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Essential Norm of Weighted Composition Operators on the Dirichlet Space

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1. Introduction

Let $H(\mathbf{D})$ denote the space of holomorphic functions on the unit disc $\mathbf{D}$. Suppose φ and ψ are holomorphic functions defined on $\mathbf{D}$ such that $\varphi(\mathbf{D}) \subseteq \mathbf{D}$. The weighted composition operator $W_{\varphi,\psi}$ is defined as follows:

$$W_{\varphi,\psi}(f)(z) = \psi(z) f(\varphi(z)) \,,$$

for all f holomorphic in $\mathbf{D}$.

If we take $\varphi \equiv I$, the identity function, or $\psi \equiv 1$, then we get the multiplication operator M_ψ and the composition operator C_φ, respectively.

Weighted composition operators are a general class of operators and appear naturally in the study of isometries on most of the function spaces, see [5]. Operators of this kind also appear in many branches of analysis; the theory of dynamical systems, semi–groups, the theory of operator algebras, the theory of solubility of equations with deviating argument and so on.

In this paper we plan to study the boundedness and compactness of weighted composition operators in the Dirichlet space. We also find the essential norm estimates for these operators. The essential norm of weighted composition operators was recently studied in [2], [4] and [8].

Fix any $a \in \mathbf{D}$ and let $\sigma_a(z)$ be the Mobius transform defined by

$$\sigma_a(z) = \frac{a - z}{1 - \overline{a} z} \,, \qquad z \in \mathbf{D} \,.$$

Further, we have

$$|\sigma_a'(z)| = \frac{1 - |a|^2}{|1 - \overline{a} z|^2}$$

and

$$1-|\sigma_a(z)|^2=\frac{(1-|a|^2)(1-|z|^2)}{|1-\overline{a}z|^2}=(1-|z|^2)|\sigma_a'(z)| \tag{1.1}$$

for all $a,z\in\mathbf{D}$.

The Dirichlet space $\mathcal{D}^2$ is the set of functions holomorphic in $\mathbf{D}$ having derivatives belonging to the Bergman space $\mathcal{A}^2$. We define the following norm on $\mathcal{D}^2$:

$$\|f\|_{\mathcal{D}^2}=\left(|f(0)|^2+\int_{\mathbf{D}}|f'(z)|^2\,\mathrm{d}A(z)\right)^{\frac{1}{2}},$$

where $\mathrm{d}A(z)$ is the Lebesgue area measure on $\mathbf{D}$. Then $\mathcal{D}^2$ is a reproducing kernel Hilbert space and the reproducing kernel function in this case is given by $K_\omega(z)=1+\log\left(\frac{1}{1-\overline{\omega}z}\right)$.

2. Essential norm

In this section we study the essential norm of weighted composition operators.

Definition 1. Take $p\geq 1$. A positive measure μ on $\mathbf{D}$ is called a bounded p–Carleson measure on $\mathbf{D}$ if

$$\sup_{I\subset\partial\mathbf{D}}\frac{\mu(S(I))}{|I|^p}<\infty\,, \tag{2.1}$$

where $S(I)=\left\{z\in\mathbf{D}\,:\,1-|I|\leq|z|\leq 1,\ \frac{z}{|z|}\in I\right\}$, where I runs through arcs on the unit circle.

Definition 2. Take $p\geq 1$. A positive measure μ on $\mathbf{D}$ is called a vanishing p–Carleson measure on $\mathbf{D}$ if

$$\lim_{I\subset\partial\mathbf{D},\,|I|\to 0}\frac{\mu(S(I))}{|I|^p}=0\,. \tag{2.2}$$

Suppose $\varphi,\psi\in\mathcal{D}^2$ be such $\varphi(\mathbf{D})\subseteq\mathbf{D}$ and $\psi\varphi'\in\mathcal{A}^2$. Define the measures $\mu_{\varphi,\psi}$ and $\nu_{\varphi,\psi}$ on $\mathbf{D}$ by

$$\mu_{\varphi,\psi}(E)=\int_{\varphi^{-1}(E)}|\psi\varphi'|^2\,\mathrm{d}A \qquad\text{and}\qquad \nu_{\varphi,\psi}(E)=\int_{\varphi^{-1}(E)}|\psi'|^2\,\mathrm{d}A\,,$$

where E is a measurable subset of the unit disk $\mathbf{D}$.

