EXTRACTA MATHEMATICAE Vol. **21**, Núm. 1, 27 – 39 (2006)

On Tauberian and Co-Tauberian Operators

S. Dutta[1] , V.P. Fonf[2]

Department of Mathematics, Ben Gurion University of the Negev
P.O.B. 653, *Beer-Sheva* 84105, *Israel*
e-mail: sudipta@math.bgu.ac.il, fonf@math.bgu.ac.il

(Presented by M. González)

AMS *Subject Class.* (2000): 46B03, 46B20 *Received November* 15, 2005

1. Introduction

A bounded linear operator $T : X \to Y$ is said to be Tauberian if $T^{**}(X^{**} \setminus X) \subseteq Y^{**} \setminus Y$.

A bounded linear operator $T : X \to Y$ is said to be co-Tauberian if T^* is Tauberian.

We call a Tauberian operator non-trivial if it is not an isomorphic embedding. We call a co-Tauberian operator non-trivial if it is not onto.

Tauberian operators appeared in [6] and were studied systematically in [1, 8, 13, 15]. A comprehensive survey on Tauberian operators and the isomorphic properties they preserve is provided in [8].

Recall that a bounded linear operator $T : X \to Y$ is called a semi-embedding (see [14]) if T is one-to-one and the image $T(B_X)$ of the unit ball B_X of X is closed in Y. It is known that to be a semi-embedding is not a hereditary property, that is, if $T : X \to Y$ is a semi-embedding then restricted to each subspace $E \subseteq X$, $T|_E$ need not necessarily be a semi-embedding. This motivated for searching a notion of embedding which is hereditary and in [3], G_δ-embedding was introduced. One could define a notion of "hereditary semi-embedding". However it turned out, as proved in [15, Theorem 2.3], that such a class of operators coincides exactly with one-to-one Tauberian operators.

Note that just the existence of a non-isomorphic semi-embedding $T : X \to Y$ already provides us with some information on X. The following result was obtained in [4, Theorem 2]:

---

[1]Partially supported by the Institute for Advanced Studies in Mathematics at Ben-Gurion University of the Negev.

[2]Partially supported by Israel Science Foundation, Grant No. 139/03.

Theorem A. *Let X be a separable Banach space. The following assertions are equivalent:*

(a) *X contains a subspace isomorphic to an infinite dimensional dual space.*

(b) *There exists a Banach space Z and a semi-embedding of X into Z which is not an isomorphic embedding.*

Our main objective in the Section 2 of this note is to obtain a result parallel to Theorem A with Tauberian and co-Tauberian operators. We show that a Banach space X contains an infinite dimensional reflexive subspace if and only if there exists a Banach space Z and a one-to-one non-trivial Tauberian operator $T : X \to Z$. For co-Tauberian operator we prove that a Banach space has an infinite dimensional reflexive quotient if and only if there exists a Banach space Z and a one-to-one dense range co-Tauberian operator from Z into X.

In Section 3 we consider Banach spaces from which there exists a Tauberian operator to c_0. In Theorem 3.1 we give a necessary and sufficient condition for existence of a Tauberian operator $T : X \to c_0$, when X is separable. We use this result to provide a generalization of a result in [12]. Another application of Theorem 3.1 connected to the set $NA(X^*)$, of all norm attaing functionals on a dual Banach space X^* is the following:

Let X^{**} is separable. Then there exits a renorming of X such that for any subspace $E \subseteq X^{**}$, satisfying $E \cap X = \{0\}$ and $E \subseteq NA(X^*)$, dim $E < \infty$ holds, that is, X is essentially the only infinite dimensional subspace contained in $NA(X^*)$.

The condition for existence of a co-Tauberian operator from c_0 to X is more stringent and we need to consider special classes of Banach spaces.

All Banach spaces in this note are real and infinite-dimensional. Our notations are standard (see [11]). For example the closed unit ball and the unit sphere of a Banach space X will be denoted by B_X and S_X respectively. All subspaces we consider are assumed to be closed.

2. Reflexive subspace and quotient

The following theorem characterizes Banach spaces containing reflexive subspaces. As mentioned in the introduction, this parallels Theorem A with Tauberian operator.

