Uniqueness of Invariant Hahn-Banach Extensions

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Abstract: Let ℓ be a linear functional on a subspace Y of a real linear space X provided with a sublinear functional p with $\ell \leq p$ on Y. If $\mathcal{G}$ is an abelian semigroup of linear transformations $T : X \to X$ such that $T(Y) \subseteq Y$, $p(Tx) \leq p(x)$ and $\ell(Ty) = \ell(y)$ for all $T \in \mathcal{G}$, $x \in X$ and $y \in Y$ respectively, then a generalization of the classical Hahn-Banach theorem asserts that there exists an extension $\widetilde{\ell}$ of ℓ, $\widetilde{\ell} \leq p$ on X and $\widetilde{\ell}$ remains invariant under $\mathcal{G}$. The present paper investigates various equivalent conditions for the uniqueness of such extensions and these are related to nested sequences of p-balls, a concept that has proved useful in recent years in dealing with such extensions. The results are illustrated by a variety of examples and applications.

Key words: Sublinear functionals, nested sequences of (p-) balls, invariant Hahn-Banach extensions.

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1. Introduction

Let p be a sublinear functional defined on a linear space X over $\mathbb{R}$ and Y a subspace with a linear functional ℓ such that $\ell(y) \leq p(y)$ for all $y \in Y$. Let $\mathcal{G}$ be an *abelian semigroup* of *linear* transformations $T : X \to X$ such that $T(Y) \subseteq Y$ and $p(Tx) \leq p(x)$ for all $T \in \mathcal{G}$ and all $x \in X$. We may assume without loss of generality that $\mathcal{G}$ contains the identity operator I. Suppose furthermore that

$$\ell(Ty) = \ell(y)$$

for all $y \in Y$ and all $T \in \mathcal{G}$. If $\mathcal{G}$ consists only of I, we are reduced to the setting of the classical Hahn-Banach (H-B for short) theorem. Under the above circumstances, the authors in [1] prove that there exists a linear extension $\widetilde{\ell} : X \to X$ of ℓ, which we shall henceforth call an invariant H-B extension, with

$$\widetilde{\ell}(Tx) = \widetilde{\ell}(x), \qquad \widetilde{\ell}(x) \leq p(x)$$

for all $x \in X$ and $T \in \mathcal{G}$. This generalization of the H-B theorem is, according to [8], "both beautiful and useful". The usefulness of the result stems from

the fact that it has been employed to discuss the existence of Banach limits [4, 6], the existence of finitely additive measures defined on the σ-algebra of all subsets of the unit circle invariant under rotations [8], the possibility of invariant norm-preserving extensions of continuous linear functionals defined on subspaces of Banach spaces [11] and etc.

The main concern of the present paper is to investigate conditions for uniqueness of such invariant extensions analogous to what was done in [3] in the setting of the standard H-B theorem. Some relevant references for this invariant extension are [7, 12, 13] where, however, the uniqueness question is not addressed.

An important concept, first defined in [9] and extensively used here, is that of a *nested sequence* of p-balls:

DEFINITION 1.1. For $x_0 \in X$ and $r > 0$, define the open p-ball of radius r around x_0 by $B_p(x_0, r) = \{x \in X : p(x_0 - x) < r\}$.

A nested sequence of p-balls is a sequence $\{B_n = B_p(x_n, r_n)\}$ of open p-balls in X such that for all $n \geq 1$, $B_n \subseteq B_{n+1}$ and $r_n \uparrow \infty$.

This was employed in [9] to characterize U-subspaces (for the definition, see [3] where various generalizations of the result in [9] are given). The necessary and sufficient conditions we give for the uniqueness of invariant extensions largely involve nested sequences of p-balls and as such, the present paper may be regarded as a continuation of [3] to which the reader is referred for unexplained notations, terminologies and results used here.

Section 2 contains a discussion of some preliminary results culminating in the main theorems. Section 3 contains a fairly liberal sprinkling of illustrative examples and applications.

2. GENERAL RESULTS

Let us sketch briefly the proof of the invariant H-B extension mentioned in the Introduction.

If $\mathcal{G}$ is the given semigroup of transformations with $I \in \mathcal{G}$, let

$$\mathcal{C} =: \text{conv}(\mathcal{G}) = \left\{ \sum_{i=0}^{n} a_i T_i : n \geq 1,\ T_i \in \mathcal{G},\ a_i \geq 0,\ \sum_{i=0}^{n} a_i = 1 \right\}.$$

Note that $S \circ T = T \circ S \in \mathcal{C}$ for $S, T \in \mathcal{C}$ and that $T(Y) \subseteq Y$ for $T \in \mathcal{C}$.

Moreover,

$$p(Sx) = p\Big(\sum_{i=0}^{n} a_i T_i x\Big) \le \sum_{i=0}^{n} a_i p(T_i x) \le \sum_{i=0}^{n} a_i p(x) = p(x)$$

for all $S \in \mathcal{C}$ and $x \in X$. Following [4] (or [8]), a new sublinear functional $\widehat{p}$ is defined on X by

$$\widehat{p}(x) = \inf\{p(Tx) : T \in \mathcal{C}\} \tag{1}$$

and it is clear that $\widehat{p} \le p$ on X. Proof of subadditivity of $\widehat{p}$ uses the fact that $\mathcal{G}$ is abelian. Since $\ell(y) = \ell(Ty) \le p(Ty)$ for all $T \in \mathcal{C}$, one immediately has $\ell(y) \le \widehat{p}(y)$ for all $y \in Y$. Extend ℓ to $\widetilde{\ell}$ on X by the ordinary H-B theorem so that $\widetilde{\ell} \le \widehat{p}$. The great advantage of the definition (1) is that it gives

$$\widehat{p}(x - Tx) \le 0 \quad \text{for all } T \in \mathcal{C}$$

from which one deduces that $\widetilde{\ell}(x - Tx) \le 0$ and hence $\widetilde{\ell}(x) = \widetilde{\ell}(Tx)$ for all $x \in X$ and $T \in \mathcal{G}$, thus proving the invariance of $\widetilde{\ell}$. A fortiori, we have $\widetilde{\ell}(x) = \widetilde{\ell}(Tx)$ for all $T \in \mathcal{C}$. We emphasise again that $\widetilde{\ell} \le \widehat{p} \le p$ on X.