Again for $\psi \in \mathcal{A}^2$, we define the measure $\nu_{\varphi,\psi,2}$ on $\mathbf{D}$ by

$$\nu_{\varphi,\psi,2}(E) = \int_{\varphi^{-1}(E)} |\psi|^2 \, \mathrm{d}A \,.$$

Using [6, page-163], we can easily prove the following lemma.

LEMMA 2.1. *Let φ be a holomorphic mapping defined on $\mathbf{D}$ such that $\varphi(\mathbf{D}) \subseteq \mathbf{D}$. Take $\psi \in \mathcal{D}^2$ such that $\psi\varphi' \in \mathcal{A}^2$. Then*

$$\int_{\mathbf{D}} g \, \mathrm{d}\mu_{\varphi,\psi} = \int_{\mathbf{D}} |\psi\varphi'|^2 (g \circ \varphi) \, \mathrm{d}A \qquad \text{and} \qquad \int_{\mathbf{D}} g \, \mathrm{d}\nu_{\varphi,\psi} = \int_{\mathbf{D}} |\psi'|^2 (g \circ \varphi) \, \mathrm{d}A \,,$$

where g is an arbitrary measurable positive function on $\mathbf{D}$.

The following characterizations of Carleson measure was given in [1].

THEOREM 2.2. *Let μ be a positive measure on $\mathbf{D}$. Then the following three statements are equivalent:*

(1) *The inclusion map $i : \mathcal{A}^2 \to L^2(\mathbf{D}, \mathrm{d}\mu)$ is bounded.*

(2) *The measure μ is bounded 2–Carleson measure, i.e., there exists a constant $K < \infty$ such that*

$$\mu(S(I)) \le K I^2 \,.$$

(3) *There exists a constant $C < \infty$ such that*

$$\int_{\mathbf{D}} |\sigma_a'(z)|^2 \, \mathrm{d}\mu(z) \le C$$

for all $a \in \mathbf{D}$.

THEOREM 2.3. *Let μ be a positive measure on $\mathbf{D}$. Then the following three statements are equivalent:*

(1) *The inclusion map $i : \mathcal{A}^2 \to L^2(\mathbf{D}, \mathrm{d}\mu)$ is compact.*

(2) *The measure μ is vanishing 2–Carleson measure, i.e.,*

$$\lim_{I \subset \partial\mathbf{D},\, |I| \to 0} \frac{\mu(S(I))}{|I|^2} = 0 \,.$$

(3) *For all $a \in \mathbf{D}$, we have*

$$\lim_{|a|\to 1} \int_{\mathbf{D}} |\sigma_a'(z)|^2 \, \mathrm{d}\mu(z) = 0 \, .$$

The following lemma is easy to establish.

Lemma 2.4. ([9, Lemma 3.8]) *Given $1 \leq p, q \leq 2$, let φ be a holomorphic mapping defined on $\mathbf{D}$ with $\varphi(\mathbf{D}) \subseteq \mathbf{D}$ and $\psi \in \mathcal{D}^2$ satisfy that $W_{\varphi,\psi} : \mathcal{D}^2 \to \mathcal{D}^2$ is bounded. Then $W_{\varphi,\psi} : \mathcal{D}^2 \to \mathcal{D}^2$ is compact (resp. weakly compact) if and only if whenever $\{f_n\}$ is a bounded sequence in $\mathcal{D}^2$ converging to zero uniformly on compact subsets of $\mathbf{D}$, then $\|W_{\varphi,\psi}(f_n)\|_{\mathcal{D}^2} \to 0$ (respectively, $\{W_{\varphi,\psi}(f_n)\}$ is a weak null sequence in $\mathcal{D}^2$).*

