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Prolongation of Linear Semibasic Tangent Valued Forms to Product Preserving Gauge Bundles of Vector Bundles

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0. Introduction

A linear semibasic tangent valued p-form on a vector bundle $E \to M$ is a section $\varphi : E \to \wedge^p T^*M \otimes TE$ such that $\varphi(X_1, \dots, X_p)$ is a linear vector field on E for any vector fields $X_1, \dots, X_p$ on M. (We recall that a vector field $X : E \to TE$ on a vector bundle $p : E \to M$ is linear if it is a vector bundle map between vector bundles $p : E \to M$ and $Tp : TE \to TM$. Equivalently, the flow $ExptX$ is formed by vector bundle (local) isomorphisms.)

A very important example of a semibasic linear tangent valued 1-form is a linear general connection Γ on a vector bundle $E \to M$. (We recall that a general linear connection on a vector bundle $E \to M$ is a semibasic linear tangent valued 1-form $\Gamma : E \to T^*M \otimes TE$ on $E \to M$ such that $\Gamma(X)$ projects onto X for any vector field X on M, [3].) Connections play important roles in differential geometry, field theories of mathematical physics, and differential equations, [3], [2], [6].

Let A be a Weil algebra and $T^A : \mathcal{M}f \to \mathcal{FM}$ be the corresponding Weil functors on the category $\mathcal{M}f$ of all manifolds and maps. Let $E \to M$ be a vector bundle. Then $T^A E \to T^A M$ is a vector bundle, too. Restricting the well know facts of lifting of tangent values forms on manifolds to Weil bundles, we obtain.

Proposition A. ([1]) *For any linear semibasic tangent valued p-form $\varphi : E \to \wedge^p T^*M \otimes TE$ there exists an unique linear semibasic tangent valued* *p-form* $\mathcal{T}^A\varphi : T^AE \to \wedge^p T^*T^AM \otimes TT^AE$ *on* $T^AE \to T^AM$ *such that*

$$(*) \qquad \mathcal{T}^A\varphi\big(\mathbf{a}f(a_1)\circ \mathcal{T}^AX_1, \dots, \mathbf{a}f(a_p)\circ \mathcal{T}^AX_p\big) = \mathbf{a}f(a_1\cdots a_p)\circ \mathcal{T}^A\big(\varphi(X_1,\dots,X_p)\big)$$

for any vector fields $X_1, \dots, X_p$ *on* M *and any* $a_1, \dots, a_p \in A$*, where we denote the flow lift of a field* Z *on* N *to* T^AN *by* $\mathcal{T}^AZ$ *and where* $\mathbf{a}f(a) : TT^AN \to TT^AN$ *is the canonical affinor on* T^AN *corresponding to* $a \in A$*.*

The Frolicher-Nijenhuis bracket $[[\varphi, \psi]]$ of linear semibasic tangent valued p- and q- forms on $E \to M$ is again a linear semibasic tangent valued $(p+q)$-form on $E \to M$.

Proposition B. ([1]) *We have*

$$(**) \qquad [[\mathcal{T}^A\varphi, \mathcal{T}^A\psi]] = \mathcal{T}^A([[\varphi, \psi]])$$

for any linear semibasic tangent valued p- *and* q-*forms* φ *and* ψ *on* $E \to M$*.*

The gauge bundle functor $T^A : \mathcal{VB} \to \mathcal{FM}$ (on the category $\mathcal{VB}$ of all vector bundles and vector bundle maps) obtained from $T^A : \mathcal{M}f \to \mathcal{FM}$ is an example of product preserving gauge bundle functors $F : \mathcal{VB} \to \mathcal{FM}$. In [5], for any Weil algebra A and any A-module V with $dim_{\mathbf{R}}(V) < \infty$ we constructed a product preserving gauge bundle functor $T^{A,V} : \mathcal{VB} \to \mathcal{FM}$, and we proved.

Proposition C. ([5]) *Any product preserving gauge bundle functor* $F : \mathcal{VB} \to \mathcal{FM}$ *is isomorphic to* $T^{A,V}$ *for some* (A, V) *in question.*

In [5], we also observed that $T^AE = T^{A,V}E$ for $V = A$, and that $T^{A,V}p : T^{A,V}E \to T^{A,V}M = T^AM$ (M is treated as the zero vector bundle over M) is a vector bundle (even A-module bundle) for any vector bundle $p : E \to M$. Thus we have the following natural problems.

Problem 1. For a product preserving gauge bundle functor $F : \mathcal{VB} \to \mathcal{FM}$ to construct canonically a linear semibasic tangent valued p-form $\mathcal{F}\varphi : FE \to \wedge^p T^*FM \otimes TFE$ on $Fp : FE \to FM$ from a linear semibasic tangent valued p-form $\varphi : E \to \wedge^p T^*M \otimes TE$ on a vector bundle $p : E \to M$ such that a formula similar to $(*)$ holds.