Conversely, if $\widetilde{\ell}$ is an invariant extension $\widetilde{\ell} \le p$, then $\widetilde{\ell}(Tx) = \widetilde{\ell}(x) \le p(Tx)$, from which it follows that $\widetilde{\ell} \le \widehat{p}$ on X.

We thus see that the uniqueness of the invariant extension is equivalent to the uniqueness of the extension dominated by the newly-defined sublinear functional $\widehat{p}$.

Let $Z = \ker(\ell) \subseteq Y$ and $y_0 \in Y$ be such that $\ell(y_0) = 1$. It follows that any $y \in Y$ can be written as $y = \alpha y_0 + z$ for some $\alpha \in \mathbb{R}$ and $z \in Z$ and $\ell(y) = \alpha$. Recall from [3, p 30-31] that a sublinear functional $\widetilde{p}$ is defined on the quotient space X/Z by

$$\widetilde{p}(x + Z) = \inf\{p(x + z) : z \in Z\}$$

$\widetilde{\widehat{p}}$ can be defined similarly. By [3, Lemma 2.6], ℓ has a unique extension from Y to X dominated by p if and only if ℓ has a unique extension from Y/Z to X/Z dominated by $\widetilde{p}$; and by the above observation, ℓ has a unique invariant extension from Y to X dominated by p if and only if ℓ has a unique extension from Y/Z to X/Z dominated by $\widetilde{\widehat{p}}$.

Since Z is also $\mathcal{C}$-invariant, for $T \in \mathcal{C}$, the map $\widetilde{T}(x + Z) = Tx + Z$ is well-defined on X/Z. And $\widetilde{\mathcal{C}} = \{\widetilde{T} : T \in \mathcal{C}\}$ is an abelian semigroup of linear transformations on X/Z. This gives rise to another sublinear functional $\widehat{\widetilde{p}}$ on X/Z, given by

$$\widehat{\widetilde{p}}(x + Z) = \inf\{\widetilde{p} \circ \widetilde{T}(x + Z) : T \in \mathcal{C}\}.$$

We now explore the relation between these sublinear functionals on X/Z.

LEMMA 2.1.   (a) $\widetilde{\widehat{p}}(x + Z) = \inf\{\widetilde{p \circ T}(x + Z) : T \in \mathcal{C}\}$.

(b) $\widetilde{p}(x + Z) \geq \widetilde{p \circ T}(x + Z) \geq \widetilde{p} \circ \widetilde{T}(x + Z)$ for all $T \in \mathcal{C}$.

(c) $\widetilde{\widehat{p}}(x + Z) \geq \widehat{\widetilde{p}}(x + Z)$.

(d) $\inf\{\widehat{\widetilde{p}} \circ \widetilde{T}(x + Z) : T \in \mathcal{C}\} = \widehat{\widetilde{p}}(x + Z)$.

Proof. (a)   Now,

$$\widetilde{\widehat{p}}(x + Z) = \inf_{z \in Z} \widehat{p}(x + z) = \inf_{z \in Z} \inf_{T \in \mathcal{C}}[p \circ T(x + z)]$$
$$= \inf_{T \in \mathcal{C}} \inf_{z \in Z}[p \circ T(x + z)] = \inf_{T \in \mathcal{C}}[\widetilde{p \circ T}(x + Z)].$$

(b)   Since $p \circ T \leq p$, we have $\widetilde{p \circ T} \leq \widetilde{p}$. And, since $TZ \subseteq Z$,

$$\widetilde{p \circ T}(x + Z) = \inf_{z \in Z}[p \circ T(x + z)] = \inf_{z \in Z}[p(Tx + Tz)]$$
$$\geq \inf_{z \in Z}[p(Tx + z)] = \widetilde{p}(Tx + Z) = \widetilde{p} \circ \widetilde{T}(x + Z).$$

(c)   By (a) and (b), $\widetilde{\widehat{p}}(x+Z) = \inf_{T \in \mathcal{C}}[\widetilde{p \circ T}(x+Z)] \geq \inf_{T \in \mathcal{C}} \widetilde{p} \circ \widetilde{T}(x+Z) = \widehat{\widetilde{p}}(x+Z)$.

(d)   Combining (a) and (b), we see that $\inf_{T \in \mathcal{C}}[\widetilde{\widehat{p}}(Tx + Z)] \geq \inf_{T \in \mathcal{C}} \widetilde{p} \circ \widetilde{T}(x + Z)$. On the other hand, from (a),

$$\widetilde{\widehat{p}}(Tx + Z) = \inf_{T \in \mathcal{C}}[\widetilde{p \circ T}(Tx + Z)]$$

and hence,

$$\inf_{T \in \mathcal{C}}[\widetilde{\widehat{p}}(Tx + Z)] = \inf_{T \in \mathcal{C}} \inf_{T \in \mathcal{C}}[\widetilde{p \circ T}(Tx + Z)] \leq \inf_{T \in \mathcal{C}}[\widetilde{p}(Tx + Z)]$$

on taking $T = I$, and we have the result.   ∎

Remark 2.2. Note that if $TZ = Z$ for all $T \in \mathcal{C}$, equality will hold in (b) and hence, by (a), we will get $\widetilde{\widehat{p}} = \widehat{\widetilde{p}}$.

The following simple observation will be needed in the sequel.

LEMMA 2.3. *If p_1, p_2 are sublinear functionals on X, and for $f \in Y^\#$, if $f \le p_1 \le p_2$ on Y, then for all $x \in X \setminus Y$,*

$$\sup\{f(y) - p_2(y - x) : y \in Y\} \le \sup\{f(y) - p_1(y - x) : y \in Y\}$$
$$\le \inf\{f(y) + p_1(x - y) : y \in Y\} \le \inf\{f(y) + p_2(x - y) : y \in Y\}$$

We can now state the following lemma which is the analogue, in the present context, of [3, Lemma 2.6].