THEOREM 2.1. *Let X be a Banach space. The following assertions are equivalent:*

(a) *X contains a reflexive subspace.*

(b) *There exist a Banach space Z and a non-trivial one-to-one Tauberian operator $T : X \to Z$.*

Proof. $(a) \Rightarrow (b)$: Let $R \subseteq X$ be a reflexive subspace. Without loss of generality we assume that R has a basis $\{x_i\}$ and $\|x_i\| = 1$. Let $Q : X \to X/R$ be the quotient map. Then $\ker Q = R$ is reflexive and Q has closed range. Thus Q is Tauberian (see [13]).

Now let $\{f_i\}$ be a bounded sequence in X^* such that $f_i(x_j) = \delta_{ij}$. Define $K : X \to R$ by

$$K(x) = \sum_{i=1}^{\infty} 2^{-i} f_i(x) x_i.$$

Then K is a compact operator from X to R. We take $Z = X/R \oplus R$ and define the operator $T : X \to Z$ by

$$T(x) = (Q(x) + K(x)).$$

We claim T is the desired one-to-one Tauberian operator.

First note that T is one-to-one. To check T is Tauberian, let us assume $x^{**} \in X^{**}$ be such that $T^{**}x^{**} \in Z$. We need to show $x^{**} \in X$. But $T^{**}x^{**} = Q^{**}x^{**} + K^{**}x^{**}$ and since K is compact, we have $K^{**}x^{**} \in R$. Thus $Q^{**}x^{**} \in X/R$ and since Q is Tauberian we have $x^{**} \in X$.

$(b) \Rightarrow (a)$: This was proved in [13]. We include a proof for completion. Suppose X does not contain a reflexive subspace. Let Y be any Banach space and $T : X \to Y$ one-to-one Tauberian. If T is not an isomorphism then it is well known that there exists a subspace $Z \subseteq X$ such that the restriction $T|_Z$ is a compact operator. But $T|_Z$ is Tauberian as well, hence Z must be reflexive. ∎

Remark 2.2. (a) A proof similar to $(b) \Rightarrow (a)$ in the Theorem 2.1 actually shows more. Namely, if $T : X \to Y$ is a Tauberian operator which is not an isomorphism on each subspace of X of finite codimension then X contains a reflexive subspace.

(b) In the proof of $(a) \Rightarrow (b)$ in the Theorem 2.1 we made a compact perturbation $Q + K$ of the quotient map Q in order to obtain a one-to-one Tauberian operator. In [9, Theorem 1], the following "perturbative" characterization of Tauberian operators is given: A continuous operator $T : X \to Y$

is Tauberian if and only if for every compact operator $K : X \to Y$, the $\ker(T + K)$ is reflexive.

We now proceed to get a characterization of Banach spaces which contain a reflexive quotient by means of co-Tauberian operators.

THEOREM 2.3. *Let X be a Banach space. The following assertions are equivalent:*

(a) *X has a reflexive quotient.*

(b) *There exists a Banach space Z and a non-trivial co-Tauberian operator $T : Z \to X$ such that $T(Z)$ is dense in X.*

Proof. $(a) \Rightarrow (b)$: Let $Y \subseteq X$ be a subspace of X such that X/Y is reflexive. We denote the quotient map from X to X/Y by Q and the inclusion map of Y into X by J. Then J is a co-Tauberian operator. Without loss of generality we assume that X/Y has a basis $\{z_i\}$ and we further assume that $\{z_i\}$ is normalized. Let $\{h_i\}$ be a bounded sequence in $(X/Y)^*$ such that $h_i(z_j) = \delta_{ij}$. We find $\{y_i\} \subseteq B_X$ such that $Q(y_i) = z_i$. Let $K : X/Y \to X$ be defined by

$$K(z) = \sum_{i=1}^{\infty} 2^{-i} h_i(z) y_i.$$

Then K is a compact operator. We now take $T : Y \oplus X/Y \to X$ as

$$T(y, z) = J(y) + K(z).$$

It is easy to check that T has dense range. An argument similar to the proof of $(a) \Rightarrow (b)$ in the Theorem 2.1 shows T^* is a one-to-one Tauberian operator. Hence T is co-Tauberian.

If T is onto, then T^* is an isomorphic embedding of X^* into $Y^* \oplus Y^{\perp}$ and so is $T^*|_{Y^{\perp}}$. But $T^*|_{Y^{\perp}}$ is compact and this contradicts that X/Y is infinite dimensional. Thus T is non-trivial.