PROBLEM 2. For a product preserving gauge bundle functor $F : \mathcal{VB} \to \mathcal{FM}$ to prove a formula similar to $(**)$.

The purpose of the present paper is to solve the above problems for all fiber product preserving gauge bundle functors $F : \mathcal{VB} \to \mathcal{FM}$. We may of course assume $F = T^{A,V}$. Given $a \in A$ we have a canonical affinor $\mathbf{a}f(a) : TT^{A,V}E \to TT^{A,V}E$ on $T^{A,V}E$. Given a linear vector field Z on E its flow $ExptZ$ is formed by (local) vector bundle isomorphisms and we have the flow prolongation $\mathcal{T}^{A,V}Z = \frac{\partial}{\partial t}_{|t=0}(T^{A,V}(ExptZ))$ of Z to $T^{A,V}E$. We prove

THEOREM A. *Given a linear semibasic tangent valued p-form $\varphi : E \to \wedge^p T^*M \otimes TE$ on a vector bundle $E \to M$ there is an unique linear semibasic tangent valued p-form $\mathcal{T}^{A,V}\varphi : T^{A,V}E \to \wedge^p T^*T^A M \otimes TT^{A,V}E$ on the vector bundle $T^{A,V}E \to T^A M$ satisfying*

$$\mathcal{T}^{A,V}\varphi\big(\mathbf{a}f(c_1) \circ \mathcal{T}^A X_1, \dots, \mathbf{a}f(c_p) \circ \mathcal{T}^A X_p\big) = \mathbf{a}f(c_1 \cdots c_p) \circ \mathcal{T}^{A,V}\big(\varphi(X^1, \dots, X^p)\big)$$

for any vector fields $X_1, \dots, X_p$ on M and any $c_1, \dots, c_p \in A$.

In the proof of Theorem A, the linear semibasic p-form $\mathcal{T}^{A,V}\varphi$ will be explicitly constructed. Next, for the Frolicher-Nijenhuis bracket we prove.

THEOREM B. *We have*

$$[[\mathcal{T}^{A,V}\varphi, \mathcal{T}^{A,V}\psi]] = \mathcal{T}^{A,V}([[\varphi, \psi]])$$

for any linear semibasic tangent valued p- and q- forms φ and ψ on $E \to M$.

In the last section we apply the above results to linear general connections on vector bundles.

All manifolds and maps are assumed to be of class C^∞.

1. WEIL BUNDLES

Let A be a Weil algebra, see [3]. Given a manifold M we have the Weil bundle

$$T^A M = \bigcup_{z \in M} Hom\big(C_z^\infty(M), A\big)$$

over M corresponding to A, where $Hom(C_z^\infty(M), A)$ is the set of all algebra homomorphisms φ from the (unital) algebra $C_z^\infty(M) = \{germ_z(g) |\ g : M \to \mathbf{R}\}$ into A. Given a map $\underline{f} : M \to N$ we have the induced (via pull-back) map $T^A\underline{f} : T^AM \to T^AN$. The correspondence $T^A : \mathcal{M}f \to \mathcal{FM}$ is a product preserving bundle functor on the category $\mathcal{M}f$ of all manifolds and maps, [3].

It is well-known that any product preserving bundle functor $F : \mathcal{M}f \to \mathcal{FM}$ is isomorphic to T^A for some Weil algebra A, [3].

2. Generalized Weil bundles

Let A be a Weil algebra and V be an A-module with $\dim_{\mathbf{R}}(V) < \infty$. In [5], similarly to Weil bundles, given a vector bundle $E = (E \xrightarrow{p} M)$ we defined an A-module bundle

$$T^{A,V}E = \big\{(\varphi, \psi) |\ \varphi \in Hom\big(C_z^\infty(M), A\big),\ \psi \in Hom_\varphi\big(C_z^{\infty,f.l.}(E), V\big),\ z \in M\big\}$$

over T^AM, where $Hom_\varphi(C_z^{\infty,f.l.}(E), V)$ is the A-module of all module homomorphisms ψ over φ from the $C_z^\infty(M)$-module $C_z^{\infty,f.l.}(E) = \{germ_z(h) \mid h : E \to \mathbf{R}$ is fibre linear$\}$ into V. Given another vector bundle $G = (G \xrightarrow{q} N)$ and a vector bundle homomorphism $f : E \to G$ over $\underline{f} : M \to N$ we have the induced A-module bundle map $T^{A,V}f : T^{A,V}E \to T^{A,V}G$ over $T^A\underline{f} : T^AM \to T^AN$ by

$$T^{A,V}f(\varphi, \psi) = (\varphi \circ \underline{f}_z^*, \psi \circ f_z^*),$$

$(\varphi, \psi) \in T_z^{A,V}E$, $z \in M$, where $\underline{f}_z^* : C_{\underline{f}(z)}^\infty(N) \to C_z^\infty(M)$ and $f_z^* : C_{\underline{f}(z)}^{\infty,f.l.}(G) \to C_z^{\infty,f.l.}(E)$ are given by the pull-back with respect to $\underline{f}$ and f. The correspondence $T^{A,V} : \mathcal{VB} \to \mathcal{FM}$ is a product preserving gauge bundle functor, see [5] (see also [4] for examples of modules over Weil algebras).

