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Substructures of Algebras with Weakly non-Negative Tits Form

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Abstract: Let $A = kQ/I$ be a finite dimensional basic algebra over an algebraically closed field k presented by its quiver Q with relations I. A fundamental problem in the representation theory of algebras is to decide whether or not A is of tame or wild type. In this paper we consider triangular algebras A whose quiver Q has no oriented paths. We say that A is essentially sincere if there is an indecomposable (finite dimensional) A-module whose support contains all extreme vertices of Q. We prove that if A is an essentially sincere strongly simply connected algebra with weakly non-negative Tits form and not accepting a convex subcategory which is either representation-infinite tilted algebra of type $\tilde{E}_p$ or a tubular algebra, then A is of polynomial growth (hence of tame type).

Key words: tame representation type, essentially sincere module, Tits form, strongly simply connected algebra.

AMS *Subject Class.* (2000): 16G20, 16G60, 16G70.

Let A be a finite dimensional algebra (associative with unity) over an algebraically closed field k. We may assume that A has a presentation $A \cong kQ/I$ where kQ is the path algebra of the Gabriel quiver $Q = Q_A$ of A and I is an admissible ideal of kQ. Equivalently, $A = kQ/I$ may be considered as a k-category with objects the vertices of Q and the space of morphism $A(x, y)$ from x to y as the quotient of the space $kQ(x, y)$, generated by the paths from x to y, by the subspace $I(x, y) = kQ(x, y) \cap I$. We denote by $\operatorname{mod} A$ the category of finite dimensional right A-modules. For basic background from representation theory of algebras we refer to [1, 4, 22, 23, 24].

From Drozd's Tame and Wild Dichotomy Theorem [10], algebras may be divided into two disjoint classes: the *tame algebras* for which indecomposable modules in each dimension occur (up to isomorphism) in a finite number of one-parametric families, and the *wild algebras* for which the representation

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theory comprises the representation theories of all algebras. One central question in the modern representation theory of algebras is the determination of the representation type.

Let $A = kQ/I$ be a *triangular* algebra, that is, Q has no oriented cycles. The *Tits form* $q_A : \mathbb{Z}^{Q_0} \to \mathbb{Z}$ is the quadratic form defined by

$$q_A(v) = \sum_{i \in Q_0} v(i)^2 - \sum_{i \to j} v(i)v(j) + \sum_{i,j} r(i,j)v(i)v(j)\,,$$

where $r(i,j)$ is the cardinality of $R \cap I(i,j)$ for a minimal set of generators $R \subset \bigcup_{i,j} I(i,j)$ of I. The Tits form plays an important role in the problem of determining the representation type of A. Indeed, if A is representation-finite (that is, A accepts, up to isomorphism, only finitely many indecomposable modules), then q_A is weakly positive, that is, $q_A(v) > 0$ for $0 \neq v \in \mathbb{N}^{Q_0}$ [5]. More generally, if A is tame, then q_A is weakly non-negative, that is, $q_A(v) \geq 0$ for $v \in \mathbb{N}^{Q_0}$ [15]. The converse implications have been shown for important families of algebras, satisfying some rigidity conditions (see for example [5, 6]), or algebras of small homological dimensions [3, 9, 11, 12, 15, 21, 28].

A throughly studied class of tame algebras are the strongly simply connected algebras. We recall that A is said to be *strongly simply connected* if, for every convex subcategory B of A, the first Hochschild cohomology group $H^1(B)$ vanishes, [26]. The modules over polynomial growth strongly simply connected algebras have been completely described [27] (see also [13] and [16]) and the critical tame strongly simply connected algebras of non-polynomial type have been classified [14]. It is a long standing conjecture that a strongly simply connected algebra A is tame if and only if q_A is weakly non-negative. The present paper answers positively the conjecture in a special case, generalizing previous results by the authors [17, 19]. This special case is shown to be essential for the solution of the conjecture as presented in [7].

We say that a strongly simply connected algebra $A = kQ/I$ is *essentially sincere* if there is an indecomposable (finite dimensional) A-module X whose support $\operatorname{supp} X = \{i \in Q_0 : X(i) \neq 0\}$ contains all extreme vertices (sinks and sources) of Q. Observe that a strongly simply connected algebra A is tame if and only if every convex subcategory B of A which is essentially sincere is tame. The main result of the paper is the following:

THEOREM. *Let A be a triangular algebra satisfying the following conditions:*

(a) *A is essentially sincere strongly simply connected;*

(b) *q_A is weakly non-negative;*

(c) *A contains a convex subcategory which is either representation-infinite tilted algebra of type $\tilde{\mathbb{E}}_p$ ($p = 6$, 7 or 8) or a tubular algebra.*

Then A is either a tilted algebra or a coil algebra. In particular, A is of polynomial growth, hence it is tame.