$(b) \Rightarrow (a)$: Let Z be a Banach space and $T : Z \to X$ be a co-Tauberian operator such that $T(Z)$ is dense in X. Suppose X does not have a reflexive quotient. By definition $T^* : X^* \to Z^*$ is Tauberian and X^* does not have a reflexive subspace. Then by Theorem 2.1 we have T^* is an isomorphic embedding and hence T has closed range. ∎

Remark 2.4. (a) As in the Tauberian case, one can show more with a proof similar to $(b) \Rightarrow (a)$ of the Theorem 2.3. Namely, let Z be a Banach space and

$T : Z \to X$ be a co-Tauberian operator. If X does not have a reflexive quotient then there exists a finite codimensional subspace of X which is isomorphic to a quotient of Z.

(b) In the proof of $(a) \Rightarrow (b)$ in the Theorem 2.3 we made a compact perturbation $J + K$ of the inclusion map J in order to obtain a dense range co-Tauberian operator. In [9, Theorem] the authors obtained the following "perturbative" characterization of co-Tauberian operators: A continuous operator $T : X \to Y$ is co-Tauberian if and only if for every compact operator $K : X \to Y$, the co-kernel $Y/\overline{(T+K)(X)}$ is reflexive.

3. Tauberian operators into c_0

In this section we give necessary and sufficient condition for the existence of Tauberian operator from a separable Banach space X to c_0. We then provide two applications of our result.

By Remark 2.2, it follows that if there exists a Tauberian operator $T : X \to c_0$, then either X contains a reflexive subspace or X is isomorphic to a subspace of c_0. Similar conclusion holds by replacing c_0 with $C(K)$ spaces, K scattered.

For a $\{g_n\} \subseteq S_{X^*}$ be a w^*-null sequence we define the following subspace of X^{**}:

$$M(\{g_n\}) = \{F \in X^{**} : \lim_n F(g_n) = 0\}.$$

Following is the main result in this section:

Theorem 3.1. *Let X be a separable Banach space. The following assertions are equivalent:*

(a) *There exists a w^*-null sequence $\{g_n\} \subseteq S_{X^*}$ such that $M(\{g_n\})$ is separable.*

(b) *There exists a Tauberian operator $T : X \to c_0$.*

To prove Theorem 3.1 we need the following two lemmas.

Lemma 3.2. *Let X be separable Banach space. Then for each $F \in X^{**} \setminus X$ there is a w^*-null sequence $\{f_i\} \subseteq S_{X^*}$ such that $\lim F(f_i) = d(F, X)$.*

Proof. Let $F \in X^{**} \setminus X$. Denote $q : X^{**} \to X^{**}/X$ a quotient map. Since $(X^{**}/X)^* = X^\perp$ there exists $G \in S_{X^\perp}$ such that $G(F) = ||q(F)|| = d(F, X)$. Next, since $w^* - clS_{X^*} = B_{X^{***}}$, there exists a net $\{g_\alpha\} \subseteq S_{X^*}$

such that $w^* \lim g_\alpha = G$. In particular, $\lim F(g_\alpha) = G(F) = d(F, X)$. Since $G \in X^\perp$, $\lim g_\alpha(x) = 0$ for each $x \in X$. We now find the required sequence $\{f_n\}$.

Let $\{x_n\} \subseteq S_X$ be a dense sequence. Define

$$V_{F,x_1,\cdots x_n} = \{\tau \in X^{***} : \max_{1 \le i \le n} |\tau(x_i)| < 1/n, \ |(\tau - G)(F)| < 1/n\}.$$

Clearly, $V_{F,x_1,\cdots x_n}$ is w^*-neighborhood of G in X^{***} (recall that $G(x_i) = 0, \ i = 1, 2, \cdots n$). We choose $g_{\alpha_n} \in V_{F,x_1,\cdots x_n}$. Then $F(g_{\alpha_n}) \to G(F)$ and $g_{\alpha_n}|_X \to 0$. Finally, put $f_n = g_{\alpha_n}$. ∎

Lemma 3.3. *Let X be a separable Banach space and $Y \subseteq X^{**}$ be a separable subspace of X^{**}. Then there exists a sequence $\{f_n\} \subseteq S_{X^*}$ with $w^* - \lim f_n = 0$ and such that for each $F \in Y$ $\limsup |F(f_n)| = d(F, X)$.*