In [5], we proved that any product preserving gauge bundle functor $F : \mathcal{VB} \to \mathcal{FM}$ is isomorphic to $T^{A,V}$ for some (A, V) in question.

3. Local description of generalized Weil bundles

A local vector bundle trivialization $(x^1, \ldots, x^m, y^1, \ldots, y^n) : E|U \tilde{=} \mathbf{R}^m \times \mathbf{R}^n$ on E induces a fiber bundle trivialization

$$(\tilde{x}^1, \ldots, \tilde{x}^m, \tilde{y}^1, \ldots, \tilde{y}^n) : T^{A,V}E|U \tilde{=} A^m \times V^n$$

by $\tilde{x}^i(\varphi,\psi) = \varphi(germ_z(x^i)) \in A$, $\tilde{y}^j(\varphi,\psi) = \psi(germ_z(y^j)) \in V$, $(\varphi,\psi) \in T_z^{A,V}E$, $z \in U$, $i = 1,\dots,m$, $j = 1,\dots,n$.

Let $f : E \to G$ be a vector bundle map. If in some vector bundle coordinates

$$f(x,y) = \Big(\varphi(x), \big(\sum_{j=1}^{n} \psi_j^k(x) y^j\big)_{k=1}^{p}\Big) \tag{1}$$

$x \in \mathbf{R}^m$, $y = (y^j) \in \mathbf{R}^n$, then in the induced coordinates we have

$$T^A f(a,w) = \Big(T^A\varphi(a), \big(\sum_{j=1}^{n} T^A\psi_j^k(a) w^j\big)_{k=1}^{p}\Big), \tag{2}$$

$a \in A^m$, $w = (w^j) \in V^n$.

4. The affinors $\mathbf{a}f(c)$

Let $c \in A$. We have an affinor $\mathbf{a}f(c) : T(A^m \times V^n) \to T(A^m \times V^n)$ on $A^m \times V^n$ given by

$$\mathbf{a}f(c)\big((a,v),(b,w)\big) = \big((a,v),(cb,cw)\big) \tag{3}$$

for $((a,v),(b,w)) \in (A^m \times V^n) \times (A^m \times V^n) = T(A^m \times V^n)$.

Lemma 1. *We have*

$$TT^{A,V}f \circ \mathbf{a}f(c) = \mathbf{a}f(c) \circ TT^{A,V}f$$

for any vector bundle map $f : \mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^q \times \mathbf{R}^p$.

Proof. The proof is standard. We propose to use (2). ∎

Thus according to the general theory of [3], for any vector bundle $E \to M$ we have a canonical affinor $\mathbf{a}f(c)$ on $T^{A,V}E$ with the form (3) in every vector bundle coordinates.

5. Linear semibasic tangent valued p-forms

Let $E \to M$ be a vector bundle. A linear semibasic tangent valued p-form on $E \to M$ is a section $\varphi : E \to \wedge^p T^* M \otimes TE$ such that $\varphi(X_1, \dots, X_p)$ is a linear vector field on E for any vector fields $X_1, \dots, X_p$ on M. Thus a linear semibasic tangent valued p-form φ on the trivial vector bundle $\mathbf{R}^m \times \mathbf{R}^n$ over $\mathbf{R}^m$ has the form

$$\varphi = \sum_{i=1}^{m} \varphi^i \otimes \frac{\partial}{\partial x^i} + \sum_{j,k=1}^{n} \varphi_j^k \otimes y^j \frac{\partial}{\partial y^k} \tag{4}$$

for some unique p-forms φ^i, $\varphi_j^k : T\mathbf{R}^m \times_{\mathbf{R}^m} \dots \times_{\mathbf{R}^m} T\mathbf{R}^m \to \mathbf{R}$ on $\mathbf{R}^m$, where $x^1, \dots, x^m, y^1, \dots, y^n$ are the usual vector bundle coordinates on $\mathbf{R}^m \times \mathbf{R}^n$. More precisely,

$$\begin{aligned}\varphi(x,y)(v_1, \dots, v_p) = &\sum_{i=1}^{m} \varphi^i(v_1, \dots, v_p) \frac{\partial}{\partial x^i}(x,y) \\ &+ \sum_{j,k=1}^{n} \varphi_j^k(v_1, \dots, v_p) y^j \frac{\partial}{\partial y^k}(x,y) \in T_{(x,y)}(\mathbf{R}^m \times \mathbf{R}^n),\end{aligned}$$

$y = (y^j) \in \mathbf{R}^n$, $x \in \mathbf{R}^m$, $v_1, \dots, v_p \in T_x\mathbf{R}^m$.

