebras are canonical structures for these brackets.

3.1. Leibniz algebras. We are going to construct a bracket $[\cdot,\cdot]^L$ on a space of multilinear maps for which Leibniz algebras are canonical structures. It will come out that $[\cdot,\cdot]^L$ is an extension of Nijenhuis-Richardson bracket.

Let V, M^p, M be as in Introduction. For $\alpha^p \in M^p$, $\beta^q \in M^q$, where $p,q \geq 0$, set $\alpha^p \circ_L \beta^q \in M^{p+q}$,

$$\begin{aligned} &\alpha^p \circ_L \beta^q \,(v_1,\ldots,v_{p+q+1}) \\ &\quad := \sum (-1)^{(J,I)}(-1)^k \alpha^p(v_{i_1},\ldots,v_{i_k},\beta^q(v_{j_1},\ldots,v_{j_{q+1}}),v_{i_{k+1}},\ldots,v_{i_p}), \end{aligned} \tag{8}$$

where the sum is over all shuffles $I=\{i_1<\ldots<i_p\}\subset\{1,\ldots,p+q+1\}=:N$. Here j's and k are defined by: $\{j_1<\ldots<j_{q+1}\}=N\setminus I$, $i_{k+1}=j_{q+1}+1$ or, in case $j_{q+1}=p+q+1$, $k:=p$ and $(-1)^{(J,I)}$ is the sign of the permutation $(J,I)=(j_1,\ldots,j_{q+1},i_1,\ldots,i_p)$ of N. If $\alpha^{-1}\in M^{-1}=V$ and $q\geq 0$ then set $\alpha^{-1}\circ_L \beta^q := 0$ and

$$\beta^q \circ_L \alpha^{-1}\,(v_1,\ldots,v_q) := \beta^q(\alpha^{-1},v_1,\ldots,v_q).$$

Now we turn $M(V)$ into a graded Lie algebra by setting

$$[\alpha^p,\beta^q]^L = (-1)^{pq}\alpha^p \circ_L \beta^q - \beta^q \circ_L \alpha^p. \tag{9}$$

Theorem 2. *The algebra $(M(V),[\cdot,\cdot]^L)$ is a graded Lie algebra. The restriction of $[\cdot,\cdot]^L$ to the space of antisymmetric maps $\mathcal{A}(V)$ is the Nijenhuis-Richardson bracket, i.e.*

$$[\alpha,\beta]^L = [\alpha,\beta]^{NR} \tag{10}$$

for $\alpha,\beta\in\mathcal{A}(V)\subset M(V)$. Moreover, the equation $[\pi,\pi]^L=0$ for $\pi\in M^1(V)$ reads as the Jacobi identity (4) *for the bracket $\{x,y\}=\pi(x,y)$.*

Remark. After we had written this article we found that this theorem was first proved in [1], Thm. 3.2.6. Nevertheless, we attach our original proof for completeness of this article.

Proof. According to (9) and (8) we have $[\pi,\pi]^L = -2\,\pi\circ_L\pi$ and

$$\pi\circ_L\pi\,(x_1,x_2,x_3) = \pi(\pi(x_1,x_2),x_3) - \pi(x_1,\pi(x_2,x_3)) + \pi(x_2,\pi(x_1,x_3)),$$

what proves the last assertion. The equality (10) is obvious. Now we are going to prove the graded Jacobi identity for the bracket $[\cdot,\cdot]^L$. We will show how to shift (in an unique way) "antisymmetric expressions" to non-antisymmetric ones and will argue that the graded Jacobi identity is satisfied for $[\cdot,\cdot]^L$ since it is so for $[\cdot,\cdot]^{NR}$.

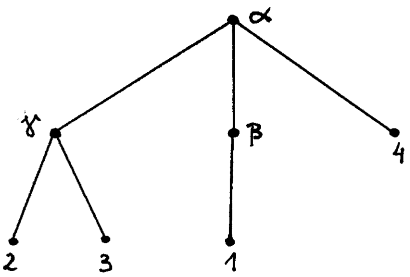

Figure 1.