Proof. Let $\{F_i\}$ be a dense sequence in $S_Y \setminus X$. By using Lemma 3.2, we find, for each i, a sequence $\{f_n^i\} \subseteq S_{X^*}$ with $w^* - \lim_n f_n^i = 0$ and such that $\lim_n F_i(f_n^i) = d(F_i, X)$. Since w^*-topology on B_{X^*} is metrizable, it follows that by throwing out a finite number of f_n^i's for each i, we can get that the set $\{f_n^i\}_{n,i=1}^\infty$ has 0 as only w^*-limit point. We enumerate $\{f_n^i\}_{n,i=1}^\infty$ in a single sequence $\{f_n\}$ and claim that it satisfies our requirement. Fix $F \in S_Y \setminus X$ and find a sequence $\{F_{i_k}\}$ such that $\lim F_{i_k} = F$. Clearly $d(F_{i_k}, X) \to d(F, X), \ k \to \infty$. For each k find an f_{n_k} such that $|F_{i_k}(f_{n_k}) - d(F_{i_k}, X)| < 1/k$. We have,

$$\begin{aligned} |F(f_{n_k}) - d(F, X)| &\le |F(f_{n_k}) - F_{i_k}(f_{n_k})| + |F_{i_k}(f_{n_k}) - d(F_{i_k}, X)| \\ &\quad + |d(F_{i_k}, X) - d(F, X)| \\ &\le ||F_{i_k} - F|| + \frac{1}{k} + |d(F_{i_k}, X) - d(F, X)| \xrightarrow[k \to \infty]{} 0. \end{aligned}$$

Therefore, $\limsup_n |F(f_n)| \ge d(F, X)$. The inverse inequality is clear since each w^*-limit point H of the set $\{f_n\} \subseteq B_{X^{***}}$ belongs to $B_{X^\perp}$ and hence $|H(F)| \le ||q(F)|| = d(F, X)$. ∎

Proof of Theorem 3.1. $(a) \Rightarrow (b)$: Taking $Y = M(\{g_n\})$ in Lemma 3.3 there is a w^*-null sequence $\{f_n\} \subseteq S_{X^*}$ such that for each $F \in M(\{g_n\}) \setminus X$ $\limsup |F(f_n)| = \text{dist}(F, X) > 0$. Put $\{h_n\} = \{f_n\} \cup \{g_n\}$ and define an operator $T : X \to c_0$ as follows

$$Tx = (h_n(x))_{n=1}^\infty \in c_0, \ x \in X.$$

It is easy to verify that T is a Tauberian operator.

$(b) \Rightarrow (a)$: If $T : X \to c_0$ Tauberian then consider $g_n = T^*e_n$ where e_n is the standard vector basis of ℓ_1. It is easy to see that $M(\{g_n\}) = X$.

The proof is complete. ∎

COROLLARY 3.4. *Let X be a Banach space such that X^{**} is separable. Then there exists a non-trivial Tauberian operator $T : X \to c_0$.*

Proof. Existence of a Tauberian operator follows from Theorem 3.1. Since X^{**} is separable X cannot be isomorphic to a subspace of c_0 and the non-triviality follows. ∎

The following corollary generalizes a result by Johnson and Rosenthal in [12, Corollary 4.1].

COROLLARY 3.5. *Let X be a separable Banach space such that for some w^*-null sequence $\{g_n\} \subseteq S_{X^*}$, $M(\{g_n\})$ is separable. Then either X contains a reflexive subspace or X is isomorphic to a subspace of c_0.*

Remark 3.6. The space $X = c_0 \oplus l_2$ shows that the class of spaces which satisfy the condition of Corollary 3.5, is wider then the class of spaces with separable bidual.

THEOREM 3.7. *Let X be a Banach space such that X^* is separable. Assume that X admits a Tauberian operator $T : X \to c_0$ (for instance, X^{**} is separable, see Corollary 3.4). Then there exist an equivalent norm $|||.|||$ on X and a countable set $B \subseteq S_{(X^*,|||.|||)}$ such that for each functional $F \in X^{**} \setminus X$ which attains its norm $|||F|||$, there is $f \in B$ with $F(f) = |||F|||$.*

Proof. Let $\{t_i\}$ be any sequence in S_{X^*} such that $||.|| - \mathrm{cl}\,\mathrm{co}\{\pm t_i\} = B_{X^*}$. Denote $\{e_i\}$ the canonical basis of $l_1 = c_0^*$ and put $g_i = \frac{1}{2}T^*e_i,\ i = 1, 2, \dots$ and

$$B = \pm\bigcup_{i=1}^{\infty}\{t_i \pm g_j\}_{j=i}^{\infty}, \quad V^* = w^* - \mathrm{cl}\,\mathrm{co}B.$$

Let $|||x||| = \max\{f(x) : f \in V^*\}$, $x \in X$. Clearly the norm $|||.|||$ is equivalent to the initial one and $B_{(X,|||.|||)^*} = V^*$.