6. The solution of Problem 1

Theorem 1. *Let A be a Weil algebra and V be an A-module,* $\dim_{\mathbf{R}}(V) < \infty$*. Let $\varphi : E \to \wedge^T E$ be a linear semibasic tangent valued p-form on a vector bundle $E \to M$. There is an unique linear semibasic tangent valued p-form $\mathcal{T}^{A,V}\varphi$ on $T^{A,V}E \to T^A M$ such that*

$$\begin{aligned}&\mathcal{T}^{A,V}\varphi\big(\mathbf{a}f(c_1) \circ \mathcal{T}^A X_1, \dots, \mathbf{a}f(c_p) \circ \mathcal{T}^A X_p\big) \\ &\qquad\qquad = \mathbf{a}f(c_1 \cdots c_p) \circ \mathcal{T}^{A,V}\big(\varphi(X_1, \dots, X_p)\big)\end{aligned} \tag{5}$$

for any vector fields $X_1, \dots, X_p$ on M and any $c_1, \dots, c_p \in A$, where $\mathcal{T}^A X$ is the flow lift of X to $T^A M$ and $\mathcal{T}^{A,V} Z$ is the flow lift of a linear vector field on E to $T^{A,V}E$.

Proof. The construction of the linear semibasic tangent valued p-form satisfying (5) will be given in Sections 7 and 8. The proof will be end in the end of Section 8.

7. Local description of $\mathcal{T}^{A,V}\varphi$

Let φ be a linear semibasic tangent valued p-form on $E = \mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^m$ of the form (4), then we define $\mathcal{T}^{A,V}\varphi$ on $T^{A,V}E = A^m \times V^n$ by

$$
\begin{aligned}
\mathcal{T}^{A,V}\varphi = &\sum_{i=1}^{m} \big(T^A\varphi^i \circ (\eta \times \cdots \times \eta)\big) \otimes_A \mathcal{T}^A \frac{\partial}{\partial x^i} \\
&+ \sum_{j,k=1}^{n} \big(T^A\varphi_j^k \circ (\eta \times \cdots \times \eta)\big) \otimes_A \mathcal{T}^{A,V}\big(y^j \frac{\partial}{\partial y^k}\big),
\end{aligned} \tag{6}
$$

where $T^A\varphi_j^k : T^A(T\mathbf{R}^m \times_{\mathbf{R}^m} \cdots \times_{\mathbf{R}^m} T\mathbf{R}^m) \to T^A\mathbf{R} = A$ is the extension of $\varphi_j^k : T\mathbf{R}^m \times_{\mathbf{R}^m} \cdots \times_{\mathbf{R}^m} T\mathbf{R}^m \to \mathbf{R}$ and $\eta : TT^A\mathbf{R}^m \to T^AT\mathbf{R}^m$ is the flow isomorphism and $\mathcal{T}^{A,V}Z$ is the flow prolongation of a linear vector field Z on $E \to M$ to $T^{A,V}E$ and where the flow lift $\mathcal{T}^A\frac{\partial}{\partial x^i}$ is the vector field on A^m and then on $A^m \times V^n$. More precisely,

$$
\begin{aligned}
(\mathcal{T}^{A,V}\varphi)(a,w)(u_1,\dots,u_p) = &\sum_{i=1}^{m} T^A\varphi^i\big(\eta(u_1),\dots,\eta(u_p)\big)\mathcal{T}^A\frac{\partial}{\partial x^i}(a,w) \\
&+ \sum_{j,k=1}^{n} T^A\varphi_j^k\big(\eta(u_1),\dots,\eta(u_p)\big)\mathcal{T}^{A,V}\big(y^j\frac{\partial}{\partial y^k}\big)(a,w),
\end{aligned}
$$

$u_1,\dots,u_p \in T_aA^m$, $a \in A^m$, $w \in V^n$.

We prove the following proposition.

Proposition 1. *The linear semibasic tangent valued p-form $\mathcal{T}^{A,V}\varphi$ on $A^m \times V^n \to A^m$ given by (6) is the unique linear tangent valued p-form satisfying (5) for any vector fields $X_1,\dots,X_p$ on $\mathbf{R}^m$ and any $c_1,\dots,c_p \in A$.*

To prove Proposition 1 we need.

Lemma 2. *We have*

$$
\mathcal{T}^{A,V}(f \otimes Z) = T^Af \otimes_A \mathcal{T}^{A,V}Z \tag{7}
$$

for any $f : \mathbf{R}^m \to \mathbf{R}$ and any linear vector field Z on $\mathbf{R}^n$, where (of course) $(f \otimes Z)(x,y) = f(x)Z(x,y) \in T_{(x,y)}(\mathbf{R}^m \times \mathbf{R}^n)$, $(x,y) \in \mathbf{R}^m \times \mathbf{R}^n$, and $(T^Af \otimes_A \mathcal{T}^{A,V}Z)(a,w) = T^Af(a)\mathcal{T}^{A,V}Z(a,w) \in V_{(a,w)}(A^m \times V^n)$, $(a,w) \in A^m \times V^n$.