Theorem 2.5. *Let $\varphi \in \mathcal{D}^2$ satisfy that $\varphi(\mathbf{D}) \subseteq \mathbf{D}$ and $\psi \in \mathcal{A}^2$. Suppose that the induced measure $\nu_{\varphi,\psi,2}$ is vanishing 2–Carleson measure. Then $W_{\varphi,\psi}$ defines a bounded operator from $\mathcal{D}^2$ into $\mathcal{A}^2$. Moreover, the operator $W_{\varphi,\psi}$: $\mathcal{D}^2 \to \mathcal{A}^2$ is compact.*

Proof. We prove compactness only. Let $\{f_n\}$ be a bounded sequence in $\mathcal{D}^2$ such that $f_n \to 0$ uniformly on compact subsets of $\mathbf{D}$. Also suppose that $f_n(0) = 0$. Since the measure $\nu_{\varphi,\psi,2}$ is a vanishing 2–Carleson measure, we have $\|f_n\|_{L^2(\mathbf{D},\nu_{\varphi,\psi,2})} \to 0$ as $n \to \infty$. Therefore, by using Theorem 2.1, we have

$$\begin{aligned}
\|W_{\varphi,\psi}(f_n)\|_{\mathcal{A}^2} &= \int_{\mathbf{D}} |\psi|^2 |f_n \circ \varphi|^2 \, \mathrm{d}A \\
&= \int_{\mathbf{D}} |f_n|^2 \, \mathrm{d}\nu_{\varphi,\psi,2} = \|f_n\|^2_{L^2(\mathbf{D},\nu_{\varphi,\psi,2})} \to 0 \quad \text{as } n \to \infty \, .
\end{aligned}$$

Thus, $W_{\varphi,\psi} : \mathcal{D}^2 \to \mathcal{A}^2$ is compact. ∎

Theorem 2.6. *Suppose $\varphi, \psi \in \mathcal{D}^2$ be such that $\varphi(\mathbf{D}) \subseteq \mathbf{D}$. Again, suppose that the induced measure $\nu_{\varphi,\psi}$ is a vanishing 2–Carleson measure. Then $W_{\varphi,\psi}$ exists as a bounded operator from $\mathcal{D}^2$ into $\mathcal{D}^2$ if and only if $W_{\varphi,\psi\varphi'}$ exists as a bounded operator from $\mathcal{A}^2$ into $\mathcal{A}^2$. Moreover, $W_{\varphi,\psi} : \mathcal{D}^2 \to \mathcal{D}^2$ is compact if and only if $W_{\varphi,\psi\varphi'} : \mathcal{A}^2 \to \mathcal{A}^2$ is compact.*

Proof. We prove compactness only. First, supppose $W_{\varphi,\psi} : \mathcal{D}^2 \to \mathcal{D}^2$ is compact. Let $\{f_n\}$ be a bounded sequence in $\mathcal{A}^2$ such that $f_n \to 0$ uniformly on compact subsets of $\mathbf{D}$. For each n, let us consider the function $g_n \in \mathcal{D}^2$ such that $g_n' = f_n$ and $g_n(0) = 0$. The sequence $\{g_n\}$ also converges to zero uniformly on compact subsets of $\mathbf{D}$ as $n \to \infty$. Further, $W_{\varphi,\psi} : \mathcal{D}^2 \to \mathcal{D}^2$ is compact, so $\|W_{\varphi,\psi}(g_n)\|_{\mathcal{D}^2} \to 0$ as $n \to \infty$. Again, by Theorem 2.5, $W_{\varphi,\psi'} : \mathcal{D}^2 \to \mathcal{A}^2$ is compact, so $\|W_{\varphi,\psi'}(g_n)\|_{\mathcal{A}^2}$ also converges to zero as $n \to \infty$. Also, we have

$$\begin{aligned}
\|W_{\varphi,\psi\varphi'}(f_n)\|_{\mathcal{A}^2} &= \|\psi\varphi' f_n \circ \varphi\|_{\mathcal{A}^2} \\
&\le \|\psi\varphi' f_n \circ \varphi + \psi' g_n \circ \varphi\|_{\mathcal{A}^2} + \|\psi' g_n \circ \varphi\|_{\mathcal{A}^2} \\
&= \|(\psi g_n \circ \varphi)'\|_{\mathcal{A}^2} + \|W_{\varphi,\psi'}(g_n)\|_{\mathcal{A}^2} \\
&\le \|W_{\varphi,\psi}(g_n)\|_{\mathcal{D}^2} + \|W_{\varphi,\psi'}(g_n)\|_{\mathcal{A}^2} \ \to 0 \quad \text{as } n \to \infty\,.
\end{aligned}$$

Thus, $W_{\varphi,\psi\varphi'} : \mathcal{A}^2 \to \mathcal{A}^2$ is compact.