We show that B satisfies the statement of the proposition.

CLAIM: For each $F \in X^{**} \setminus X$, $|||F||| > ||F||$.

To see this, without loss of generality we may assume that $||F|| = 1$. Put $\gamma = \limsup |F(g_i)|$ and find j so that $F(t_j) > 1 - \gamma/4$. Next there exists $k \geq j$ such that $|F(g_k)| > \frac{3}{4}\gamma$. We then have

$$|F(t_j + \text{sign}F(g_k)g_k)| = |F(t_j) + (\text{sign}F(g_k))F(g_k)| \geq 1 - \frac{\gamma}{4} + \frac{3}{4}\gamma = 1 + \frac{\gamma}{2}$$

which proves that claim.

Assume that $F \in X^{**} \setminus X$ attains its norm $|||F|||$. The dual space X^*, being separable, has the Krein-Milman property and hence, F attains its norm on some extreme point of $B_{(X,\,|||.|||)^*}$. By the Milman theorem

$$\text{ext}B_{(X,\,|||.|||)^*} \subseteq w^* - \text{cl}B = B \cup (w^* - \text{cl}B \setminus B).$$

However, F cannot attain the norm $|||F|||$ on any point of $w^* - \text{cl}B \setminus B$. Indeed, since $\{g_i\}$ is w^*-null, each such point belongs to B_{X^*} and as we proved above $|||F||| > ||F||$. Therefore F attains its norm $|||F|||$ on some point of B which completes the proof. ∎

For a Banach space X the set $NA(X)$ of all norm-attaining functionals on B_X has been studied extensively. In [2] the authors considered the "spaceability" of the set $NA(X)$, that is, whether $NA(X) \cup \{0\}$ contains a linear subspace. The following corollary shows that if X^{**} is separable, then there exists a renorming of X such that X is essentially the only subspace contained in $NA(X^*)$.

Corollary 3.8. *Suppose X^{**} is separable. Then there exists an equivalent norm $|||\cdot|||$ on X such that if $E \subseteq NA(X^*)$ is a closed subspace then dim $E/(E \cap X) < \infty$.*

Proof. Suppose X^{**} is separable. By [12, Corollary 4.1], X^{**} is saturated by reflexive subspaces. Now consider the norm $|||\cdot|||$ constructed in the Theorem 3.7. Let $E \subseteq NA(X^*)$. If dim $E/(E \cap X) = \infty$, then there exists a subspace $Z \subseteq E$, $Z \cap X = \{0\}$. But by Theorem 3.7, there exists a countable set $B \subseteq S(X^*)$ such that for each $F \in Z$ there is $f \in B$ with $F(f) = |||F|||$. Also any $f \in B$ acts naturally as a linear functional on Z. Hence Z has a countable boundary and by [5], Z is saturated by c_0 and cannot contain any reflexive subspace. This contradiction proves the corollary. ∎

4. Co-Tauberian operators from c_0

In this section we consider co-Tauberian operators from c_0 to X. Analogous to the Tauberian case, it follows from Remark 2.4 that if $T : c_0 \to X$ is co-Tauberian then either X has a reflexive quotient or X is isomorphic to a quotient of c_0 and in the later case, it is well known that X is isomorphic to a subspace of c_0.

Recall that a series $\sum x_n$ in X is called weakly unconditionally convergent (wuC for short) if for every $x^* \in X^*$, $\sum |x^*(x_n)|$ is convergent. It is well known that if $\sum x_n$ is wuC then there is an $M > 0$ such that $\| \sum_{j=1}^n \alpha_j x_j \| \leq M \max_{1\leq j\leq n} |\alpha_j|$ for all n and for all scalars α_j. In [7], the following property was considered: Let X be a Banach space. Denote by $\mathcal{A}$ the set of all series $\sum f_n$ in X^* such that $\sum |f_n(x)|$ is convergent for each $x \in X$. Let $\sum^* f_n$ denotes the w^*-limit point in X^* of the series $\sum f_n$. The space X is said to have Property $(\mathcal{X})$ if

$$\{F \in X^{**} : \sum F(f_n) = F(\textstyle\sum^* f_n) \quad \forall\ \sum f_n \in \mathcal{A}\} = X.$$

We need to consider the following weak*-version of Property $(\mathcal{X})$. For a wuC-series $\sum x_n$ we denote by $\sum^* x_n$ the w^*-limit point in X^{**}.