The paper is organized as follows. In Section 1 we present some remarks on *essentially present* modules, that is, indecomposable modules X such that $\operatorname{supp} X$ contains all the extreme vertices of the quiver of the algebra. In Section 2 we recall concepts and results needed for the proof of the Theorem. The proof presented in Section 3 depends heavely on the arguments given in [17, 19].

1. Essentially present modules

1.1. Let $A = kQ/I$ be a finite dimensional k-algebra. For each vertex $i \in Q_0$, we denote by e_i the corresponding primitive idempotent of A, hence $P_i = e_iA$ is the projective cover of the simple module $S_i = e_iA/e_i \operatorname{rad} A$ and $I_i = DAe_i$ the injective envelope of S_i. By $D = \operatorname{Hom}_k(-, k)$ we denote the usual duality on $\operatorname{mod} A$.

For a module $X \in \operatorname{mod} A$, $i \in \operatorname{supp} X$ if $\operatorname{Hom}_A(P_i, X) \neq 0$ (equivalently, $\operatorname{Hom}_A(X, I_i) \neq 0$). We say that X is *omnipresent* (resp. *essentially present*) if $\operatorname{supp} X = Q_0$ (resp. each source or sink in Q belongs to $\operatorname{supp} X$). Clearly, X is essentially present if and only if for every simple projective A-module S we have $\operatorname{Hom}_A(S, X) \neq 0$ and for every simple injective A-module T we have $\operatorname{Hom}_A(X, T) \neq 0$.

We consider the Grothendieck group $K_0(A) = \mathbb{Z}^{Q_0}$ and the classes $\mathbf{dim}\, X = (\dim_k X(i))_{i \in Q_0}$ of modules $X \in \operatorname{mod} A$. We recall that the *homological form* defined by Ringel [22] for algebras A of finite global dimension is given by

$$\langle \mathbf{dim}\, X, \mathbf{dim}\, Y \rangle_A = \sum_{s=0}^{\infty} (-1)^s \dim_k \operatorname{Ext}_A^s(X, Y)\,.$$

1.2. We denote by Γ_A the Auslander-Reiten quiver of A with translation $\tau_A = D\mathrm{Tr}$. By a *component* of Γ_A we mean a connected component. The structure of preprojective, preinjective and tubular components may be seen in [1, 22, 23, 24]. A *path* in $\operatorname{mod} A$ is a sequence $X_0 \to X_1 \to \cdots \to X_t$ of non-zero non-isomorphisms between indecomposable A-modules; it is a cycle if X_0 and X_t are isomorphic.

We say that an indecomposable A-module X is *directing* if it does not belong any cycle in $\operatorname{mod} A$.

Given a component $\mathcal{C}$ of Γ_A we say that $\mathcal{C}$ is *convex* in $\operatorname{mod} A$ if any path $X_0 \to X_1 \to \cdots \to X_t$ in $\operatorname{mod} A$ with extremes X_0 and X_t in $\mathcal{C}$, has all $X_i \in \mathcal{C}$, $i = 1, \ldots, t-1$. We shall consider also the *support* $\operatorname{supp} \mathcal{C} := \bigcup_{X \in \mathcal{C}} \operatorname{supp} X$ of $\mathcal{C}$.

Proposition. *Let $A = kQ/I$ be a triangular algebra and let X be an essentially present indecomposable A-module in a component $\mathcal{C}$ of Γ_A.*

(a) *If $i \in Q_0 \setminus \operatorname{supp} X$, then there is a cycle in* $\operatorname{mod} A$ *passing through X and S_i.*

(b) *If $\mathcal{C}$ is convex in* $\operatorname{mod} A$, *then* $\operatorname{supp} \mathcal{C} = Q_0$.

Proof. (Following [5]) (a) Assume $i \notin Q_0 \setminus \operatorname{supp} X$. Since X is essentially present and A is triangular, there is a path γ of the form $i_0 \to i_1 \to \cdots \to i_s$ in Q with $i_0, i_s \in \operatorname{supp} X$ and $i_1, \ldots, i_{s-1} \notin \operatorname{supp} X$ with $i = i_t$ for some $1 \le t \le s-1$. Let $\bar{A}$ be the quotient of A by all paths $x \xrightarrow{\alpha} y \xrightarrow{\beta} z$ with exactly one arrow in γ. Then there is a cycle in $\operatorname{mod} \bar{A}$

$$X \to \bar{I}_{i_s} \to S_{i_s} \to \begin{pmatrix} i_{s-1} \\ i_s \end{pmatrix} \to S_{i_{s-1}} \to \cdots \to \begin{pmatrix} i_1 \\ i_2 \end{pmatrix} \to S_{i_1} \to \bar{P}_{i_0} \to X$$

where $\bar{P}_x$ (resp. $\bar{I}_x$) denotes the indecomposable projective (resp. injective) $\bar{A}$-module associated to x and $\begin{pmatrix} x \\ y \end{pmatrix}$ is the indecomposable module of dimension two with socle S_y and top S_x.