Proof. We can prove (7) as follows. Let $\psi_t = (\psi^k_l(t)) \in GL(\mathbf{R}^n)$ be the flow of Z. Then the flow of $f \otimes Z$ is $\Psi_t : \mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^m \times \mathbf{R}^n$, $\Psi_t(x,y) = (x, \psi_{tf(x)}(y))$. Then (by (2)) we have

$$T^{A,V}\Psi_t(a,w) = \Big(a, \big(\sum_{l=1}^n T^A\psi^k_l(tT^Af(a))w^l\big)^n_{k=1}\Big),$$

$a \in A^m$, $w = (w^l) \in V^n$. Therefore

$$\begin{aligned}
\mathcal{T}^{A,V}(f\otimes Z)(a,w) &= \frac{d}{dt}_{|t=0}\big(T^{A,V}\Psi_t(a,w)\big)\\
&= \Big(0, \frac{d}{dt}_{|t=0}\big(\sum_{l=1}^n T^A\psi^k_l(tT^Af(a))w^k\big)^n_{k=1}\Big)\\
&= \Big(0, T^Af(a)\frac{d}{dt}_{|t=0}\big(\sum_{l=1}^n \psi^k_l(t)w^k\big)^n_{k=1}\Big)\\
&= T^Af(a)\frac{d}{dt}_{|t=0}T^{A,V}(id_{\mathbf{R}^m}\times\psi_t)(a,w)\\
&= T^Af(a)\mathcal{T}^{A,V}Z(a,w) = (T^Af\otimes_A\mathcal{T}^{A,V}Z)(a,w).
\end{aligned}$$

The proof of Lemma 2 is complete. ∎

Proof of Proposition 1. We prove (5) as follows. By (6) and (7), by the **R**-linearity of the flow lift of linear vector fields and the well-known formulas for the flow lift $\mathcal{T}^A$ of vector fields to T^AM, we have

$$\begin{aligned}
&\mathcal{T}^{A,V}\varphi\big(\mathbf{a}f(c_1)\circ\mathcal{T}^AX_1,\dots,\mathbf{a}f(c_p)\mathcal{T}^AX_p\big)\\
&\quad= \sum_{i=1}^m T^A\varphi^i\big(\eta(\mathbf{a}f(c_1)\circ\mathcal{T}^AX_1),\dots,\eta(\mathbf{a}f(c_p)\circ\mathcal{T}^AX_p)\big)\otimes_A\mathcal{T}^A\frac{\partial}{\partial x^i}\\
&\qquad+ \sum_{j,k=1}^n T^A\varphi^k_j\big(\eta(\mathbf{a}f(c_1)\circ\mathcal{T}^AX_1),\dots,\eta(\mathbf{a}f(c_p)\circ\mathcal{T}^AX_p)\big)\otimes_A\mathcal{T}^{A,V}\big(y^j\frac{\partial}{\partial y^k}\big)\\
&\quad= \sum_{i=1}^m c_1\cdots c_pT^A\big(\varphi^i(X_1,\dots,X_p)\big)\otimes_A\mathcal{T}^A\frac{\partial}{\partial x^i}\\
&\qquad+ \sum_{j,k=1}^n c_1\cdots c_pT^A\big(\varphi^k_j(X_1,\dots,X_p)\big)\otimes_A\mathcal{T}^{A,V}\big(y^j\frac{\partial}{\partial y^k}\big)\\
&\quad= \mathbf{a}f(c_1\cdots c_p)\circ\mathcal{T}^{A,V}\big(\varphi(X_1,\dots,X_p)\big).
\end{aligned}$$

The uniqueness of $\mathcal{T}^{A,V}\varphi$ follows from the fact that the $\mathbf{a}f(c) \circ \mathcal{T}^A X$ for all vector fields X and $\mathbf{R}^m$ and all $c \in A$ generates (over $C^\infty(A^m)$) the space of all vector fields on A^m, see [3]. ∎

8. Global description of $\mathcal{T}^{A,V}\varphi$

Let φ be a linear tangent valued p-form on $E \to M$. Using vector bundle coordinates we can define $\mathcal{T}^{A,V}\varphi$ locally by (6). According to respective theory of [3], to define $\mathcal{T}^{A,V}\varphi$ globally on $T^{A,V}E \to T^A M$ it remains to show

Proposition 2. *The construction $\mathcal{T}^{A,V}$ given by (6) is invariant with respect to vector bundle isomorphisms $f : \mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^m \times \mathbf{R}^n$. It means, we have*

$$\mathcal{T}^{A,V}(f_*\varphi) = (T^{A,V}f)_*\mathcal{T}^{A,V}\varphi \tag{8}$$

for any f as above.