Definition 4.1. A dual Banach space X^* is said to have Property $(\mathcal{X}^*)$ if any $F \in X^{***}$ which satisfies $\sum F(x_n) = F(\sum^* x_n)$ for every wuC-series $\sum x_n$ in X, must be in X^*.

The proof of the following lemma is straightforward.

Lemma 4.2. *Suppose $T : c_0 \to X$ is a co-Tauberian operator. Then X^* has Property $(\mathcal{X}^*)$.*

We now consider a natural class of Banach spaces satisfying Property $(\mathcal{X}^*)$. Recall that a subspace $Y \subseteq X$ is called an L-summand if there exists $E \subseteq X$ such that $X = Y \oplus_1 E$. A Banach space X which is L-summand in X^{**} is called an L-embedded space. X is called M-embedded if $X^\perp$ as a subspace of X^{***} is an L-summand. The book [10] is a standard reference for $M-$ and L-embedded spaces.

If X is a separable M-embedded space, its dual X^* is a separable L-embedded space (see [10]).

Lemma 4.3. *Suppose X is a separable M-embedded space. Then for each $F \in X^{***} \setminus X^*$ there exists a wuC-series $\sum x_n$ in X such that $F(\sum^* x_n) - \sum F(x_n) > \frac{1}{2}\mathrm{dist}(F, X^*)$ and $\| \sum^* x_n\| = 1$.*

Proof. Let $F \in X^{***} \setminus X^*$. Since $X^{***} = X^* \oplus_1 X^\perp$ we can write $F = x^* + \tau$, $x^* \in X^*$, $\tau \in X^\perp$, $\tau \neq 0$. Thus $\operatorname{dist}(F, X^*) = \|\tau\|$. Take $0 < \varepsilon < \frac{1}{4}\|\tau\|$. Then we can find an $x^{**} \in S_{X^{**}}$ such that $\|F\| - \varepsilon < F(x^{**}) = x^{**}(x^*) + \tau(x^{**})$. But $\|F\| = \|x^*\| + \|\tau\|$ and hence $\tau(x^{**}) > \frac{1}{2}\|\tau\|$. Since X is M-embedded, by [10, Theorem I. 2. 10] for each $x^{**} \in X^{**}$ there exists wuC-series $\sum x_n$ in X such that $x^{**} = \sum^* x_n$. Observe that $\sum \tau(x_n) = 0$. ∎

Lemma 4.4. *Let X be a separable M-embedded space and $N \subseteq X^{***}$ be a separable subspace. Then there exists a wuC-series $\sum x_n$ in X such that if $F \in N$, satisfies $\sum a_n F(x_n) = F(\sum^* a_n x_n)$ for all bounded sequence (a_n), then $F \in X^*$.*

Proof. Let (F_n) be a dense sequence in $S_N \setminus X^*$. Since X^* is L-embedded, we can write $X^{***} = X \oplus_1 X^\perp$ and thus each F_n can be decomposed as $F_n = x_n + \tau_n$ with $\|F_n\| = \|x_n^*\| + \tau_n\|$. Then $\operatorname{dist}(F_n, X) = \|\tau_n\|$.

By the Lemma 4.3, for each n there exists a wuC-series $\sum_k x_{nk}$ in X such that $\tau_n(\sum_k^* x_{nk}) > \frac{1}{2}\|\tau_n\| > 0$, $\sum_k \tau_n(x_{nk}) = 0$ and $\|\sum_k^* x_{nk}\| = 1$.

We get $\mathbb{N}_n$ infinite disjoint subsets of $\mathbb{N}$ such that $\mathbb{N} = \cup \mathbb{N}_n$. Ordering appropriately, we assume $\mathbb{N}_n = \{m_i^n\}_{i=1}^\infty$ where $m_1^n < m_2^n < \cdots$. We now take $y_{nk} = 2^{-n} x_{nm_k^n}$, $k \in \mathbb{N}_n$. Then $\sum_{nk} y_{nk}$ is a wuC-series in X.