(b) Since $X \in \mathcal{C}$, by (a), for every $i \notin Q_0 \setminus \operatorname{supp} X$, the simple module S_i belongs to $\mathcal{C}$. Hence $\operatorname{supp} \mathcal{C} = Q_0$. ∎

1.3. We recall that an algebra A is *tame* [10] if, for each $d \in \mathbb{N}$, there is a finite number of $k[t]-A$-bimodules M_i, $1 \le i \le n_d$, which are finitely generated free as left $k[t]$-modules and such that all but finitely many isoclasses of indecomposable A-modules of dimension d are of the form $k[t]/(t-\lambda) \otimes_{k[t]} M_i$ for some i and some $\lambda \in k$. Let $\mu_A(d)$ be the minimal n_d in the definition. Then A is said to be of *polynomial growth* [25] if there is a number m such that $\mu_A(d) \le d^m$ for every $d \ge 1$.

The following proposition on the behaviour of the Auslander-Reiten components of strongly simply connected algebras of polynomial growth has been proved in [27, Theorem 4.1].

Proposition. *Let A be a strongly simply connected algebra of polynomial growth. Then every component of Γ_A is convex in* $\operatorname{mod} A$.

1.4. A useful construction is the *one-point extension* $B[M]$ of an algebra B by a B-module M, given as the matrix algebra

$$B[M] = \begin{pmatrix} k & M_B \\ 0 & B \end{pmatrix}.$$

One-point coextensions $[M]B$ are defined dually. The following extension of a result in [17] yields necessary conditions for an algebra to be essentially sincere.

Splitting Lemma. *Let A be a triangular algebra and $B = B_0, B_1, \ldots, B_s = A$ a family of convex subcategories of A such that, for each $0 \le i \le s$ with $B_{i+1} = B_i[M_i]$ or $B_{i+1} = [M_i]B_i$ for some indecomposable B_i-module M_i. Assume that the category of indecomposable B-modules admits a splitting* $\operatorname{ind} B = \mathcal{P} \vee \mathcal{J}$, *where $\mathcal{P}$ and $\mathcal{J}$ are full subcategories of* $\operatorname{ind} B$ *satisfying the following conditions:*

(S1) $\operatorname{Hom}_B(\mathcal{J}, \mathcal{P}) = 0$;

(S2) *for each i such that $B_{i+1} = B_i[M_i]$, the restriction $M_{i|B}$ belongs to* $\operatorname{add} \mathcal{J}$;

(S3) *for each i such that $B_{i+1} = [M_i]B_i$, $M_{i|B}$ belongs to* $\operatorname{add} \mathcal{P}$;

(S4) *there is an index i with $B_{i+1} = B_i[M_i]$ and $M_i \in \mathcal{J}$ and an index j with $B_{j+1} = [M_j]B_j$ and $M_j \in \mathcal{P}$.*

Then A is not essentially sincere.

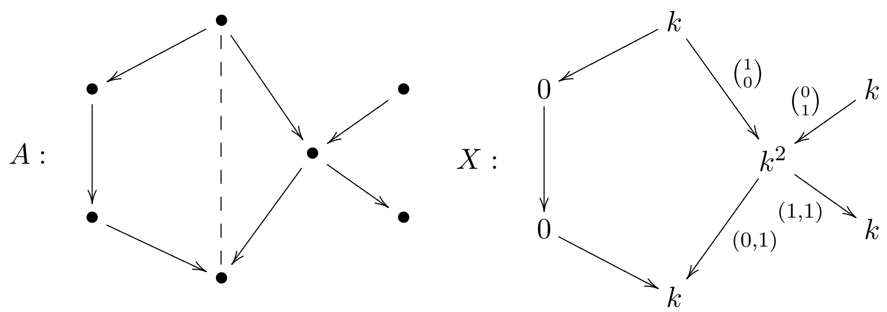
Proof. Let $x_1, \ldots, x_r$ (resp. $y_1, \ldots, y_t$) be those vertices at the quiver Q of A being sources (resp. targets) or arrows with target (resp. source) in B. For each i, denote by B_i^+ the maximal convex subcategory of B_i not containing any $y_1, \ldots, y_t$ (resp. $x_1, \ldots, x_r$). Let $\mathcal{P}_i$ (resp. $\mathcal{J}_i$) be the full subcategory of $\operatorname{ind} B_i^-$ (resp. of $\operatorname{ind} B_i^+$) consisting of modules X such that $X|_B \in \operatorname{add} \mathcal{P}_i$ (resp. $X|_B \in \operatorname{add} \mathcal{J}_i$). We claim that $\operatorname{ind} B_i = \mathcal{P}_i \vee \mathcal{J}_i$ and $\operatorname{Hom}_{B_i}(\mathcal{J}_i, \mathcal{P}_i) = 0$. The proof of the claim follows by induction as in [17, page 1022].