LEMMA 2.4. *Let X, Y, $\mathcal{C}$, p be as above. Let $\ell \in Y^\#$ be invariant and $\ell \le p$ on Y. Let $Z = \ker(\ell) \subseteq Y$ and $y_0 \in Y$ be such that $\ell(y_0) = 1$. The following are equivalent:*

(a) *ℓ has a unique $\mathcal{C}$-invariant extension $\widetilde{\ell}$ dominated by p.*

(b) *$\sup\{\ell(y) - \widehat{p}(y - x) : y \in Y\} = \inf\{\ell(y) + \widehat{p}(x - y) : y \in Y\}$ for all $x \in X \setminus Y$.*

(c) *For any $x \in X \setminus Y$ and $\varepsilon > 0$, there exists $T = T(x, \varepsilon) \in \mathcal{C}$ such that*

$$\inf\{\ell(y) + p \circ T(x - y) : y \in Y\} \le \sup\{\ell(y) - p \circ T(y - x) : y \in Y\} + \varepsilon.$$

(d) *ℓ has a unique extension from Y/Z to X/Z dominated by $\widetilde{\widehat{p}}$.*

(e) *For any $x \in X \setminus Y$,*

$$\sup\{\alpha - \widetilde{\widehat{p}}(\alpha y_0 - x + Z) : \alpha \in \mathbb{R}\} = \inf\{\alpha + \widetilde{\widehat{p}}(x - \alpha y_0 + Z) : \alpha \in \mathbb{R}\}.$$

(f) *For any $x \in X \setminus Y$ and $\varepsilon > 0$, there exists $T = T(x, \varepsilon) \in \mathcal{C}$ such that*

$$\inf\{\alpha + \widetilde{p \circ T}(x - \alpha y_0 + Z) : \alpha \in \mathbb{R}\} \le \sup\{\alpha - \widetilde{p \circ T}(\alpha y_0 - x + Z) : \alpha \in \mathbb{R}\} + \varepsilon$$

(g) *ℓ has a unique $\widetilde{\mathcal{C}}$-invariant extension $\widetilde{\ell}$ from Y/Z to X/Z dominated by $\widetilde{p}$.*

(h) *For any $x \in X \setminus Y$,*

$$\sup\{\alpha - \widehat{\widetilde{p}}(\alpha y_0 - x + Z) : \alpha \in \mathbb{R}\} = \inf\{\alpha + \widehat{\widetilde{p}}(x - \alpha y_0 + Z) : \alpha \in \mathbb{R}\}.$$

(i) *For any $x \in X \setminus Y$ and $\varepsilon > 0$, there exists $T = T(x, \varepsilon) \in \mathcal{C}$ such that*

$$\inf\{\alpha + \widetilde{p} \circ \widetilde{T}(x - \alpha y_0 + Z) : \alpha \in \mathbb{R}\} \le \sup\{\alpha - \widetilde{p} \circ \widetilde{T}(\alpha y_0 - x + Z) : \alpha \in \mathbb{R}\} + \varepsilon.$$

Proof. From our earlier discussion, it follows that (a) is equivalent to

(a′) ℓ has a unique extension $\widetilde{\ell}$ dominated by $\widehat{p}$.

From [3, Lemma 2.6], we get (a′) $\Leftrightarrow$ (b) $\Leftrightarrow$ (d) $\Leftrightarrow$ (e) and also (g) $\Leftrightarrow$ (h).

(b) $\Rightarrow$ (c)   Let $x \in X \setminus Y$ and $\varepsilon > 0$. By (b), there exist $y_1, y_2 \in Y$ such that

$$\ell(y_1) - \widehat{p}(y_1 - x) > \sup\{\ell(y) - \widehat{p}(y - x) : y \in Y\} - \varepsilon/2$$

and

$$\begin{aligned}
\ell(y_2) + \widehat{p}(x - y_2) &< \inf\{\ell(y) + \widehat{p}(x - y) : y \in Y\} + \varepsilon/2 \\
&= \sup\{\ell(y) - \widehat{p}(y - x) : y \in Y\} + \varepsilon/2 \\
&< \ell(y_1) - \widehat{p}(y_1 - x) + \varepsilon
\end{aligned}$$

and hence
$$\widehat{p}(y_1 - x) + \widehat{p}(x - y_2) < \ell(y_1 - y_2) + \varepsilon.$$

Now choose $T_1, T_2 \in \mathcal{C}$ such that

$$p(T_1(y_1 - x)) + p(T_2(x - y_2)) < \widehat{p}(y_1 - x) + \widehat{p}(x - y_2) + 2\varepsilon.$$

Taking $T = T_1 T_2 \in \mathcal{C}$, we see that

$$p(T(y_1 - x)) + p(T(x - y_2)) < \ell(y_1 - y_2) + 3\varepsilon.$$

This implies

$$\ell(y_2) + p(T(x - y_2)) \le \ell(y_1) - p(T(y_1 - x)) + 3\varepsilon,$$

and hence, (c) follows.

(c) $\Rightarrow$ (b)   Since $\widehat{p} \le p \circ T$ for any $T \in \mathcal{C}$, by Lemma 2.3, we have

$$\begin{aligned}
\sup\{\ell(y) - p \circ T(y - x) : y \in Y\} &\le \sup\{\ell(y) - \widehat{p}(y - x) : y \in Y\} \\
\le \inf\{\ell(y) + \widehat{p}(x - y) : y \in Y\} &\le \inf\{\ell(y) + p \circ T(x - y) : y \in Y\}.
\end{aligned}$$

Thus (c) $\Rightarrow$ (b).

From Lemma 2.1 (a), we see that in (e), $\widetilde{\widehat{p}}$ could be replaced by $\inf\{\widetilde{p \circ T} : T \in \mathcal{C}\}$. Now, (e) $\Leftrightarrow$ (f) and (h) $\Leftrightarrow$ (i) follow from similar arguments.