Proof. The formula (8) is clear because of the uniqueness case of Proposition 1, the formula (5) for any vector fields $X_1, \dots, X_p$ on $\mathbf{R}^m$ and $c_1, \dots, c_p \in A$ (see Proposition 1), and the naturality of the flow operators and the naturality of the affinors $\mathbf{a}f(c)$. ∎

The proof of Theorem 1 is complete. ∎

9. Some natural properties of $\mathcal{T}^{A,V}\varphi$

From the uniqueness of $\mathcal{T}^{A,V}\varphi$ satisfying (5) we have

Proposition 3. *Let φ_1 and φ_2 be linear semibasic tangent valued p-forms on $E \to M$ and $G \to N$. If they are f-related by a local vector bundle isomorphism $f : E \to G$, then $\mathcal{T}^{A,V}\varphi_1$ and $\mathcal{T}^{A,V}\varphi_2$ are $T^{A,V}f$-related. In other words, the correspondence $\varphi \to \mathcal{T}^{A,V}\varphi$ is a $\mathcal{VB}_{m,n}$-natural operator in the sense of* [3].

Proposition 4. *Let φ be a linear semibasic tangent valued p-form on $E \to M$. Let (A_1, V_1) and (A_2, V_2) be two pairs in question. Suppose that $\nu : V_1 \to V_2$ is a module isomorphism over an algebra isomorphism $\mu : A_1 \to A_2$. Let $\eta^{\nu,\mu} : T^{A_1,V_1}E \to T^{A_2,V_2}E$ be the corresponding vector bundle isomoprphism, see* [4]. *Then $\mathcal{T}^{A_1,V_1}\varphi$ and $\mathcal{T}^{A_2,V_2}\varphi$ are $\eta^{\nu,\mu}$-related.*

By the same arguments we easily see that

PROPOSITION 5. *Let V_1 and V_2 be A modules (finite dimensional over $\mathbf{R}$). Let $\nu : V_1 \to V_2$ be an A-module homomorphism (not necessarily isomorphism) over $id_A : A \to A$. Then $\mathcal{T}^{A,V_1}\varphi$ and $\mathcal{T}^{A,V_2}\varphi$ are $\eta^{id_A,\nu}$-related.*

10. The bracket formula

Let (A, V) be in question. Let U and W be linear vector fields on $E \to M$. Then $[U, W]$ is a linear vector field on E, too. Let $a, b \in A$.

LEMMA 3. *The following formula*

$$[\mathbf{a}f(a) \circ \mathcal{T}^{A,V}U, \mathbf{a}f(b) \circ \mathcal{T}^{A,V}W] = \mathbf{a}f(ab) \circ \mathcal{T}^{A,V}([U, W]) \tag{9}$$

holds.

Proof. Because of the $\mathbf{R}$-bilinearity of booth sides of (9) with respect to U and W, we can assume that U is not vertical. Then using vector bundle coordinate invariance of booth sides of (9) we can assume $E = \mathbf{R}^m \times \mathbf{R}^n$ and $U = \frac{\partial}{\partial x^1}$. Then because of the $\mathbf{R}$-linearity of both sides of (9) with respect to W we can assume that $W = f(x)\frac{\partial}{\partial x^i}$ or $W = f(x)y^j\frac{\partial}{\partial y^k}$.

In the first case the formula (9) is the well-known (for Weil bundles) one

$$\big[\mathbf{a}f(a) \circ \mathcal{T}^A \frac{\partial}{\partial x^1}, \mathbf{a}f(b) \circ \mathcal{T}^A\big(f(x)\frac{\partial}{\partial x^i}\big)\big] = \mathbf{a}f(ab) \circ \mathcal{T}^A\big([\frac{\partial}{\partial x^1}, f(x)\frac{\partial}{\partial x^i}]\big) \ .$$

If $U = \frac{\partial}{\partial x^1}$ and $W = f(x)y^j\frac{\partial}{\partial y^k}$, then because of formula (7) and the fact that $[\mathbf{a}f(a) \circ \mathcal{T}^{A,V}\frac{\partial}{\partial x^1}, \mathcal{T}^{A,V}(y^j\frac{\partial}{\partial y^k})] = 0$ (as $\mathbf{a}f(a) \circ \mathcal{T}^{A,V}\frac{\partial}{\partial x^1}$ is a vector field on A^m and $\mathcal{T}^{A,V}(y^j\frac{\partial}{\partial y^k})$ is a vector field on V^n) we have

$$\begin{aligned}
&\big[\mathbf{a}f(a) \circ \mathcal{T}^{A,V}\frac{\partial}{\partial x^1}, \mathbf{a}f(b) \circ \mathcal{T}^{A,V}\big(f(x)y^j\frac{\partial}{\partial y^k}\big)\big] \\
&= [\mathbf{a}f(a) \circ \mathcal{T}^{A,V}\frac{\partial}{\partial x^1}, bT^A f\mathcal{T}^{A,V}(y^j\frac{\partial}{\partial y^k})] \\
&= (\mathbf{a}f(a) \circ \mathcal{T}^A\frac{\partial}{\partial x^1})(bT^A f)\mathcal{T}^{A,V}(y^j\frac{\partial}{\partial y^k}) \\
&= \big(bTT^A f \circ \mathbf{a}f(a) \circ \mathcal{T}^A\frac{\partial}{\partial x^1}\big)\mathcal{T}^{A,V}(y^j\frac{\partial}{\partial y^k})
\end{aligned}$$