We get that $\operatorname{ind} A = \mathcal{P}_s \vee \mathcal{J}_s$ with $\operatorname{Hom}_A(\mathcal{J}_s, \mathcal{P}_s) = 0$, $\mathcal{P}_s$ consists of B_s^+-modules and $\mathcal{J}_s$ consists of B_s^--modules. Moreover, by (S4), $B \neq B_s^+$ and $B \neq B_s^-$. Let $X \in \mathcal{P}_s$ and let y be a sink in Q which is a successor of y_1. Since B_s^+ is convex in A, then y is not in B_s^+, hence $X(y) = 0$. That is, X is not essentially present. Similarly, any module $Y \in \mathcal{J}_s$ is not essentially present. We conclude that A is not essentially sincere. ∎

Observe that, for a strongly simply connected algebra A and a convex subcategory B of A, there exists a chain $B = B_0, B_1, \dots, B_s = A$ of convex subcategories of A such that $B_{i+1} = B_i[M_i]$ or $B_{i+1} = [M_i]B_i$ for some indecomposable B_i-module M_i (see [17, Proposition 2.2]).

1.5. The following are typical *examples* of strongly simply connected algebras A and essentially present (not omnipresent) indecomposable A-modules X. (We indicate, the relations in A by dotted edges: given $i - - j$, the sum of all paths from i to j in Q is zero).

(1)



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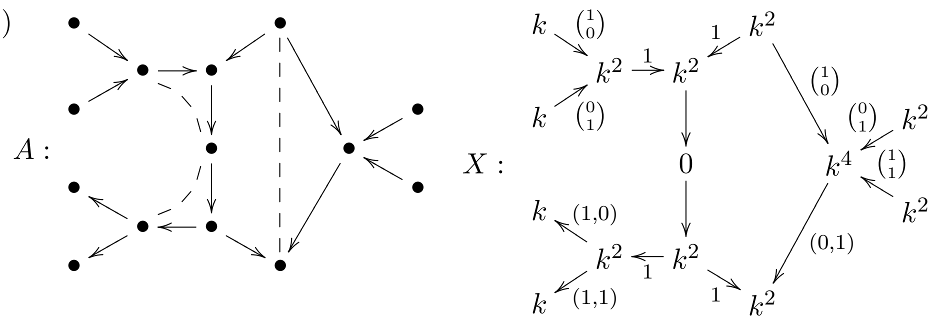


We note that, in the first case, A is a tame concealed algebra, and hence is of polynomial growth.

On the other hand, in the second case, A is a tame algebra of non-polynomial growth and there is an infinite family of pairwise nonisomorphic indecomposable A-modules $(Y_\lambda)_{\lambda \in k}$ with $\mathbf{dim}\, Y_\lambda = \mathbf{dim}\, X$.

Proposition. *Let A be a strongly simply connected algebra. Assume $v \in \mathbb{N}^{Q_0}$ is an essentially present vector which is not omnipresent and such that there exists an infinite family $(Y_\lambda)_\lambda$ of pairwise nonisomorphic indecomposable A-modules with $\mathbf{dim}\, Y_\lambda = v$. Then A is not of polynomial growth.*

Proof. Assume that A is tame of polynomial growth. Since A is tame, by a result of Crawley-Boevey [8], some module Y in the family $(Y_\lambda)_\lambda$ satisfies $\tau_A Y \cong Y$, and hence lies in a stable tube $\mathcal{C}$ of rank one in Γ_A. Further, since A is of polynomial growth, applying 1.3, we conclude that $\mathcal{C}$ is convex in $\operatorname{mod} A$. Hence, applying 1.2, we obtain $\operatorname{supp} \mathcal{C} = Q_0$. Finally, since every module $X \in \mathcal{C}$ has $\mathbf{dim}\, X = qv$ for certain rational number $q > 0$, we conclude that the vector v is omnipresent, a contradiction. ∎