We begin with introducing a rather formal language in which we treat expressions we deal with as elements of a vector space. Consider a tree in which leaves are labelled by positive integers and each interior node has a label which is the name of a k-linear map $V\times\ldots\times V\to V$, where k is the number of its children (see Figure 1). Every such a tree determines and is determined by a simple expression. For example the tree on Figure 1 determines the simple expression $\alpha(\gamma(x_2,x_3),\beta(x_1),x_4)$, where x_1,x_2,x_3,x_4 are its free variables. All formulas in the definitions of the brackets $[\cdot,\cdot]^L$ or $[\cdot,\cdot]^{NR}$ are linear combinations of such simple expressions. We call them expressions. A simple expression is called homogeneous if its associated tree has leaves labelled injectively. We say that an expression is homogeneous if it is a linear combination of homogeneous simple expressions with the same set of free variables.

Let ϕ be a label function from the set of leaves of T to $\mathbb{N}$, where T is a tree of above form. Define the weight $h(w)\in\mathbb{N}$ of a node w inductively by

$$\begin{cases} h(w) := \phi(w), & \text{if } w \text{ is a leaf,}\\ h(w) := h(w_r), & \text{otherwise, where } w_r \text{ is the right most child of } w. \end{cases}$$

In brief, $h(w)$ is a label of the right most leaf in a subtree rooted at w. We distinguish the set of trees of above form satisfying the following:

3.2. Weight Rule (WR): For every node v, if $w_1, \ldots, w_r$ are all children of a node v then $h(w_1) < \ldots < h(w_r)$.

Remark. The Weight Rule makes sense also in the case when $\mathbb{N}$ is replaced by any ordered set as a set of possible labels of a considered tree.

We say that a homogeneous expression satisfies the Weight Rule if it is a linear combination of simple expressions associated with a tree satisfying WR. For example, $\alpha(x_1, \beta(x_2, x_3))$, unlike $\alpha(\beta(x_2, x_3), x_1)$, satisfies WR.

Two homogeneous simple expressions are said to be similar if its associated trees are the same up to branch reordering. For example, $\alpha(x_1, \beta(x_3, x_2), x_4)$ is similar to $\alpha(x_4, x_1, \beta(x_2, x_3))$. It is clear that:

3.3. Unique shift: Every simple homogeneous expression is similar to an unique one satisfying WR.

We begin reordering from the bottom of the associated tree and move up to the root. In a similar way we can reorder a homogeneous expression, remembering of changing the sign when necessary as if all occurring maps in the expression were antisymmetric. For example the expression $\alpha(\gamma(X_2, X_3), \beta(X_1), X_4)$ turns to $-\alpha(\beta(X_1), \gamma(X_2, X_3), X_4)$, which satisfies WR. In fact the "unique shift" is a linear map from the space spanned freely by all simple expressions to one spanned by simple expressions satisfying WR.

Now we are going to finish the proof. As we have already mentioned, $[\cdot,\cdot]^{NR}$ satisfies the graded Lie identity and the proof of this assertion can be made by direct computation. This means that the Jacobi identity for the bracket $[\cdot,\cdot]^{NR}$, i.e.

$$\begin{aligned}([[\alpha^p, \beta^q]^{NR}, \gamma^r]^{NR} - [\alpha^p, [\beta^q, \gamma^r]^{NR}]^{NR} \\ + (-1)^{pq}[\beta^q, [\alpha^p, \gamma^r]^{NR}]^{NR})(x_1, \ldots, x_{p+q+r+1}),\end{aligned}$$

is in the kernel of the "unique shift" map.

Note that $\alpha^p \circ_L \beta^q\,(x_1, \ldots, x_{p+q+1})$, and hence $[\alpha^p, \beta^q]^L(x_1, \ldots, x_{p+q+1})$, are homogeneous expressions which satisfy WR. Hence the Jacobi identity for the bracket $[\cdot,\cdot]^{NR}$ shifts, by the "unique shift", to the same identity but for $[\cdot,\cdot]^L$. This means that the latter is zero and so the Jacobi identity for $[\cdot,\cdot]^L$ is satisfied. ∎