(d) $\Rightarrow$ (g)   Follows from Lemma 2.1 (c) and Lemma 2.3.

(g) $\Rightarrow$ (a)   Suppose λ_1 and λ_2 are two distinct $\mathcal{C}$-invariant extensions of ℓ dominated by p. Then $\lambda_i \leq \widehat{p}$. Define Λ_i on X/Z by $\Lambda_i(x + Z) = \lambda_i(x)$. Notice that if $x_1 + Z = x_2 + Z$, then $x_1 - x_2 \in Z$ and therefore, $\lambda_i(x_1 - x_2) = \ell(x_1 - x_2) = 0$, showing that Λ_i's are well-defined. It is also clear that $\Lambda_i \leq \widetilde{\widehat{p}}$ and Λ_i's are $\widetilde{\mathcal{C}}$-invariant. By (g), $\Lambda_1 = \Lambda_2$, and hence, $\lambda_1 = \lambda_2$. $\blacksquare$

Remark 2.5. Can one replace (c) above by the following: *For any $x \in X \setminus Y$, there exists $T \in \mathcal{C}$ such that*

$$\sup\{\ell(y) - p \circ T(y - x) : y \in Y\} = \inf\{\ell(y) + p \circ T(x - y) : y \in Y\}?$$

As Example 3.4 shows, this equality can hold sometimes.

We will now tie up the above results with nested sequences of balls and present the main theorems of this paper. We adhere to the notations introduced so far.

THEOREM 2.6. *Let $X, Y, \mathcal{C}, p$ be as above. Let $\ell \in Y^\#$ be invariant and $\ell \leq p$ on Y. Let $Z = \ker(\ell) \subseteq Y$. The following are equivalent:*

(a) $\ell \in Y^\#$ *has a unique $\mathcal{C}$-invariant extension $\widetilde{\ell}$ dominated by p.*

(b) *If $x \in X$ and $\{B_{\widetilde{p}}(y_n + Z, r_n)\}$ is a nested sequence of balls in X/Z with $\{y_n\} \subseteq Y$, $0 \in B_{\widetilde{p}}(y_1 + Z, r_1)$, $\widetilde{p}(x + Z) \leq 1$, then there exists $T \in \mathcal{C}$ such that*
$$\inf_n \frac{d_{T,n}}{r_n} < 2,$$
where
$$d_{T,n} = \inf\{\widetilde{p \circ T}(y_n - x - y + Z) + \widetilde{p \circ T}(y_n + x + y + Z) : y + Z \in Y/Z\}.$$

(c) *If $x \in X$ and $\{B_{\widetilde{p}}(y_n + Z, r_n)\}$ is a nested sequence of balls in X/Z with $\{y_n\} \subseteq Y$, $0 \in B_{\widetilde{p}}(y_1 + Z, r_1)$, $\widetilde{p}(x + Z) \leq 1$, then there exist $T \in \mathcal{C}$, $y \in Y$ and $n_0 \geq 1$ such that*
$$\widetilde{p \circ T}(y_{n_0} \pm (x - y) + Z) < r_{n_0}.$$

(d) *If $x \in X$ and $\{B_p(y_n, r_n)\}$ is a nested sequence of balls in X with $\{y_n\} \subseteq Y$, $0 \in B_p(y_1, r_1)$, $p(x) \leq 1$ then there exist $T \in \mathcal{C}$, $n_0 \geq 1$ and $y \in Y$ such that*
$$\widetilde{p \circ T}(y_{n_0} \pm (x - y) + Z) < r_{n_0}.$$

Proof. (a) $\Rightarrow$ (b)   As before, (a) is equivalent to

(a$'$) ℓ has a unique extension $\widetilde{\ell}$ dominated by $\widehat{p}$.

Let $\{B_{\widetilde{p}}(y_n + Z, r_n)\}$ be a nested sequence of balls in X/Z with $\{y_n\} \subseteq Y$, $0 \in B_{\widetilde{p}}(y_1 + Z, r_1)$, $\widetilde{p}(x + Z) \leq 1$. Since $\widetilde{\widehat{p}} \leq \widetilde{p}$,

$$\widetilde{\widehat{p}}(y_{n+1} - y_n + Z) \leq \widetilde{p}(y_{n+1} - y_n + Z) \leq r_{n+1} - r_n,$$

consequently, $\{B_{\widetilde{\widehat{p}}}(y_n + Z, r_n)\}$ is also a nested sequence of $\widetilde{\widehat{p}}$-balls in X/Z with $\{y_n\} \subseteq Y$, $0 \in B_{\widetilde{\widehat{p}}}(y_1 + Z, r_1)$ and $\widetilde{\widehat{p}}(x + Z) \leq 1$. By [3, Theorem 2.12], (a$'$) implies

$$\inf_n \frac{1}{r_n} D_n < 2,$$

where

$$D_n = \inf\{\widetilde{\widehat{p}}(y_n - x - y + z) + \widetilde{\widehat{p}}(y_n + x + y + Z) : y + Z \in Y/Z\}.$$

This implies, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{1}{r_{n_0}} D_{n_0} < 2 \quad \text{or}, \quad D_{n_0} < 2r_{n_0}$$

Therefore, we can find $y' + Z \in Y/Z$ such that

$$\widetilde{\widehat{p}}(y_{n_0} - x - y' + Z) + \widetilde{\widehat{p}}(y_{n_0} + x + y' + Z) < 2r_{n_0}.$$

So there are $z_1, z_2 \in Z$ such that

$$\widehat{p}(y_{n_0} - x - y' + z_1) + \widehat{p}(y_{n_0} + x + y' + z_2) < 2r_{n_0}.$$

Again by the definition of $\widehat{p}$, there exist $T_1, T_2 \in \mathcal{C}$ such that

$$p \circ T_1(y_{n_0} - x - y' + z_1) + p \circ T_2(y_{n_0} + x + y' + z_2) < 2r_{n_0}.$$

Taking $T = T_1 T_2 \in \mathcal{C}$, we see that

$$p \circ T(y_{n_0} - x - y' + z_1) + p \circ T(y_{n_0} + x + y' + z_2) < 2r_{n_0},$$

whence it follows that

$$\widetilde{p \circ T}(y_{n_0} - x - y' + Z) + \widetilde{p \circ T}(y_{n_0} + x + y' + Z) < 2r_{n_0}.$$

Hence

$$d_{T,n_0} < 2r_{n_0}, \quad \text{or,} \quad \frac{1}{r_{n_0}} d_{T,n_0} < 2.$$

This proves (b).