2. Algebras of polynomial growth

2.1. Let C be a tame concealed algebra, that is, $A = \operatorname{End}_H(T)$ for a preprojective tilting module T over a tame hereditary algebra H, and let $(\mathcal{T}_\lambda)_{\lambda\in\mathbb{P}_1(k)}$ be the unique family of stable tubes in Γ_C. Let $E = (E_1, \ldots, E_s)$ be a sequence of pairwise non-isomorphic C-modules which are simple among the regular modules and a family $K = (K_1, \ldots, K_s)$ of branches. In [22], the *tubular extension* $B = C[E, K](= C[E_i, K_i]_{i=1}^s)$ is defined and has tubular type $n_B = (n_\lambda)_\lambda$ with $n_\lambda = \operatorname{rank} \mathcal{T}_\lambda + \sum_{E_i\in\mathcal{T}_\lambda} |K_i|$. Since almost all $n_\lambda = 1$, we write instead of $n_B = (n_\lambda)_\lambda$ the finite sequence consisting of at least two n_λ, keeping those which are larger than 1, and arranged in non-decreasing order. We recall that B is a *domestic tubular* (resp. *tubular*) algebra if n_B is (p, q), $1 \le p \le q$, $(2, 2, r)$, $2 \le r$, $(2, 3, 3)$, $(2, 3, 4)$, $(2, 3, 5)$ (resp. $(3, 3, 3)$, $(2, 4, 4)$, $(2, 3, 6)$ or $(2, 2, 2, 2)$).

The following fact is well known (see [15, 22]).

Proposition. *Let B be a tubular extension of a tame concealed algebra C. Then the following statements are equivalent:*

(a) *B is tame;*
(b) *B is domestic tubular or a tubular algebra;*
(c) *q_B is weakly non-negative.*

2.2. For the definitions of *admissible operations* and the construction of *coils*, we refer the reader to [2, 3].

Following [3], an algebra B is said to be a *coil enlargement* of a tame concealed algebra C if there is a finite sequence of algebras $C = B_0, B_1, \ldots, B_m = B$ such that B_{j+1} is obtained from B_j by an admissible operations (ad 1), (ad 2) or (ad 3) (resp. (ad 1*), (ad 2*), (ad 3*)) with a pivot (resp. a copivot) on a stable tube of Γ_C or in a component of Γ_{B_j} obtained from a stable tube of Γ_C by a sequence of admissible operations done so far. By a *coil algebra* we mean a tame strongly simply connected algebra obtained as a coil enlargement of a tame concealed algebra.

The following structure result has been proved in [3].

PROPOSITION. *Let B be a coil enlargement of a tame concealed algebra C. Then:*

(a) *There exists a unique maximal tubular extension B^+ of C which is a convex subcategory of B such that B is obtained from B^+ as a sequence of algebras $B^+ = B_0, B_1, \dots, B_m = B$ such that B_{j+1} is obtained from B_j by an admissible operation* (ad 1*), (ad 2*) *or* (ad 3*) *with a copivot on a coil component of Γ_{B_j}.*
(b) *There exists a unique maximal tubular coextension B^- of C which is a convex subcategory of B such that B is obtained from B^- as a sequence of algebras $B^- = B_0, B_1, \dots, B_n = B$ such that B_{j+1} is obtained from B_j by an admissible operation* (ad 1), (ad 2) *or* (ad 3) *with a pivot on a coil component of Γ_{B_j}.*
(c) *There is a splitting $\operatorname{ind} B = \mathcal{P} \vee \mathcal{J}$, where $\mathcal{P}$ is formed by components of Γ_{B^-} and some coils obtained by admissible operations as in* (b), *and $\mathcal{J}$ is formed by components of Γ_{B^+}. The splitting satisfies conditions* (S1), (S2) *and* (S3) *in* 1.4. *It satisfies* (S4) *if and only if B^+ is a proper subcategory of B (equivalently B^- is a proper subcategory of B).*
(d) *B is tame if and only if B^+ and B^- are tame.*

As a consequence of the splitting of $\operatorname{ind} B$ for a coil enlargement B of a tame concealed algebra, we get the following result [27, 18].

PROPOSITION 2.3. *Let A be a polynomial growth strongly simply connected algebra and X be a essentially present indecomposable A-module. Then one of the following situations occur:*

(a) *A is a tilted algebra of tame representation type, X is a directing module and $q_A(\mathbf{dim}\, X) = 1$.*
(b) *A is a coil algebra and X belongs to a coil component of Γ_A.*

3. The proof of the Theorem

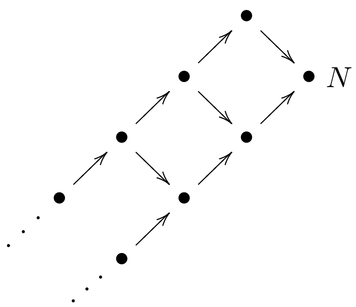
We start with some technical considerations.