Let $(\mathfrak{g}, \{\cdot,\cdot\})$ be a Leibniz algebra and set $\pi(x,y) = \{x,y\}$. Hence $\pi \in M^1(\mathfrak{g})$, $[\pi,\pi]^L = 0$ and the coboundary operator $\delta_\pi = [\pi,\cdot]^L$ from Theorem 1 is defined. Actually, $\delta_\pi^{p-1} : M^{p-1} \to M^p$ coincides with the coboundary operator d^p in the cohomology complex built for Leibniz algebras (see [2]), in which the space of p-cochains is $CL^p(\mathfrak{g},\mathfrak{g}) := M^{p-1}(\mathfrak{g})$ and the formula for d^p is as follows:

$$\begin{aligned}(d^p\alpha)(x_1,\dots,x_{p+1}) = {} & \sum_{i=1}^{p}\sum_{j=i+1}^{p+1}(-1)^i\alpha(x_1,\dots,\widehat{x_i},\dots,x_{j-1},\{x_i,x_j\},\dots,x_{p+1}) \\ & + \sum_{i=1}^{p}(-1)^{i+1}\{x_i,\alpha(x_1,\dots,\widehat{x_i},\dots,x_{p+1})\} \\ & + (-1)^{p+1}\{\alpha(x_1,\dots,x_p),x_{p+1}\}\,.\end{aligned}$$

3.4. n-Leibniz algebras. First recall the definition of the cohomology complex for n-Leibniz algebra (see [5, 13]).

Let A be an n-Leibniz algebra. Let $\mathfrak{g} := A^{\otimes(n-1)}$ be the Leibniz algebra with the bracket $\{\cdot,\cdot\}$ defined in Example 3. The p-cochain of A ($p \geq 1$) with coefficients in A is a linear map from $\mathfrak{g}^{\otimes(p-1)} \otimes A$ to A. Set also $\Gamma L^0(A,A) := \mathfrak{g}$ for the space of 0-cochains. The space of p-cochains is denoted by $\Gamma L^p(A,A)$. The coboundary map is given by

$$d^p : \Gamma L^p(A,A) \to \Gamma L^{p+1}(A,A),$$
$$(d^0(x_1 \otimes \ldots \otimes x_{n-1}))(x) = -[x_1,\dots,x_{n-1},x],$$

$$\begin{aligned}&(d^p\alpha)(X_1,\dots,X_{p-1},Y) \\ &= \sum_{i=1}^{p-2}\sum_{j=i+1}^{p-1}(-1)^i\,\alpha(X_1,\dots,\widehat{X_i},\dots,X_{j-1},\{X_i,X_j\},X_{j+1},\dots,X_{p-1},Y) \\ &+ \sum_{i=1}^{p-1}(-1)^i\alpha(X_1,\dots,\widehat{X_i},\dots,X_{p-1},\{X_i,Y\}) \\ &+ (-1)^p\alpha(X_1,\dots,X_{p-1},[y_1,\dots,y_n]) \\ &+ \sum_{i=1}^{p-1}(-1)^{i+1}\{X_i,\alpha(X_1,\dots,\widehat{X_i},\dots,X_p)\} \\ &+ (-1)^{p+1}\sum_{i=1}^{n}[y_1,\dots,y_{i-1},\alpha(X_1,\dots,X_{p-1},y_i),\dots,y_n]\,,\end{aligned} \tag{11}$$

where $X_i \in \mathfrak{g}$ for $i = 1, \dots, p-1$, $Y = y_1 \otimes \dots \otimes y_n \in A^{\otimes n}$, and for $X \in \mathfrak{g}$ of the form $X = x_1 \otimes \dots \otimes x_{n-1}$ we set $\{X, Y\} := \sum_{i=1}^{n} [y_1, \dots, [x_1, \dots, x_{n-1}, y_i], \dots, y_n]$.

It turns out that the space of cochains for an n-Leibniz algebra can be embedded into the space of cochains for its associated Leibniz algebra $\mathfrak{g}$. At the same time the coboundary map is preserved. In detail, let $\triangle : \mathrm{Hom}\,(A, A) \to \mathrm{Hom}\,(\mathfrak{g}, \mathfrak{g})$ be given by (see [2, 5])

$$\triangle(f)(y_1 \otimes \dots \otimes y_{n-1}) = \sum_{i=1}^{n-1} y_1 \otimes \dots \otimes y_{i-1} \otimes f(y_i) \otimes \dots \otimes y_{n-1}\,. \tag{12}$$