$$= baTT^A f(T^A \frac{\partial}{\partial x^1}) \mathcal{T}^{A,V}(y^j \frac{\partial}{\partial y^k}) = abT^A(\frac{\partial}{\partial x^1} f) \mathcal{T}^{A,V}(y^j \frac{\partial}{\partial y^k})$$
$$= \mathbf{a} f(ab) \circ \mathcal{T}^{A,V}\big(\frac{\partial}{\partial x^1} f(x) y^j \frac{\partial}{\partial y^k}\big) = \mathbf{a} f(ab) \circ \mathcal{T}^{A,V}\big([\frac{\partial}{\partial x^1}, f(x) y^j \frac{\partial}{\partial y^k}]\big).$$

The proof of Lemma 3 is complete. ∎

11. Solution of Problem 2

By using the pull-back with respect to $p : E \to M$, a linear semibasic tangent valued p-form $K : E \to \wedge^p T^*M \otimes TE$ on $p : E \to M$ can be treated as the tangent valued p-form $K \in \Omega^p(E, TE)$ on manifold E. Given $K \in \Omega^p(E, TE)$ and $L \in \Omega^q(E, TE)$ we have the Frolicher-Nijenhuis bracket $[[K, L]] \in \Omega^{p+q}(E, TE)$ given by

$$\begin{aligned}
&[[K,L]](Z_1, \dots, Z_{p+q}) \\
&\quad = \frac{1}{p!q!} \sum_\sigma \operatorname{sign}\sigma \big[K(Z_{\sigma 1}, \dots, Z_{\sigma p}), L\big(Z_{\sigma(p+1)}, \dots, Z_{\sigma(p+q)}\big)\big] \\
&\quad + \frac{-1}{p!(q-1)!} \sum_\sigma \operatorname{sign}\sigma L\big(\big[K(Z_{\sigma 1}, \dots, Z_{\sigma p}), Z_{\sigma(p+1)}\big], Z_{\sigma(p+2)}, \dots\big) \\
&\quad + \frac{(-1)^{pq}}{(p-1)q!} \sum_\sigma \operatorname{sign}\sigma K\big(\big[L(Z_{\sigma 1}, \dots, Z_{\sigma q}), Z_{\sigma(q+1)}\big], Z_{\sigma(q+2)}, \dots\big) \\
&\quad + \frac{(-1)^{p-1}}{(p-1)!(q-1)!2!} \sum_\sigma \operatorname{sign}\sigma L\big(K\big([Z_{\sigma 1}, Z_{\sigma 2}], Z_{\sigma 3}, \dots\big), Z_{\sigma(p+2)}, \dots\big) \\
&\quad + \frac{(-1)^{p-1)q}}{(p-1)!(q-1)!2!} \sum_\sigma \operatorname{sign}\sigma K\big(L\big([Z_{\sigma_1}, Z_{\sigma 2}], Z_{\sigma 3}, \dots\big), Z_{\sigma(q+2)}, \dots\big)
\end{aligned}$$

for any vector fields $Z_1, \dots, Z_{p+q}$ on manifold E, see [3].

Then easily seen that for linear semibasic tangent valued p- and q- forms φ and ψ on $E \to M$, $[[\varphi, \psi]]$ is again a linear semibasic tangent valued $(p+q)$-form on $E \to M$.

Theorem 2. *Let (A, V) be in question. We have*

$$[[\mathcal{T}^{A,V}\varphi, \mathcal{T}^{A,V}\psi]] = \mathcal{T}^{A,V}([[\varphi, \psi]]) \tag{10}$$

for any linear semibasic tangent valued p- and q- forms φ and ψ on a vector bundle $E \to M$.

Proof. Because of the invariance of both sides of (10) with respect to vector bundle charts we may assume that $E \to M$ is the trivial vector bundle $\mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^m$. Using many times of formulas (5) and (9) and the formula defining the Frolicher-Nijenhuis bracket we easily verify

$$\begin{aligned}&[[\mathcal{T}^{A,V}\varphi, \mathcal{T}^{A,V}\psi]]\big(\mathbf{a}f(c_1)\circ \mathcal{T}^A X_1, \dots, \mathbf{a}f(c_{p+q})\circ T^A X_{p+q}\big)\\ &= \mathcal{T}^{A,V}\big([[\varphi,\psi]]\big)\big(\mathbf{a}f(c_1)\circ \mathcal{T}^A X_1, \dots, \mathbf{a}f(c_{p+q})\circ T^A X_{p+q}\big)\end{aligned}$$

for any vector fields $X_1, \dots, X_{p+q}$ on $\mathbf{R}^m$ (treated also as linear vector fields on $\mathbf{R}^m \times \mathbf{R}^n$) and any $c_1, \dots, c_{p+q} \in A$. ∎