Proposition 3.1. *Let A be an essentially sincere strongly simply connected algebra such that q_A is weakly non-negative. Let $B \subset D = [X]B$ be two convex subcategories of A such that B is a coil enlargement of a tame concealed algebra C and X is an indecomposable module lying on a coil Γ of Γ_B such that $\mathrm{Hom}_B(Z, X) \neq 0$ for a non-directing Z in Γ. Then D is either a coil algebra or B^- is a tilted algebra of type $\tilde{\mathbb{D}}_n$ with an indecomposable Y in the preprojective component of Γ_{B^-} satisfying $\dim_k \mathrm{Hom}_B(Y, X) = 2$.*

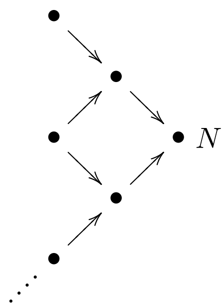
Proof. Let $F = B^-$ and N be the restriction of X to F. Then $[N]F$ is a convex subcategory of $D = [X]B$. By 2.1, F is a domestic tubular or a tubular algebra which is a tubular extension of C. Assume, in order to ge a contradiction, that F is a tubular algebra. Then X belongs to the inserted family of coils in Γ_F. If X is copivoting, then $D = [X]B$ is a coil algebra. Suppose now that X is not copivoting. We distinguish two situations.

Assume first that the support of $\mathrm{Hom}_F(-, N)|_{\mathcal{T}}$ contains the k-linear category of a subquiver $\mathcal{S}$ of the component $\mathcal{T}$ of Γ_F with $N \in \mathcal{T}$, where $\mathcal{S}$ has the shape (1).

(1)



(2)



In this case, F is a tubular extension of the tame concealed algebra C of type $\tilde{\mathbb{D}}_n$. Then there is a component $\mathcal{T}' \neq \mathcal{T}$ of Γ_F containing projective modules. A simple application of the Splitting Lemma implies that A is not essentially sincere, a contradiction. Since X is not copivoting, then $\mathrm{supp}\,\mathrm{Hom}_F(-, N)|_{\mathcal{T}}$ contains a k-linear category of a poset of type (2). If C is of type $\tilde{\mathbb{D}}_n$ we obtain a contradiction as above. Otherwise, $\mathrm{Hom}_F(\mathrm{mod}\, F, N)$

contains a full subcategory given by a poset

$$\begin{array}{cccc} & & & \mathrm{Hom}_F(Z_5,N) \\ & & & \downarrow \\ \mathrm{Hom}_F(Z_1,N) & \mathrm{Hom}_F(Z_2,N) & \mathrm{Hom}_F(Z_3,N) & \mathrm{Hom}_F(Z_4,N) \end{array}$$

of type $(1,1,1,2)$ where Z_1, Z_2 lie in $\mathcal{T}$ and Z_3, Z_4, Z_5 lie in the preprojective component of Γ_F. Considering the coextension vertex t of $[N]F$, and the vector

$$v = 4e_t + 2\sum_{i=1}^{4} \mathbf{dim}\, Z_i + \mathbf{dim}\, Z_5 \in K_0([N]F)$$

evaluating the Tits form $q_{[N]F}$ at v (using that $\operatorname{gldim} F \le 2$) we get

$$\begin{aligned} q_{[N]F}(v) &= \langle v, v\rangle_F + 8\sum_{i=1}^{4} \dim_k \operatorname{Ext}^3_{[N]F}(Z_i, S_t) + 4\dim_k \operatorname{Ext}^3_{[N]F}(Z_5, S_t) \\ &= -1 + 8\sum_{i=1}^{4} \dim_k \operatorname{Ext}^2_F(Z_i, N) + 4\dim_k \operatorname{Ext}^2_F(Z_5, N) = -1\,. \end{aligned}$$

The last equality due to the fact that $\operatorname{pdim}_F Z_i \le 1$ for $i = 3,4,5$ and $\operatorname{Ext}^2_F(Z_i, N) = 0$, $i = 1,2$, from the structure of $\mathcal{T}$. This contradicts the weak non-negativity of q_A and shows that F is tilted of type $\tilde{\mathbb{D}}_n$ or $\tilde{\mathbb{E}}_p$ ($p = 6$, 7 or 8).

If X is copivoting, then the vector space category $\mathrm{Hom}_B(\operatorname{mod} B, X)$ is tame. Indeed, if it is not linear, say $\dim_k \mathrm{Hom}_B(M, X) \ge 2$ for an indecomposable B-module M, then every object $Y \in \Gamma_B$ is comparable with X (that is, there is $0 \ne f \in \mathrm{Hom}_B(X, Y)$ or $0 \ne f \in \mathrm{Hom}_B(Y, X)$ with $\mathrm{Hom}_B(f, X) \ne 0$). Then F is tilted of type $\tilde{\mathbb{D}}_n$ and M is preprojective in Γ_F. Assume $\mathrm{Hom}_B(\operatorname{mod} B, X)$ is linear.