Then $\triangle$ induces a map $\triangle : \Gamma L^p(A, A) \to CL^p(\mathfrak{g}, \mathfrak{g}) = M^{p-1}(\mathfrak{g})$ given by

$$\begin{aligned} &(\triangle\alpha)(X^1 \otimes \dots \otimes X^p) \\ &\qquad = \sum_{i=1}^{n-1} x_1^p \otimes \dots x_{i-1}^p \otimes \alpha(X^1 \otimes \dots \otimes X^{p-1} \otimes x_i^p) \otimes \dots \otimes x_{n-1}^p\,, \end{aligned} \tag{13}$$

where $\alpha \in \Gamma L^p(A, A)$ and $X^p = x_1^p \otimes \dots \otimes x_{n-1}^p \in \mathfrak{g}$. For $p = 0$ it is assumed that $\triangle(X) = X$ for $X \in \Gamma L^0(A, A) = \mathfrak{g}$. The embedding theorem can be stated in the following form

THEOREM 3. ([2]) *The following diagram is commuting*

$$\begin{array}{ccc} CL^p(\mathfrak{g},\mathfrak{g}) & \xrightarrow{d^p} & CL^{p+1}(\mathfrak{g},\mathfrak{g}) \\ \uparrow{\scriptstyle\triangle} & & \uparrow{\scriptstyle\triangle} \\ \Gamma L^p(A,A) & \xrightarrow{d^p} & \Gamma L^{p+1}(A,A) \end{array}$$

COROLLARY 1. *The map* (11) *induce a coboundary operator, i.e.* $d^p \circ d^{p-1} = 0$.

Now we are going to introduce the structure of a graded Lie algebra on the space of cochains $\Gamma L^*(A, A)$. We only assume A to be a vector space. Set $\mathfrak{g} := A^{\otimes(n-1)}$, $L^p = L^p(A) := \Gamma L^{p+1}(A, A)$ and $L := \bigoplus_{p \geq -1} L^p$. Let $\alpha \in L^p$, $\beta \in L^q$, $p, q \geq 0$. For each subset $J = \{j_1 < \dots < j_{q+1}\} \subset N := \{1, \dots, p+q+1\}$ set $I = \{i_1 < \dots < i_p\} = N \setminus J$. Let $X^i = x_1^i \otimes \dots \otimes x_{n-1}^i \in \mathfrak{g}$ for $i = 1, \dots, p+q$ and let $x \in A$. Set $T := X^1 \otimes \dots \otimes X^{p+q} \otimes x \in \mathfrak{g}^{\otimes(p+q)} \otimes A$. Define $Z_J^{\alpha,\beta}(T) \in L^{p+q}(A)$ in the following way:

1. If $j_{q+1} < p+q+1$ (it implies that $i_p = p+q+1$) then set

$$\begin{aligned} Z_J^{\alpha,\beta} =&(-1)^k \sum_{s=1}^{n} \alpha(X^{i_1} \otimes \ldots \otimes X^{i_k} \otimes \\ &\qquad x_1^{j_{q+1}} \otimes \ldots \otimes x_{s-1}^{j_{q+1}} \otimes \beta(X^{j_1} \otimes \ldots \otimes X^{j_q} \otimes x_s^{j_{q+1}}) \\ &\otimes x_{s+1}^{j_{q+1}} \otimes \ldots \otimes x_{n-1}^{j_{q+1}} \otimes X^{i_{k+1}} \otimes \ldots \otimes X^{i_{p-1}} \otimes x), \end{aligned} \tag{14}$$

where k is chosen in the unique way in which $Z_J^{\alpha,\beta}$ satisfies the Weight Rule, where the order of the free variables in the above formula is the following: $x_1^1 \prec x_2^1 \prec \ldots \prec x_{n-1}^1 \prec x_1^2 \prec \ldots \prec x_{n-1}^q \prec x$. Hence $i_k < j_{q+1}$ and in case $k+1 \leq p$ we have $k+1 < p$.

2. If $j_{q+1} = p+q+1$ then set

$$Z_J = (-1)^p \, \alpha(X^{i_1} \otimes \ldots \otimes X^{i_p} \otimes \beta(X^{j_1} \otimes \ldots \otimes X^{j_q} \otimes x)). \tag{15}$$

Define

$$\alpha \circ_{nL} \beta \, (T) = \sum_J (-1)^{(J,I)} \, Z_J^{\alpha,\beta}(T) \;, \tag{16}$$

where the sum is over all $(q+1)$-shuffles $J \subset N$.