Conversely, suppose $W_{\varphi,\psi\varphi'} : \mathcal{A}^2 \to \mathcal{A}^2$ is compact. Again, by Theorem 2.5, $W_{\varphi,\psi'} : \mathcal{D}^2 \to \mathcal{A}^2$ is compact. Let g_n be defined as above. Then, we have

$$\begin{aligned}
\|W_{\varphi,\psi}(g_n)\|_{\mathcal{D}^2} &= \|(\psi g_n \circ \varphi)'\|_{\mathcal{A}^2} \\
&= \|\psi\varphi' g_n' \circ \varphi + \psi' g_n \circ \varphi\|_{\mathcal{A}^2} \\
&\le \|W_{\varphi,\psi\varphi'}(f_n)\|_{\mathcal{A}^2} + \|W_{\varphi,\psi'}(g_n)\|_{\mathcal{A}^2} \ \to 0 \quad \text{as } n \to \infty\,.
\end{aligned}$$

Thus, $W_{\varphi,\psi} : \mathcal{D}^2 \to \mathcal{D}^2$ is compact. ∎

By using Theorem 2.2, Theorem 2.5 and Theorem 2.6, we can prove the following result.

THEOREM 2.7. *Let $\varphi, \psi \in \mathcal{D}^2$ be such that $\varphi(\mathbf{D}) \subseteq \mathbf{D}$. Suppose that the induced measure $\nu_{\varphi,\psi}$ is a bounded 2–Carleson measure. Then the weighted composition operator $W_{\varphi,\psi}$ is bounded on $\mathcal{D}^2$ if and only if the functions Φ and Ψ, defined by*

$$\Phi(z) = \int_{\mathbf{D}} \frac{(1-|z|^2)^2}{|1-\overline{z}\omega|^4}\, \mathrm{d}\mu_{\varphi,\psi}(\omega) \qquad \text{and} \qquad \Psi(z) = \int_{\mathbf{D}} \frac{(|1-|z|^2)^2}{|1-\overline{z}\omega|^4}\, \mathrm{d}\nu_{\varphi,\psi}(\omega)\,,$$

belong to $L^\infty(\mathbf{D})$.

COROLLARY 2.8. *Let $\varphi, \psi \in \mathcal{D}^2$ be such that $\varphi(\mathbf{D}) \subseteq \mathbf{D}$. Suppose that the induced measure $\nu_{\varphi,\psi}$ is a bounded 2–Carleson measure. Suppose the weighted composition operator $W_{\varphi,\psi}$ is bounded on $\mathcal{D}^2$. Then*

$$\sup_{z\in\mathbf{D}} \frac{(1-|z|^2)|\psi\varphi'(z)|}{1-|\varphi(z)|^2} < \infty \qquad \text{and} \qquad \sup_{z\in\mathbf{D}} \frac{(1-|z|^2)|\psi'(z)|}{1-|\varphi(z)|^2} < \infty\,.$$

Proof. First, suppose $W_{\varphi,\psi}$ is bounded on $\mathcal{D}^2$. Then by Theorem 2.7, we have

$$\sup_{a\in\mathbf{D}}\int_{\mathbf{D}}\frac{(1-|a|^2)^2|\psi\varphi'(z)|^2}{|1-\overline{a}\varphi(z)|^4}\,\mathrm{d}A(z)<\infty \tag{2.3}$$

and

$$\sup_{a\in\mathbf{D}}\int_{\mathbf{D}}\frac{(|1-|a|^2)^2|\psi'(z)|^2}{|1-\overline{a}\varphi(z)|^4}\,\mathrm{d}A(z)<\infty\,. \tag{2.4}$$