If it is not of tame type, then it contains a full subposet L belonging to the Nazarova's list. We identify each point $a \in L$ with an indecomposable X_a in the preprojective component $\mathcal{P}$ of Γ_F. Moreover, since the orbit graph of $\mathcal{P}$ is a tree (since A is strongly simply connected), we may choose L such that any subchain H yields a sectional path in $\mathcal{P}$. Let v be a positive vector such that $\chi_L(v) = -1$ for the graphical form χ_L associated to L (see [22]). Then using that $\operatorname{gldim} D \le 2$ we get

$$q_D\left(\sum_{a\in L} v(a)\,\mathbf{dim}\, X_a + v(w)e_t\right) = \chi_L(v) = -1\,,$$

for t the extension vertex of D such that $I_t/\operatorname{soc} I_t = X$. This leads to a contradiction with the weak non-negativity of q_A, showing that $\operatorname{Hom}_B(\operatorname{mod} B, X)$ is tame. Hence D is a tame coil enlargement of C.

If X is not copivoting, then $\operatorname{supp}\operatorname{Hom}_B(-,X)|_\Gamma$ contains one of the posets (1) or (2).

In the first case, as above, $F = B^-$ is of type $\tilde{\mathbb{D}}_n$. In the second case, if F is not of type $\tilde{\mathbb{D}}_n$ we find a full subposet of $\operatorname{Hom}_F(\operatorname{mod} F, X)$ of type $(1,1,1,2)$ and, as above, we get a contradiction against the weak non-negativity of q_A. In both cases, there is a preprojective module Y in Γ_F with $\dim_k \operatorname{Hom}_F(Y,X) = 2$. ∎

Proposition 3.2. *Let A be an essentially sincere strongly simply connected algebra with q_A weakly non-negative. Let B be a convex subcategory of A satisfying the following conditions:*

- (i) *B is a representation-infinite tilted algebra of type $\tilde{\mathbb{E}}_p$ ($p = 6$, 7 or 8) having a complete slice in its preinjective component;*
- (ii) *A admits not a convex subcategory of the form $[N]B$ for any indecomposable B-module N;*
- (iii) *for any convex subcategory $B[M]$ of A, M is an indecomposable preinjective B-module.*

Then A is a tame tilted algebra.

Proof. We know that Γ_B consists of a preprojective component $\mathcal{P}$, a family $\mathcal{T}_\lambda$ of inserted tubes and a preinjective component $\mathcal{J}$ having a section of type $\tilde{\mathbb{E}}_p$. We may choose Σ a section of $\mathcal{J}$ such that any indecomposable M such that $B[M]$ is a convex subcategory of A, is a successor of Σ (in order of paths in $\mathcal{J}$).

Choose a sequence of categories $B = B_0, B_1, \dots, B_s = A$ such that $B_{j+1} = B_j[M_j]$ or $B_{j+1} = [M_j]B_j$ for an indecomposable B_j-module M_j. We claim that for each j, there is a component $\mathcal{C}_j$ in Γ_{B_j} satisfying:

(a) $\mathcal{C}_j$ is a directing component (that is, $\mathcal{C}_j$ is convex in $\operatorname{mod} B_j$ and without cycles);

(b) $\mathcal{C}_j$ has a complete slice Σ_j which is a tree.

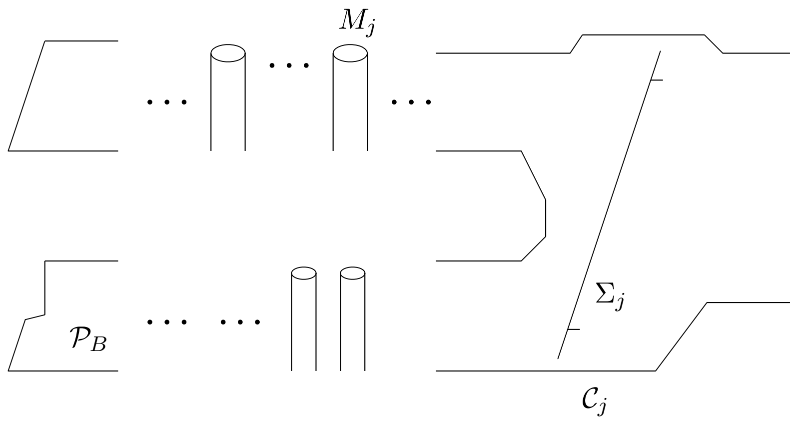
In particular, this shows that $A = B_s$ is a tilted algebra. Then, by [11], A is tame.