The bilinear bracket $[\cdot,\cdot]^{nL} : L^p(A) \times L^q(A) \to L^{p+q}(A)$ is defined by

$$[\alpha,\beta]^{nL} = (-1)^{pq} \alpha \circ_{nL} \beta - \beta \circ_{nL} \alpha. \tag{17}$$

Note that the expressions which define the bracket $[\cdot,\cdot]^{nL}$ satisfies WR. The fundamental role in the proof of the graded Jacobi identity for $[\cdot,\cdot]^{nL}$ plays the following:

LEMMA 1. *Let $M^p, L^p, A, \mathfrak{g}, \triangle$ be as usual. Then the following diagram commutes*

$$\begin{array}{ccc} M^p(\mathfrak{g}) \otimes M^q(\mathfrak{g}) & \xrightarrow{[\cdot,\cdot]^L} & M^{p+q}(\mathfrak{g}) \\ {\scriptstyle \triangle\otimes\triangle}\big\uparrow & & \big\uparrow{\scriptstyle \triangle} \\ L^p(A) \otimes L^q(A) & \xrightarrow{[\cdot,\cdot]^{nL}} & L^{p+q}(A) \end{array} .$$

Proof. Let $\alpha \in L^p(A)$, $\beta \in L^q(A)$. Compare $(\triangle Z_J^{\alpha,\beta})(X^1, \ldots, X^{p+q+1})$ occurring as a summand in $(\triangle \circ [\alpha,\beta]^{nL})(X^1, \ldots, X^{p+q+1})$ with

$$(\triangle\alpha)(X^{i_1}, \ldots, X^{i_k}, (\triangle\beta)(X^{j_1}, \ldots, X^{j_{q+1}}), X^{i_{k+1}}, \ldots, X^{i_p})$$

which occurs when expanding $(\triangle\alpha)\circ_L(\triangle\beta)$ (see (8), (13)), where $J=\{j_1<\ldots<j_{q+1}\}$ is fixed. Start with the case $j_{q+1}<p+q+1$. It is ease to see that they are identical. Note also that the sign they follow is the same (equal to $(-1)^k(-1)^{(J,I)}$). Next turn to the case $j_{q+1}=p+q+1=:m$. The difference between $(\triangle\alpha)(X^{i_1},\ldots,X^{i_p},(\triangle\beta)(X^{j_1},\ldots,X^{j_{q+1}}))$ and $(\triangle Z_J^{\alpha,\beta})(X^1,\ldots,X^{p+q+1})$ is easily seen to be

$$
\begin{aligned}
&\pm\sum_{\substack{u,v\\u\neq v}} x_1^m\otimes\ldots\otimes x_{u-1}^m\otimes\\
&\qquad\alpha(X^{i_1},X^{i_2},\ldots,X^{i_p},x_u^m)\otimes x_{u+1}^m\otimes\ldots\otimes x_{v-1}^m\otimes\\
&\qquad\beta(X^{j_1},\ldots,X^{j_q},x_v^m)\otimes\ldots\otimes x_{n-1}^m\ ,
\end{aligned}
\tag{18}
$$

where the sign $\pm$ is equal

$$(-1)^p\operatorname{sgn}(j_1,\ldots,j_q,m,i_1,\ldots,i_p)=\operatorname{sgn}(j_1,\ldots,j_q,i_1,\ldots,i_p).$$

Hence $(\triangle\alpha)\circ_L(\triangle\beta)\neq\triangle\circ(\alpha\circ_{nL}\beta)$. The difference between $\triangle\circ(\alpha\circ_{nL}\beta)$ and $(\triangle\alpha)\circ_L(\triangle\beta)$ is just the sum of expressions (18) over all q-shuffles $\{j_1,\ldots,j_q\}\subset\{1,\ldots,p+q\}$. By symmetry, the difference between $\triangle\circ(\beta\circ_{nL}\alpha)$ and $(\triangle\beta)\circ_L(\triangle\alpha)$ is the same but now the sign is $\operatorname{sgn}(i_1,\ldots,i_p,j_1,\ldots,j_q)=(-1)^{pq}\operatorname{sgn}(j_1,\ldots,j_q,i_1,\ldots,i_p)$. This agree with the signs in the definition (17) and ends the proof. ∎

Lemma 2. *The equation* $[\pi,\pi]^{nL}$ *for* $\pi\in L^1(A)=\operatorname{Hom}(A^{\otimes n},A)$ *reads as the fundamental identity* (5) *for the bracket* $[x_1,\ldots,x_n]=\pi(x_1,\ldots,x_n)$.