Condition (2.3) can be written as

$$\sup_{\alpha\in\mathbf{D}}\int_{\mathbf{D}}\frac{(1-|\varphi(\alpha)|^2)^2|\psi\varphi'(z)|^2}{|1-\overline{\varphi(\alpha)}\varphi(z)|^4}\,\mathrm{d}A(z)<\infty\,. \tag{2.5}$$

Now, fix some $\alpha\in\mathbf{D}$, and take $G_\alpha=\Big\{z\in\mathbf{D}\,:\,|z-\alpha|\le\frac{(1-|\alpha|^2)}{2}\Big\}$. Then $G_\alpha\subset\mathbf{D}$. So, by subharmonicity of the function $\frac{|\psi\varphi'(z)|^2}{|1-\overline{\varphi(\alpha)}\varphi(z)|^4}$, we have

$$\begin{aligned}\frac{4}{(1-|\alpha|^2)^2}&\int_{G_\alpha}\frac{(1-|\varphi(\alpha)|^2)^2|\psi\varphi'(z)|^2}{|1-\overline{\varphi(\alpha)}\varphi(z)|^4}\,\mathrm{d}A(z)\\&\ge\frac{(1-|\varphi(\alpha)|^2)^2|\psi\varphi'(\alpha)|^2}{(1-|\varphi(\alpha)|^2)^4}=\frac{|\psi\varphi'(\alpha)|^2}{(1-|\varphi(\alpha)|^2)^2}\,.\end{aligned}$$

Thus, we have

$$\frac{(1-|\alpha|^2)^2|\psi\varphi'(\alpha)|^2}{(1-|\varphi(\alpha)|^2)^2}\le 4\int_{G_\alpha}\frac{(1-|\varphi(\alpha)|^2)^2|\psi\varphi'(z)|^2}{|1-\overline{\varphi(\alpha)}\varphi(z)|^4}\,\mathrm{d}A(z)\,. \tag{2.6}$$

Similarly, we have

$$\frac{(1-|\alpha|^2)^2|\psi'(\alpha)|^2}{(1-|\varphi(\alpha)|^2)^2}\le 4\int_{G_\alpha}\frac{(1-|\varphi(\alpha)|^2)^2|\psi'(z)|^2}{|1-\overline{\varphi(\alpha)}\varphi(z)|^4}\,\mathrm{d}A(z)\,. \tag{2.7}$$

Thus, the result follows from (2.6) and (2.7). ∎

The following two lemmas are proved in [4].

LEMMA 2.9. *Take* $0<r<1$ *and denote* $\mathbf{D}_r=\{z\in\mathbf{D}\,:\,|z|<r\}$. *Let* μ *be a positive Borel measure on* $\mathbf{D}$. *Take*

$$\|\mu\|_r=\sup_{|I|\le 1-r}\frac{\mu(S(I))}{|I|^2}\qquad\text{and}\qquad\|\mu\|=\sup_{I\subset\partial\mathbf{D}}\frac{\mu(S(I))}{|I|^2}\,,$$

where I *runs through all arcs on the unit circle. Let* μ_r *denote the restriction of the measure* μ *to the set* $\mathbf{D}\setminus\mathbf{D}_r$. *Then, if* μ *is a Carleson measure for the Dirichlet space, so is* μ_r *and* $\|\mu_r\|\le 4\|\mu\|_r$.

LEMMA 2.10. *For* $0 < r < 1$, *let*

$$\|\mu\|_r^* = \sup_{|a|\geq r} \int_{\mathbf{D}} |\sigma_a'(z)|^2 \, \mathrm{d}\mu(z)\,.$$

If μ *is a Carleson measure for the Dirichlet space, then* $\|\mu_r\| \leq K\|\mu\|_r^*$, *where* K *is an absolute constant.*

Take $f(z) = \sum_{s=0}^{\infty} a_s z^s$ holomorphic on $\mathbf{D}$. For a positive integer n, define the operators $R_n f(z) = \sum_{s=n+1}^{\infty} a_s z^s$ and $Q_n = I - R_n$, where I is the identity map.