(b) $\Rightarrow$ (c)   If (c) does not hold, then there exist $x \in X$ and a nested sequence of balls $\{B_{\widetilde{p}}(y_n + Z, r_n)\}$ in X/Z with $\{y_n\} \subseteq Y$, $0 \in B_{\widetilde{p}}(y_1 + Z, r_1)$, $\widetilde{p}(x+Z) \le 1$, such that for all $T \in \mathcal{C}$, $y \in Y$ and $n \ge 1$,

$$\max\{\widetilde{p \circ T}(y_n + x - y + Z), \widetilde{p \circ T}(y_n - x + y + Z)\} \ge r_n.$$

Thus,

$$\inf_n \frac{\inf\{\max\{\widetilde{p \circ T}(y_n + x - y + Z), \widetilde{p \circ T}(y_n - x + y + Z)\} : y + Z \in Y/Z\}}{r_n} \ge 1.$$

As before, $\{B_{\widetilde{p \circ T}}(y_n + Z, r_n)\}$ is also a nested sequence of $\widetilde{p \circ T}$-balls in X/Z with $\{y_n\} \subseteq Y$, $0 \in B_{\widetilde{p \circ T}}(y_1 + Z, r_1)$ and $\widetilde{p \circ T}(x + Z) \le 1$. Now use [3, Lemma 2.10] to conclude that

$$\inf_n \frac{d_{T,n}}{r_n} \ge 2,$$

contradicting (b).

(c) $\Rightarrow$ (d) is clear.

(d) $\Rightarrow$ (a)   We show that (d) $\Rightarrow$ Lemma 2.4 (f).

As in Lemma 2.4, let $y_0 \in Y$ be such that $\ell(y_0) = 1$. Let $x \in X \setminus Y$. We may assume that $p(x) \le 1$. Let $\varepsilon > 0$.

Let $\alpha_n = n + \varepsilon/(n+2) - \varepsilon/2$ for $n \ge 1$. And as in the proof of [3, Theorem 2.12] (d) $\Rightarrow$ (a), inductively construct a sequence $\{y_n\}$ such that $y_n \in \alpha_n y_0 + Z$ and $0 < p(y_1) < 1$ and $0 < p(y_{n+1} - y_n) < 1$ for all $n \ge 1$. Then $\{B_p(y_n, n)\}$ is a nested sequence of p-balls such that the centers $\{y_n\} \subseteq Y$. Hence, by (d), there exist $T \in \mathcal{C}$, $n_0 \ge 1$ and $y \in Y$ such that

$$\widetilde{p \circ T}(y_{n_0} \pm (x - y) + Z) < n_0.$$

Let α_0 be such that $y \in \alpha_0 y_0 + Z$. Now arguing as in the proof of [3, Theorem 2.12] (d) $\Rightarrow$ (a), we get,

$$[(\alpha_0 - \alpha_{n_0}) + \widetilde{p \circ T}(x - (\alpha_0 - \alpha_{n_0})y_0 + Z)]$$
$$- [(\alpha_0 + \alpha_{n_0}) - \widetilde{p \circ T}((\alpha_0 + \alpha_{n_0})y_0 - x + Z)] < \varepsilon.$$

Hence,

$$\inf\{\alpha + \widetilde{p \circ T}(x - \alpha y_0 + Z) : \alpha \in \mathbb{R}\} \le \sup\{\alpha - \widetilde{p \circ T}(\alpha y_0 - x + Z) : \alpha \in \mathbb{R}\} + \varepsilon,$$

as was to be shown. ∎

So far we have confined ourselves to the (local) problem of finding conditions for invariant extensions of *single* linear functionals. Let us now turn our attention to the global problem and for that it would be convenient to make the following:

DEFINITION 2.7. Let Y be a subspace of X and $\mathcal{C}$ be an abelian semigroup of linear transformations on X under which Y remains invariant. We say that Y is a $\mathcal{C}$-invariant p-U-subspace if every $\mathcal{C}$-invariant $\ell \in Y^{\#}$ with $\ell \le p$ has a unique $\mathcal{C}$-invariant extension $\widetilde{\ell}$ with $\widetilde{\ell} \le p$.

Such subspaces are characterized by

THEOREM 2.8. *Let Y be a subspace of X and $\mathcal{C}$ be an abelian semigroup of linear transformations on X under which Y remains invariant. Then the following statements are equivalent:*

(a) *Y is a $\mathcal{C}$-invariant p-U-subspace.*

(b) *If $x \in X$ and $\{B_p(y_n, r_n)\}$ is a nested sequence of p-balls in X with centres in Y, $0 \in B_p(y_1, r_1)$ and $p(x) \le 1$, then there exists $T \in \mathcal{C}$ such that*

$$\inf_n \frac{1}{r_n}[\inf\{p \circ T(y_n - x - y) + p \circ T(y_n + x + y) : y \in Y\}] < 2.$$

(c) *If $x \in X$ and $\{B_p(y_n, r_n)\}$ is a nested sequence of p-balls in X with centres in Y, $0 \in B_p(y_1, r_1)$, $p(x) \le 1$, then there exist $T \in \mathcal{C}$, $y \in Y$ and $n_0 \ge 1$ such that*

$$p \circ T(y_{n_0} \pm (x - y)) < r_{n_0}.$$

Proof. (a) $\Rightarrow$ (b) It follows from our earlier discussion that $\ell \in Y^{\#}$ is $\mathcal{C}$-invariant with $\ell \le p$ if and only if $\ell \le \widehat{p}$. Thus, by [3, Theorem 2.14], (a) implies:

If $x \in X$ and $\{B_{\widehat{p}}(y_n, r_n)\}$ is a nested sequence of $\widehat{p}$-balls in X with centres in Y, $0 \in B_{\widehat{p}}(y_1, r_1)$ and $\widehat{p}(x) \leq 1$, then

$$\inf_n \frac{1}{r_n}[\inf\{\widehat{p}(y_n - x - y) + \widehat{p}(y_n + x + y) : y \in Y\}] < 2.$$

Now arguing as in the proof of (a) $\Rightarrow$ (b) of Theorem 2.6, we get (b).