Proof. We have $[\pi,\pi]^{nL}=-2\,\pi\circ_{nL}\pi$. Let $T=X^1\otimes X^2\otimes x\in\mathfrak{g}\otimes\mathfrak{g}\otimes A$, $X^i=x_1^i\otimes\ldots\otimes x_{n-1}^i\in\mathfrak{g}$ for $i=1,2$. According to the definitions (14) and (15), for $Z=Z^{\pi,\pi}$, we have

$$
\begin{aligned}
Z_{\{1,2\}}(T)&=\sum_{s=1}^{n}\pi(x_1^2,\ldots,x_{s-1}^2,\pi(X^1,x_s^2),x_{s+1}^2,\ldots,x_{n-1}^2,x),\\
Z_{\{1,3\}}(T)&=-\pi(X^2,\pi(X^1,x))=-\pi(x_1^2,\ldots,x_{n-1}^2,\pi(x_1^1,\ldots,x_{n-1}^1,z)),\\
Z_{\{2,3\}}(T)&=-\pi(X^1,\pi(X^2,x)).
\end{aligned}
$$

Moreover,

$$\pi\circ_{nL}\pi=Z_{\{1,2\}}-Z_{\{1,3\}}+Z_{\{2,3\}},$$

so $\pi\circ_{nL}\pi=0$ reads as the fundamental identity for the bracket π. ∎

The direct implication of Theorem 1 and Lemmas 1 and 2 is the following

COROLLARY 2. *n-Leibniz algebras are canonical structures for the bracket $[\cdot,\cdot]^{nL}$. Moreover, $[\cdot,\cdot]^{nL}$ turns the cohomology space $H\Gamma L^*(A,A)$ for an n-Leibniz algebra A into a graded Lie algebra.*

The above lemmas and the corollary adapt easily to the case of n-Lie algebras, where we, like in the classical Chevalley-Eilenberg complex for Lie algebras, impose some antisymmetric condition for cochains. In detail, let A be an n-Lie algebra and set $\mathfrak{g} := \bigwedge^{n-1} A$ for its associated Leibniz algebra with the bracket given by an analog of formula (6). We set the cohomology complex $(\Gamma^p(A,A), d^p)$ as follows:

$$\Gamma^p(A,A) := \mathrm{Hom}\,(\mathfrak{g}^{\otimes p-2} \otimes \bigwedge^{n} A, A) \;\text{ for } p \geq 2,$$
$$\Gamma^1(A,A) := \mathrm{Hom}\,(A,A) \text{ and } \Gamma^0(A,A) := \bigwedge^{n-1} A.$$

Of course, $\Gamma^p(A,A) \subset \Gamma L^p(A,A)$. Moreover, the following is true

LEMMA 3. $[\Gamma^*(A,A), \Gamma^*(A,A)]^{nL} \subseteq \Gamma^*(A,A)$.

Proof. Let $\alpha \in \Gamma^{p+1}(A,A)$ and $\beta \in \Gamma^{q+1}(A,A)$. Then $Z_J^{\alpha,\beta}(T)$, where $T = X^1 \otimes \ldots \otimes X^{p+q} \otimes x$, in the formulas (14) and (15) is antisymmetric with respect to $x_1^i, \ldots, x_{n-1}^i$ for each $i = 1, \ldots, p+q-1$. Moreover, $\alpha \circ_{nL} \beta\,(X^1, \ldots, X^{p+q}, x)$ is also antisymmetric with respect to $x_1^{p+q}, \ldots, x_{n-1}^{p+q}, x$, although $Z_J^{\alpha,\beta}(T)$ alone is not. ∎

The last lemma implies that the space $\Gamma^*(A,A)$ is a graded Lie subalgebra of graded Lie algebra $\Gamma L^*(A,A)$. Hence n-Lie algebras are canonical structures for the bracket being the restriction of $[\cdot,\cdot]^{nL}$ to $\Gamma^*(A,A)$.

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