Recall that the essential norm of an operator T is defined as:

$$\|T\|_e = \inf\big\{\|T - K\| \;:\; \text{where } K \text{ is compact operator}\big\}.$$

Now we have the following lemma.

LEMMA 2.11. *Suppose* $W_{\varphi,\psi}$ *is bounded on* $\mathcal{D}^2$. *Then*

$$\|W_{\varphi,\psi}\|_e = \lim_{n\to\infty} \|W_{\varphi,\psi} R_n\|_{\mathcal{D}^2}\,.$$

The proof is similar to the proof of Lemma given in [3, Lemma 3.16].

In the following theorem we give the upper and lower estimates for the essential norm of weighted composition operator.

THEOREM 2.12. *Let* $\varphi, \psi \in \mathcal{D}^2$ *be such that* $\varphi(\mathbf{D}) \subseteq \mathbf{D}$. *Suppose that the induced measure* $\nu_{\varphi,\psi}$ *is a bounded* 2*–Carleson measure. Again, suppose* $W_{\varphi,\psi}$ *is bounded on* $\mathcal{D}^2$. *Then there are absolute constants* $M_1, M_2 \geq 1$ *such that*

$$\limsup_{|a|\to 1} \|(W_{\varphi,\psi})\sigma_a\|_{\mathcal{D}^2}^2 \;\leq\; \|W_{\varphi,\psi}\|_e^2 \;\leq\; M_1 \limsup_{|a|\to 1} \Phi(a) \;+\; M_2 \limsup_{|a|\to 1} \Psi(a)\,,$$

where the functions $\Phi(a)$ *and* $\Psi(a)$ *are defined in Theorem* 2.7.

Proof. First we prove the upper estimate. By Lemma 2.11, we have

$$\|W_{\varphi,\psi}\|_e^2 \;=\; \lim_{n\to\infty} \|W_{\varphi,\psi} R_n\|_{\mathcal{D}^2}^2 \;=\; \lim_{n\to\infty} \sup_{\|f\|_{\mathcal{D}^2}\leq 1} \|(W_{\varphi,\psi} R_n) f\|_{\mathcal{D}^2}^2\,.$$

Also, we have

$$\|(W_{\varphi,\psi} R_n) f\|_{\mathcal{D}^2}^2 = |\psi(0)(R_n f(\varphi(0)))|^2 + \int_{\mathbf{D}} |(\psi(z)(R_n f(\varphi(z)))'|^2 \, \mathrm{d}A(z)\,.$$

The term $|\psi(0)(R_nf(\varphi(0)))|$ is bounded as $n \to \infty$. So, by using Lemma 2.1, we have

$$\begin{aligned}
\|(W_{\varphi,\psi}R_n)f\|_{\mathcal{D}^2}^2 &\leq \int_{\mathbf{D}} |\psi\varphi'(z)|^2|(R_nf)'(\varphi(z))|^2\,\mathrm{d}A(z) \\
&\qquad + \int_{\mathbf{D}} |\psi'(z)|^2|(R_nf)(\varphi(z))|^2\,\mathrm{d}A(z) \\
&= \int_{\mathbf{D}} |(R_nf)'(\omega)|^2\,\mathrm{d}\mu_{\varphi,\psi}(\omega) \\
&\qquad + \int_{\mathbf{D}} |(R_nf)(\omega)|^2\,\mathrm{d}\nu_{\varphi,\psi}(\omega) \;=\; I_1 \;+\; I_2 \;<\; \infty\,, \qquad (2.8)
\end{aligned}$$

where the last condition follows from Theorem 2.5 and Theorem 2.6. Now, for the integral I_1, we have

$$\begin{aligned}
\int_{\mathbf{D}} |(R_nf)'(\omega)|^2\,\mathrm{d}\mu_{\varphi,\psi}(\omega) &= \int_{\mathbf{D}\setminus\mathbf{D}_r} |(R_nf)'(\omega)|^2\,\mathrm{d}\mu_{\varphi,\psi}(\omega) \\
&\qquad + \int_{\mathbf{D}_r} |(R_nf)'(\omega)|^2\,\mathrm{d}\mu_{\varphi,\psi}(\omega)\,.
\end{aligned}$$