(b) $\Rightarrow$ (c) follows from the proof of (b) $\Rightarrow$ (c) of Theorem 2.6, putting $Z = \{0\}$.

(c) $\Rightarrow$ (a) is easy. ∎

Let us now specialize to the case when X is a normed linear space. Thus, $p(x) = \|x\|$. Let Y be a subspace of X and $\mathcal{C}$ be an abelian semigroup of linear transformations on X under which Y remains invariant. Then since $p(Tx) = \|Tx\| \leq p(x) = \|x\|$, we have $\|T\| \leq 1$ for all $T \in \mathcal{C}$. Let $\ell \in Y^*$ with $\|\ell\| = 1$ be $\mathcal{C}$-invariant. Then, $|\ell(Ty)| = |\ell(y)| \leq \|\ell\|\|Ty\| = \|Ty\|$ for any $y \in Y$. Hence $1 = \|\ell\| \leq \|T|_Y\| \leq \|T\|$ and we see that $\|T\| = 1$ for all $T \in \mathcal{C}$. Since an extension $\widetilde{\ell}$ dominated by $\widehat{p}$ always exists, we have proved [11, Theorem 5.24], which was proved as an application of Kakutani's fixed point theorem:

THEOREM 2.9. *Suppose Y is a subspace of a normed linear space X and $\ell \in Y^*$ with $T(Y) \subseteq Y$, $\|T\| = 1$ and $\ell(Ty) = \ell(y)$ for all $y \in Y$, where $T \in \mathcal{C}$, an abelian semigroup of linear transformations on X, then there always exists a norm-preserving extension $\widetilde{\ell}$ of ℓ which is also $\mathcal{C}$-invariant.*

The uniqueness of such an extension is given by:

THEOREM 2.10. *Let Y be a subspace of a normed linear space X and $\mathcal{C}$ be an abelian semigroup of linear transformations on X under which Y remains invariant. Then the following statements are equivalent:*

(a) *Y is a $\mathcal{C}$-invariant U-subspace of X.*

(b) *If $x \in X$ and $\{B(y_n, r_n)\}$ is a nested sequence of balls in X with centres in Y, $0 \in B(y_1, r_1)$ and $\|x\| \leq 1$, then there exists $T \in \mathcal{C}$ such that*

$$\inf_n \frac{1}{r_n}[\inf\{\|T(y_n - x - y)\| + \|T(y_n + x + y)\| : y \in Y\}] < 2.$$

(c) *If $x \in X$ and $\{B(y_n r_n)\}$ is a nested sequence of balls in X with $\{y_n\} \subseteq Y$, $0 \in B(y_1, r_1)$ and $\|x\| \leq 1$, then there exist $T \in \mathcal{C}$, $y \in Y$ and $n_0 \geq 1$ such that*

$$\|T(y_{n_0} \pm (x - y))\| < r_{n_0}.$$

Remark 2.11. If p is a norm, clearly $\widehat{p}$ is a seminorm. It is generally not necessarily a norm. However, since any invariant functional on Y and its invariant extensions vanish on $W = \{x \in X : \widehat{p}(x) = 0\}$, for the uniqueness question, it suffices to work on the quotient space X/W, where $\widehat{p}$ is a norm.

3. Examples and applications

We will now consider some examples pertaining to Theorems 2.6 - 2.10 and give a few applications of these results.

EXAMPLE 3.1. We first present an example of a subspace Y that is not an invariant U-subspace.

Let $X = B[0,1]$, the space of bounded Borel functions on $[0,1]$ with the sup-norm and $Y = \{f \in X : \int_0^1 f(x)dx = 0\}$. If $T : X \to X$ is defined by $Tg(x) = g(1-x)$, then it is immediate that $T(Y) \subseteq Y$. Also since $T^2 = I$, $\mathcal{C} = \{a_0 I + a_1 T : a_0, a_1 \geq 0, \ a_0 + a_1 = 1\}$. Let

$$\phi = \chi_{[0,\,1/4]} - \chi_{[1/4,\,3/4]} + \chi_{[3/4,\,1]},$$

where χ_A denotes the indicator function of the Borel set A, and let

$$\ell(f) = \int_0^1 f(x)\phi(x)dx, \quad f \in Y.$$

Note that $\phi(x) = \phi(1-x)$ for all $x \in [0,1]$. Hence,

$$\ell(Tf) = \int_0^1 f(1-x)\phi(x)dx = \int_0^1 f(1-x)\phi(1-x)dx = \int_0^1 f(x)\phi(x)dx = \ell(f),$$

so ℓ is $\mathcal{C}$-invariant. It is easy to check that

$$\|\ell\| = \sup\left\{\left|\int_0^1 f(x)\phi(x)dx\right| : f \in Y, \|f\|_\infty \leq 1\right\} = 1.$$