Let $F \in S_N \setminus X^*$. Choose $0 < \varepsilon < \operatorname{dist}(F, X^*)/10$. Since $\{F_n\}$ is dense in $S_N \setminus X^*$, it follows that there exists an n such that $\|F - F_n\| < \varepsilon$. Take $a_{nk} = 2^n$, if $k \in \mathbb{N}_n$ and $a_{nk} = 0$ otherwise.

We now estimate

$$\begin{aligned} F({\textstyle\sum_{nk}^*} a_{nk} y_{nk}) - {\textstyle\sum_{nk}} a_{nk} F(y_{nk}) &= F({\textstyle\sum_k^*} x_{nk}) - {\textstyle\sum_k} F(x_{nk}) \\ &> F_n({\textstyle\sum_k^*} x_{nk}) - \varepsilon - {\textstyle\sum_k} F_n(x_{nk}) - \varepsilon \\ &\geq \frac{1}{2}\operatorname{dist}(F, X) - 3\varepsilon > 0. \end{aligned}$$

This completes the proof. ∎

Let X be a Banach space and $\sum x_n$ is a wuC-series in X^*. We define the following subspace of X^{***},

$$N(\textstyle\sum x_n) = \{F \in X^{***} : F(\sum^* a_n x_n) = \sum a_n F(x_n) \text{ for all bounded sequence } (a_n)\}.$$

THEOREM 4.5. *Let X be a separable M-embedded space. The following assertions are equivalent:*

(a) *There exists a wuC-series $\sum x_n$ in X such that $N(\sum x_n)$ is separable.*

(b) *There exists a co-Tauberian operator $T : c_0 \to X$.*

Proof. $(a) \Rightarrow (b)$: Taking $N = N(\sum x_n)$ in Lemma 4.4, there exits wuC-series $\sum y_n$ in X such that for each $F \in N \setminus X^*$ there exists a bounded sequence (b_n) with the property $\sum b_n F(y_n) \neq F(\sum^* b_n y_n)$. Take $z_n = y_n$ for n odd and $z_n = x_n$ for n even.

We now define the required co-Tauberian operator $T : c_0 \to X$. Let $\{u_n\}$ be the standard unit vector basis of c_0. We first take $Tu_n = z_n$. Now for any $u \in c_0$, there exists $\{\alpha_n\}$ scalars and $\alpha_n \to 0$ such that $u = \sum \alpha_n u_n$. Extend T to whole of c_0 by taking $Tu = \sum \alpha_n z_n$. Note that $\sum z_n$ is wuC-series in X, and therefore, T is well-defined.

We now verify that T is co-Tauberian, that is T^* is Tauberian. Suppose on the contrary, there exists $F \in X^{***} \setminus X^*$ such that $T^{***}F \in \ell_1$. By Lemma 4.4, we fix a bounded sequence $\{b_n\}$ such that $\sum b_n F(y_n) \neq F(\sum^* b_n y_n)$. Since $T^{***}F \in \ell_1$, we have $(T^{***}F)(\sum^* a_n u_n) = \sum a_n F(T^{**}(u_n))$ for all bounded sequences (a_n). This implies $F(\sum^* a_n(z_n) = \sum a_n F(z_n)$. Taking $a_n = 0$ for n odd we observe that $F \in N$. But taking $a_n = b_n$ for n odd and 0 for n even we get the contradiction to the choice of $\sum y_n$. Thus T^* is Tauberian.

$(b) \Rightarrow (a)$: If $T : c_0 \to X$ is co-Tauberian take $x_n = Tu_n$. It is easy to see $N(\sum x_n) = X^*$ and X being an M-embedded space, X^* is separable. ∎

Let X be a separable L-embedded space. In a recent work [16], H. Pfitzner has shown that X has Property $(\mathcal{X})$. Moreover, from his proof it follows:

LEMMA 4.6. *Let X be a separable L-embedded space and $X^{**} = X \oplus_1 E$. Let $F \in X^{**}$. If $F = x + \tau,\ x \in X, \tau \in E$, there exists $\sum f_n$ in X^*, satisfying $\sum |f_n(x)| < \infty$ for each $x \in X$, such that $F(\sum^* f_n) - \sum F(f_n) = \operatorname{dist}(F, X) = \|\tau\|$ and $\|\sum^* f_n\| = 1$.*