Also, the measure $\mu_{\varphi,\psi}$ is a bounded 2–Carleson measure, because the operator $W_{\varphi,\psi}$ is bounded on $\mathcal{D}^2$. Let $k_\omega = 1 + \log(\frac{1}{1-\overline{\omega}z})$ be the kernel for evaluation of ω. Then, by using [3, Proposition 3.15, page 133] and [7], we can show that, for a fixed r,

$$\sup_{\|f\|_{\mathcal{D}^2}\leq 1} \int_{\mathbf{D}_r} |(R_nf)'(\omega)|^2\,\mathrm{d}\mu_{\varphi,\psi}(\omega) \;\to 0 \quad \text{as } n \to \infty\,.$$

Let $\mu_{\varphi,\psi,r}$ denotes the restriction of measure $\mu_{\varphi,\psi}$ to the set $\mathbf{D}\setminus\mathbf{D}_r$. So by using Lemma 2.10 and Theorem 2.2, we have

$$\begin{aligned}
\int_{\mathbf{D}\setminus\mathbf{D}_r} |(R_nf)'(\omega)|^2\,\mathrm{d}\mu_{\varphi,\psi,r}(\omega) &\leq K\|\mu_{\varphi,\psi,r}\|\|(R_nf)'\|_{\mathcal{A}^2}^2 \\
&\leq KM\|\mu_{\varphi,\psi}\|_r^*\|f'\|_{\mathcal{A}^2}^2 \leq KM\|\mu_{\varphi,\psi}\|_r^*\,,
\end{aligned}$$

where K and M are absolute constants and $\|\mu_{\varphi,\psi}\|_r^*$ is defined as in Lemma 2.10.

Following similar techniques to the above ones, we can show that the integral I_2 is also bounded by $K_1\,M_1\,\|\nu_{\varphi,\psi}\|_r^*$, where K_1 and M_1 are absolute constants. Therefore,

$$\lim_{n\to\infty}\sup_{\|f\|_{\mathcal{D}^2}\leq 1}\|(W_{\varphi,\psi}R_n)f\|_{\mathcal{D}^2}^2 \leq \lim_{n\to\infty} KM\|\mu_{\varphi,\psi}\|_r^* + \lim_{n\to\infty} K_1M_1\|\nu_{\varphi,\psi}\|_r^*\,.$$

Thus, $\|W_{\varphi,\psi}\|_e^2 \le KM\|\mu_{\varphi,\psi}\|_r^* + K_1M_1\|\nu_{\varphi,\psi}\|_r^*$. Taking $r \to 1$, we have

$$\begin{aligned}\|W_{\varphi,\psi}\|_e^2 &\le KM \lim_{r\to 1} \|\mu_{\varphi,\psi}\|_r^* + K_1M_1 \lim_{r\to 1} \|\nu_{\varphi,\psi}\|_r^* \\ &= KM \limsup_{|a|\to 1} \int_{\mathbf{D}} |\sigma_a'(\omega)|^2 \,\mathrm{d}\mu_{\varphi,\psi}(\omega) \\ &\quad + K_1M_1 \limsup_{|a|\to 1} \int_{\mathbf{D}} |\sigma_a'(\omega)|^2 \,\mathrm{d}\nu_{\varphi,\psi}(\omega) \\ &= KM \limsup_{|a|\to 1} \int_{\mathbf{D}} \frac{(1-|a|^2)^2}{|1-\bar{a}\omega|^4} \,\mathrm{d}\mu_{\varphi,\psi}(\omega) \\ &\quad + K_1M_1 \limsup_{|a|\to 1} \int_{\mathbf{D}} \frac{(1-|a|^2)^2}{|1-\bar{a}\omega|^4} \,\mathrm{d}\nu_{\varphi,\psi}(\omega) \\ &= KM \limsup_{|a|\to 1} \Phi(a) + K_1M_1 \limsup_{|a|\to 1} \Psi(a)\,,\end{aligned}$$

which is the desired upper bound.