12. Applications to linear general connections

A linear general connection Γ on $E \to M$ is a linear semibasic tangent valued 1-form $\Gamma : E \to T^*M \otimes TE$ such that $\Gamma(X)$ covers X, [3]. One can observe

COROLLARY 1. *For a linear general connection Γ on $E \to M$ its lifting $\mathcal{T}^{A,V}\Gamma$ is a linear general connection on $T^{A,V}E \to T^A M$.*

A curvature of Γ is a linear semibasic (vertical) tangent valued 2-form

$$\mathcal{R}_\Gamma := \frac{1}{2} P \circ [[\Gamma, \Gamma]],$$

where $P : TTE \to VTE$ is the projection in direction given by the horizontal distribution of Γ, [3]. From Theorem 2 and (6) we have.

COROLLARY 2. *It holds*

$$\mathcal{R}_{\mathcal{T}^{A,V}\Gamma} = \mathcal{T}^{A,V}(\mathcal{R}_\Gamma)$$

for any linear general connection Γ on a vector bundle $E \to M$.

13. Final remarks

We give briefly another purposes, why we could make the constructions.

Remark 1. Let A be a Weil algebra and V be an A-module in question. Let $E \to M$ be a vector bundle. One can observe that we have $\mathcal{VB}$-natural equivalence $T^{A,V}E = T^A E \otimes_A V$ (tensor product of the A-module bundles $T^A E \to T^A M$ and (trivial) $T^A M \times V \to T^A M$).

Remark 2. Let Γ be a linear general connection on a vector bundle $E \to M$. The connection $\mathcal{T}^A\Gamma$ (from [3] or [1]) on the A-module bundle $T^AE \to T^AM$ is A-linear. It means that the horizontal lift $\mathcal{T}^A\Gamma(Y)$ of a vector field Y on T^AM is an A-linear vector field on $T^AE \to T^AM$ (i.e., with the flow formed by A-module bundle local isomorphisms). On the trivial A-module bundle $T^AM \times V$ over T^AM we have the trivial A-linear general connection $\Gamma_{T^AM\times V}$. Thus we have the tensor product connection $\mathcal{T}^A\Gamma \otimes_A \Gamma_{T^AM\times V}$ on $T^{A,V}E = T^AE \otimes_A V \to T^AM$, defined quite similarly as tensor product of ($\mathbf{R}$-)linear general connections (see Proposition 47.14 in [3]).

Remark 3. Similarly, let $\varphi : E \to \wedge^p T^*M \otimes TE$ be a semibasic linear tangent valued p-form on a vector bundle $E \to M$, and let $\underline{\varphi} : M \to \wedge^p T^*M \otimes TM$ be its underlying tangent valued p-form. By [1], we have the semibasic (A-)linear tangent valued p-form $\mathcal{T}^A\varphi : T^AE \to \wedge^p T^*T^AM \otimes TT^AE$ on $T^AE \to T^AM$ with the underlying tangent valued p-form $\mathcal{T}^A\underline{\varphi} : T^AM \to \wedge^p T^*T^AM \otimes TT^AM$. The A-linearity means that given vector fields $Y_1, \dots, Y_p$ on T^AM, $\mathcal{T}^A\varphi(Y_1, \dots, Y_p)$ is an A-linear vector field on $T^AE \to T^AM$ with the underlying vector field $\mathcal{T}^A\underline{\varphi}(Y_1, \dots, Y_p)$. Let V be an A-module in question. Clearly, $\mathcal{T}^A\underline{\varphi}(Y_1, \dots, Y_p) \times 0$ (where 0 is the zero vector field on V) is an A-linear vector field (on the trivial A-module bundle $T^AM \times V$ over T^AM) with the underlying vector field $\mathcal{T}^A\underline{\varphi}(Y_1, \dots, Y_p)$, too. Thus we have A-linear vector field $\mathcal{T}^{A,V}\varphi(Y_1, \dots, Y_p) := \mathcal{T}^A\varphi(Y_1, \dots, Y_p) \otimes_A (\mathcal{T}^A\underline{\varphi}(Y_1, \dots, Y_p) \times 0)$ on $T^{A,V}E = T^AE \otimes_A V$, defined similarly as tensor product of linear vector fields covering some vector field. (More precisely, its flow is the tensor product over A of the flows of $\mathcal{T}^A\varphi(Y_1, \dots, Y_p)$ and $\mathcal{T}^A\underline{\varphi}(Y_1, \dots, Y_p) \times 0$.) Consequently, we have semibasic (A-)linear tangent valued p-form $\mathcal{T}^{A,V}\varphi : T^{A,V}E \to \wedge^p T^*T^AM \otimes TT^{A,V}E$ on $T^{A,V}E = T^AE \otimes_A V \to T^AM$.

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