A slight modification of the proof of Lemma 4.4 now gives,

LEMMA 4.7. *Let X be a separable L-embedded space and $N \subseteq X^{**}$ be a separable subspace. Then there exists a series $\sum f_n$ in X^* such that if $F \in N$, satisfies $\sum a_n F(f_n) = F(\sum^* a_n f_n)$ for all bounded sequence (a_n), then $F \in X$.*

Following the same line of proof as in Theorem 4.5, we have, THEOREM 4.8. *Let X be a separable L-embedded space. The following assertions are equivalent:*

(a) *There exists a series $\sum f_n$ in X^* satisfying $\sum |f_n(x)| < \infty$ for each $x \in X$, such that the following subspace of X^{**} is separable:*

$$N(\sum f_n) = \{F \in X^{**} : \ F(\sum\nolimits^* a_n f_n) = \sum a_n F(f_n) \text{ for all bounded sequence } (a_n)\}.$$

(b) *There exists a Tauberian operator $T : X \to \ell_1$.*

Remark 4.9. The notion of M-embedded Banach space is an isometric property. However, it is clear that Theorem 4.5 holds true with the assumption that X is isomorphic to a separable M-embedded space.

Acknowledgements

We would like to thank the referee for his valuable suggestions which improved the paper.

References

[1] Alvarez, T., Gonzalez, M., Some examples of Tauberian operators, *Proc. Amer. Math. Soc.,* **111**(4) (1991), 1023 – 1027.

[2] Bandyopadhyay, P., Godefroy, G., Linear structure in the set of norm-attaining functionals on a Banach space, Preprint 2005.

[3] Bourgain, J., Rosenthal, H.P., Application of the Theory of semi-embeddings to Banach space theory, *J. Func. Anal.,* **52** (1983), 149 – 188.

[4] Fonf, V.P., Semi-embeddings and G_δ-embeddings of Banach spaces, *Mat. Zametki,* **39**(4) (1986), 550 – 561; translation in *Math. Notes,* **39**(3-4) (1986), 302 – 307.

[5] Fonf, V.P., A property of Lindenstrauss-Phelps spaces, *Funktsional. Anal. i Prilozhen.,* **13**(1) (1979), 79 – 80; translation in *Functional Anal. Appl.,* **13**(1) (1979), 66 – 67.

[6] Garling, D.J.H., Wilansky, A., On a summability theorem of Berg, Crawford and Whitley, *Proc. Cambridge Philos. Soc.,* **71** (1972), 495 – 497.

[7] Godefroy, G., Talagrand, M., Nouvelles classes d'espaces de Banach à predual unique, *Séminaire d'Analyse Fonctionelle de l'Ecole Polytechnique*, (1980-81), Exp. No. VI, 29 pp., École Polytech., Palaiseau, 1981.

[8] González, M., Properties and application of Tauberian operators, *Extracta Math.,* **5**(3) (1990), 91 – 107.

[9] González, M., Onieva, V.M., Charcateriztions of tauberian operators and other semigroups of operators, *Proc. Amer. Math. Soc.,* **108** (1990), 399 – 405.

[10] Harmand, H., Werner, D., Werner, W., "M-Ideals in Banach Spaces and Banach Algebras", Lecture Notes in Mathematics 1547, Springer-Verlag, Berlin, 1993.

[11] Johnson, W.B., Lindenstrauss, J., "Basic Concepts in the Geometry of Banach Spaces", in Handbook of the Geometry of Banach Spaces, Vol. I, 1 – 84, North-Holland, Amsterdam, 2001.

[12] Johnson, W.B., Rosenthal, H.P., On weak*-basic sequence and their applications to the study of Banach space, *Studia Math.,* **43** (1972), 77 – 92.

[13] Kalton, N., Wilansky, A., Tauberian operators on Banach spaces, *Proc. Amer. Math. Soc.,* **57**(2) (1976), 251 – 255.

[14] Lotz, H.P., Peck, N.T., Porta, H., Semi-embeddings of Banach spaces, *Proc. Edinburg Math. Soc.,* **22** (1979), 233 – 240.

[15] Neidinger, R., Rosenthal, H.P., Norm-attainment of linear functionals on subspaces and characterization of Tauberian operators, *Pacific J. Math.,* **118**(1) (1985), 215 – 228.

[16] Pfitzner, H., A separable L-embedded Banach space has property (X) and is therefore the unique predual of its dual, Preprint.

[17] Rosenthal, H.P., On wide-(s) sequences and their applications to certain classes of operators, *Pacific J. Math.,* **189**(2) (1999), 311 – 338.