For $\alpha \in \mathbb{R}$, define $\widetilde{\ell}$ on X by

$$\widetilde{\ell}(g) = \int_0^1 g(x)(\phi(x) + \alpha)dx, \qquad g \in X.$$

Then for any $f \in Y$,

$$\widetilde{\ell}(f) = \int_0^1 f(x)(\phi(x) + \alpha)dx = \int_0^1 f(x)\phi(x)dx = \ell(f),$$

and for any $g \in X$,

$$\widetilde{\ell}(Tg) = \int_0^1 g(1-x)[\alpha + \phi(x)]dx = \int_0^1 g(1-x)[\alpha + \phi(1-x)]dx = \widetilde{\ell}(g),$$

thus $\widetilde{\ell}$ is a $\mathcal{C}$-invariant extension of ℓ. Moreover,

$$\|\widetilde{\ell}\| = \int_0^1 |\alpha + \phi(x)|dx.$$

One checks easily that for $|\alpha| \geq 1$,

$$\int_0^1 |\alpha + \phi(x)|dx = \left|\int_0^1 [\alpha + \phi(x)]dx\right| = |\alpha| \geq 1.$$

For $0 \leq \alpha \leq 1$,

$$\int_0^1 |\alpha + \phi(x)|dx = \int_0^{1/4}(\alpha+1)dx + \int_{1/4}^{3/4}(1-\alpha)dx + \int_{3/4}^1 (\alpha+1)dx$$

$$= \frac{1}{2}(1+\alpha) + \frac{1}{2}(1-\alpha) = 1,$$

and similarly, for $-1 \leq \alpha \leq 0$,

$$\int_0^1 |\alpha + \phi(x)|dx = \frac{1}{2}(1+\alpha) + \frac{1}{2}(1-\alpha) = 1.$$

Hence all choices of α in $[-1, 1]$ give invariant norm-preserving extensions.

We will interpret this result in terms of Lemma 2.4 and Theorem 2.10. Note that here $p(g) = \|g\|_\infty$. We will show, for the constant function $1 \in X$, that

$$-1 = \sup\{\ell(f) - p(f-1) : f \in Y\} = \sup\{\ell(f) - \widehat{p}(f-1) : f \in Y\}$$

$$< 1 = \inf\{\ell(f) + p(1-f) : f \in Y\} = \inf\{\ell(f) + \widehat{p}(1-f) : f \in Y\},$$

thus proving non-uniqueness of $\widetilde{\ell}$ by Lemma 2.4 ((a) $\Leftrightarrow$ (b)). Since both sup and inf are attained at $f = 0 \in Y$, it suffices to show for all $f \in Y$,

$$\ell(f) + 1 \leq \widehat{p}(f+1) \leq p(f+1) \quad \text{and} \quad \ell(f) + 1 \leq \widehat{p}(f-1) \leq p(f-1).$$

Now, writing $A = [0, 1/4] \cup [3/4, 1]$ for convenience,

$$
\begin{aligned}
\ell(f) + 1 &= \int_0^1 f(x)\phi(x)dx + 1 = \int_0^1 [f(x)\phi(x) + 1]dx \\
&= \int_A (f(x) + 1)dx + \int_{1/4}^{3/4} (1 - f(x))dx \\
&= \int_A (f(x) + 1)dx + \frac{1}{2} - \int_{1/4}^{3/4} f(x)dx \\
&= \int_A (f(x) + 1)dx + \frac{1}{2} + \int_A f(x)dx \qquad \left(\text{as } \int_0^1 f(x)dx = 0\right) \\
&= 2\int_A (f(x) + 1)dx \leq \|f + 1\|_\infty.
\end{aligned}
$$

Similarly,

$$
\ell(f) + 1 = \int_A (f(x) + 1)dx + \int_{1/4}^{3/4} (1 - f(x))dx = 2\int_{1/4}^{3/4} (1 - f(x))dx \leq \|f - 1\|_\infty.
$$

Thus we have proved that for all $f \in Y$,

$$
\ell(f) + 1 \leq p(f + 1) \quad \text{and} \quad \ell(f) + 1 \leq p(f - 1). \tag{2}
$$

Since Y, ℓ and the constant function 1 are $\mathcal{C}$-invariant, we have by (2),

$$
\ell(f) + 1 = \ell(Sf) + 1 \leq p(Sf + 1) \leq p(S(f + 1))
$$

and similarly, $\ell(f) + 1 \leq p(S(f - 1))$. Since this is true for all $S \in \mathcal{C}$, it follows that

$$
\ell(f) + 1 \leq \widehat{p}(f + 1) \quad \text{and} \quad \ell(f) + 1 \leq \widehat{p}(f - 1),
$$

as claimed. Now, let $f(x) = x - 1/2$ on $[0, 1]$. Then $f \in Y$ and

$$
\begin{aligned}
\widehat{p}(f - 1) &= \inf\left\{ \left\| a_0\left(x - \frac{3}{2}\right) + a_1\left(-x - \frac{1}{2}\right) \right\|_\infty : a_0, a_1 \geq 0,\ a_0 + a_1 = 1 \right\} \\
&= \inf\left\{ \left\| (a_0 - a_1)x - \left(\frac{3}{2}a_0 + \frac{1}{2}a_1\right) \right\|_\infty : a_0, a_1 \geq 0,\ a_0 + a_1 = 1 \right\} \\
&= \inf\left\{ \left\| (1 - 2a_1)x - \left(\frac{3}{2} - a_1\right) \right\|_\infty : a_1 \in [0, 1] \right\} \\
&= \inf\left\{ \max\left\{ \frac{3}{2} - a_1, a_1 + \frac{1}{2} \right\} : a_1 \in [0, 1] \right\} \\
&= \inf\left\{ 1 + \left| a_1 - \frac{1}{2} \right| : a_1 \in [0, 1] \right\} = 1,
\end{aligned}
$$

but $p(f - 1) = \|x - 3/2\|_\infty = 3/2$. Hence $\hat{p} \neq p$ in general, as it should be, despite the various equalities proved above.

To link the above with nested sequences of balls, let $Z = \ell^{-1}(0) \subseteq Y$. It is trivial to check that $Z = \{f \in Y : \int_{1/4}^{3/4} f(x)dx = 0\}$, and that if $f_0 \in Y$, $\ell(f_0) = 1$ if and only if

$$\int_{1/4}^{3/4} f_0(x)dx = -1/2.$$

In particular, $\phi \in Y$ and $\ell(\phi) = 1$. Also, notice that $\widetilde{p}(1 + Z) \leq 1$.