Indeed, $\mathcal{C}_0 = \mathcal{J}$ and $\Sigma_0 = \Sigma$. Assume $\mathcal{C}_j$ is a directing component of Γ_{B_j} with a complete slice Σ_j such that any indecomposable M, such that $B[M]$

is a convex subcategory of A, is a successor of Σ_j –maybe not in $\mathcal{C}_j$ (observe that Σ_j may be selected this way as an application of the Splitting Lemma).

Suppose $B_{j+1} = B_j[M_j]$ for an indecomposable. We claim that $M_j \in \mathcal{C}_j$. Otherwise by the Splitting Lemma, there are no injective modules in $\mathcal{C}_j$. Since q_{B_j} is weakly non-negative, then Σ_j is of extended Dynkin type and $j = 0$. In that case $\mathcal{C}_0 = \mathcal{J}$ is a preinjective component, a contradiction showing that $M_j \in \mathcal{C}_j$. By [20], M_j lies in a directing component of $\Gamma_{B_{j+1}}$ with a (complete) slice Σ_{j+1} which is a tree (extending Σ_j).

Suppose $B_{j+1} = [M_j]B_j$. By hypothesis, we have $M_{j|B} = 0$. If $M_j \notin \mathcal{C}_j$, then the Splitting Lemma implies that A is not essentially sincere as illustrated in the following picture:



Hence $M_j \in \mathcal{C}_j$ and there should exists Σ_j preceeding M_j (apply Splitting Lemma again!). Then M_j belongs to a directing component $\mathcal{C}_{j+1}$ of $\Gamma_{B_{j+1}}$ with a complete slice Σ_{j+1}. ∎

The case complementary Proposition 3.2 goes as follows:

Proposition 3.3. *Let A be an essentially sincere strongly simply connected algebra with q_A weakly non-negative. Assume A contains a full convex subcategory B satisfying the conditions:*

(i) *B is either a representation-infinite algebra of type $\tilde{\mathbb{E}}_p$ ($p = 6$, 7 or 8) with a complete slice in the preinjective component and some projective outside the preprojective component or B is a tubular algebra;*

(ii) *there is a convex subcategory A of the form $[N]B$ for some indecomposable B-module N.*

Then A is a coil algebra.

Proof. Choose B maximal satisfying (i) and (ii). Let D be a maximal coil enlargement of B in A. We want to prove that $D = A$.

Let $\Gamma_D = \mathcal{P}_\infty \vee \mathcal{C} \vee \mathcal{J}_0$ where $\mathcal{J}_0$ is the preinjective component of B, $\mathcal{C} = (\mathcal{C}_\lambda)_\lambda$ is a family of coils such that, for certain λ_0, $\mathcal{C}_{\lambda_0}$ contains a projective module and $\mathcal{P}_\infty$ is formed by D^--modules. By Proposition 2.3, D^- is a tilted algebra or a tubular algebra.

Observe that the maximality of B implies that $N \notin \mathcal{J}_0$. Hence $N \in \mathcal{C}$. The Splitting Lemma implies that $\mathcal{C}_{\lambda_0}$ is the only component in $\mathcal{C}$ that may contains projective or injective modules, and in fact contains both types (in particular, $N \in \mathcal{C}_{\lambda_0}$). If D is properly contained in A, then there is a convex subcategory D' of D of the form $D[X]$ or $[X]D$ for an indecomposable D-module X. Maximality of B and the Splitting Lemma imply that $X \in \mathcal{C}_{\lambda_0}$. Since $q_{D'}$ is weakly non-negative, D' is a coil algebra by Proposition 3.1. Then $D' \subset D$ which is a contradiction. Therefore, $A = D$ is a coil algebra. ∎

Proof of the Theorem. We may assume that A admits a maximal proper convex subcategory B which is a tubular extension of a tame concealed algebra C and such that B is either a tubular algebra or a representation-infinite tilted algebra of type $\tilde{\mathbb{E}}_p$ ($p = 6$, 7 or 8) having a complete slice in its preinjective component. Therefore, for any convex subcategory of A of the form $B[M]$, M is a preinjective B-module, since $q_{B[M]}$ is weakly non-negative, M is not preprojective, and the maximality of B and Proposition 3.1 imply that M is not in a coil component). Hence the Splitting Lemma implies that B is not a tubular algebra.

By the maximality of B we may assume that either the hypothesis of Proposition 3.2 or those of Proposition 3.3 hold. Then either A is a tilted algebra or a coil algebra. ∎

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