Now we prove the lower estimate. Note that set $\{\sigma_a : a \in \mathbf{D}\}$ is bounded in $\mathcal{D}^2$. Also, $\sigma_a - a \to 0$ as $|a| \to 1$ uniformaly on compact sets in $\mathbf{D}$ since

$$|\sigma_a(z) - a| = |z| \frac{1-|a|^2}{|1-\bar{a}z|}\,.$$

Also, fix a compact operator K on $\mathcal{D}^2$. Then $\|K(\sigma_a - a)\|_{\mathcal{D}^2} \to 0$ as $|a| \to 1$. Thus, $\|K(\sigma_a)\|_{\mathcal{D}^2} \to 0$ as $|a| \to 1$. Therefore,

$$\begin{aligned}\|W_{\varphi,\psi} - K\|_{\mathcal{D}^2} &\ge \limsup_{|a|\to 1} (\|(W_{\varphi,\psi} - K)\sigma_a\|_{\mathcal{D}^2}) \\ &\ge \limsup_{|a|\to 1} (\|(W_{\varphi,\psi})\sigma_a\|_{\mathcal{D}^2} - \|K\sigma_a\|_{\mathcal{D}^2}) \\ &= \limsup_{|a|\to 1} \|(W_{\varphi,\psi})\sigma_a\|_{\mathcal{D}^2}\,.\end{aligned}$$

Thus,

$$\|W_{\varphi,\psi}\|_e^2 \ge \|W_{\varphi,\psi} - K\|_{\mathcal{D}^2}^2 \ge \limsup_{|a|\to 1} \|(W_{\varphi,\psi})\sigma_a\|_{\mathcal{D}^2}^2\,.$$ ∎

COROLLARY 2.13. *Let $\varphi, \psi \in \mathcal{D}^2$ be such that $\varphi(\mathbf{D}) \subseteq \mathbf{D}$. Suppose that the induced measure $\nu_{\varphi,\psi}$ is a bounded 2–Carleson measure. Then $W_{\varphi,\psi}$ is compact on $\mathcal{D}^2$ if and anly if*

$$\limsup_{|a|\to 1} \int_{\mathbf{D}} \frac{(1-|a|^2)^2}{|1-\bar{a}\omega|^4} \,\mathrm{d}\mu_{\varphi,\psi}(\omega) = 0$$

and

$$\limsup_{|a|\to 1} \int_{\mathbf{D}} \frac{(1-|a|^2)^2}{|1-\overline{a}\omega|^4}\, \mathrm{d}\nu_{\varphi,\psi}(\omega) = 0\,.$$

THEOREM 2.14. *Let $\varphi, \psi \in \mathcal{D}^2$ be such that $\varphi(\mathbf{D}) \subseteq \mathbf{D}$. Suppose $W_{\varphi,\psi}$ is compact on $\mathcal{D}^2$, then*

$$\lim_{|z|\to 1} \frac{|\psi(z)|\left\{1+\log\left(\frac{1}{1-|\varphi(z)|^2}\right)\right\}}{1+\log\left(\frac{1}{1-|z|^2}\right)} = 0\,. \tag{2.9}$$

Proof. We know that $(W_{\varphi,\psi})^* K_a(z) = \overline{\psi(a)} K_{\varphi(a)}(z)$. So, we have

$$\|(W_{\varphi,\psi})^* K_a\|_{\mathcal{D}^2} = \|\overline{\psi(a)} K_{\varphi(a)}\|_{\mathcal{D}^2} = |\psi(a)| \left\{1+\log\left(\frac{1}{1-|\varphi(a)|^2}\right)\right\}.$$

Let k_a be the normalised reproducing kernel in $\mathcal{D}^2$. Then we have

$$\|(W_{\varphi,\psi})^* k_a\|_{\mathcal{D}^2} = \frac{|\psi(a)|\left\{1+\log\left(\frac{1}{1-|\varphi(a)|^2}\right)\right\}}{1+\log\left(\frac{1}{1-|a|^2}\right)}\,.$$

Further, since $W_{\varphi,\psi}$ is compact on $\mathcal{D}^2$, and so $(W_{\varphi,\psi})^*$ is also compact. Again, $k_a \to 0$ weakly in $\mathcal{D}^2$ as $|a| \to 1$, and so condition (2.9) is satisfied. ∎

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