Define $\{f_n\} \subseteq Y$ by $f_n = \alpha_n \phi$, where $\alpha_n \geq 0$, $\alpha_1 < 1$ and $\alpha_n \uparrow \infty$. Let $r_n = \alpha_n + 1$ for $n \geq 1$. Then one checks easily that $\{B_{\widetilde{p}}(f_n + Z, r_n)\}$ is a nested sequence of $\widetilde{p}$-balls. For $S = a_0 I + a_1 T \in \mathcal{C}$, we wish to compute $\inf_n d_{S,n}/n$ (See Theorem 2.6).

If $f = \alpha\phi + h$ for some $\alpha \in \mathbb{R}$ and $h \in Z$, by (2), we have for any $h \in Z$,

$$p(S(f_n - 1 - f + h)) = p(S(f_n - f + h) - 1) \geq \ell(f_n - f + h) + 1 = \alpha_n - \alpha + 1$$

Thus,

$$\widetilde{p \circ S}(f_n - 1 - f + Z) \geq \alpha_n - \alpha + 1.$$

Similarly,

$$\widetilde{p \circ S}(f_n + 1 + f + Z) \geq \alpha_n + \alpha + 1.$$

Consequently,

$$\widetilde{p \circ S}(f_n - 1 - f + Z) + \widetilde{p \circ S}(f_n + 1 + f + Z)$$
$$\geq (\alpha_n - \alpha + 1) + (\alpha_n + \alpha + 1) = 2(\alpha_n + 1) = 2r_n$$

and we conclude that

$$\inf_n \frac{1}{r_n} d_{S,n} \geq 2,$$

once again showing non-uniqueness of $\widetilde{\ell}$. Moreover,

$$\widetilde{p}(f_n - 1 - f + Z) = \widetilde{p}((\alpha_n - \alpha)\phi - 1 + Z) \geq \alpha_n - \alpha + 1 > r_n \quad \text{if } \alpha < 0,$$

and

$$\widetilde{p}(f_n + 1 + f + Z) \geq \alpha_n + \alpha + 1 \geq r_n \quad \text{if } \alpha \geq 0.$$

We see therefore that there are no $y \in Y$ and $n_0 \in \mathbb{N}$ such that

$$\widetilde{p}(f_{n_0} \pm (1 + f + Z)) < r_{n_0} = \alpha_{n_0} + 1$$

showing once again non-uniqueness in view of Theorem 2.6 (c).

Remark 3.2. The simple construction above of a nested sequence of balls in $B[0,1]$ which fails to meet the conditions of Theorem 2.6 does not seem to be available in $C[0,1]$.

EXAMPLE 3.3. Clearly, a U-subspace is necessarily an invariant U-subspace. The following is a very simple example of an invariant U-subspace which is not a U-subspace. Let $X = C[0,1]$ and

$$Y = \{f \in C[0,1] : f(0) = -f(1)\} = \ker(\delta_0 + \delta_1),$$

where δ_x is the Dirac measure at $x \in [0,1]$. For $x \in [0,1]$, let ϕ_x denote the restriction of δ_x to Y. Since the functional ϕ_0 has two distinct norm-preserving extensions, namely, δ_0 and $-\delta_1$, Y is not an U-subspace.

Define $T : X \to X$ by $Tg(x) = -g(1-x)$. It is immediate that $T(Y) \subseteq Y$.

Let $\ell \in Y^*$, $\|\ell\| = 1$ and $\ell(Tf) = \ell(f)$ for all $f \in Y$. Suppose ℓ has two invariant norm-preserving extensions represented by μ and ν respectively. Then $\mu - \nu = \alpha(\delta_0 + \delta_1)$ for some $\alpha \in \mathbb{R}$.

Let $\Phi(x) = 1 - x$, $x \in [0,1]$. By the T-invariance of μ and ν,

$$\mu + \mu \circ \Phi^{-1} = \nu + \nu \circ \Phi^{-1} = 0.$$

It follows that

$$\alpha(\delta_0 + \delta_1) = \mu - \nu = -(\mu - \nu) \circ \Phi^{-1} = -\alpha(\delta_0 + \delta_1) \circ \Phi^{-1} = -\alpha(\delta_0 + \delta_1).$$

Therefore, $\alpha = 0$ and hence, $\mu = \nu$.

We will now give a few applications of our results.

EXAMPLE 3.4. BANACH LIMITS. Recall briefly what a Banach limit is. If $x = (x_n) \in \ell^\infty$, let $(Tx)_n = x_{n+1}$. Let $\mathcal{C} = \{\sum_{k=0}^n a_k T^k : n \geq 1,\ a_i \geq 0,\ \sum_{i=0}^n a_i = 1\}$, $p(x) = \limsup_{n\to\infty} x_n$, a sublinear functional on ℓ^∞.

Note that $p(Tx) = p(x)$ for all $x \in \ell^\infty$. If $y = (y_n) \in c$, i.e. $\lim_n y_n$ exists (in $\mathbb{R}$) and $\ell(y) = p(\{y_n\}) = \lim_n y_n$, then a Banach limit on ℓ^∞ is a linear functional $\widetilde{\ell}$ on ℓ^∞ that extends ℓ and

$$\widetilde{\ell}(x) \leq p(x), \qquad \widetilde{\ell}(Tx) = \widetilde{\ell}(x) \text{ for all } x \in \ell^\infty.$$

In other words, $\widetilde{\ell}$ is an invariant H-B extension. We showed in [3] the non-uniqueness of $\widetilde{\ell}$ by checking that p is not linear on ℓ^∞ (as required by [3, Proposition 4.4]). Here we treat the problem by the methods of this paper,

viz. by showing that $\widetilde{\ell}$ does not satisfy the conditions of Lemma 2.4 nor does it satisfy those of Theorems 2.6-2.10. The proofs will be somewhat sketchy since the arguments are reminiscent of those used in Example 3.1.

We first show that the invariant extension is unique from c to $x_0 = (1, 0, 1, 0, 1, 0, \ldots) \in \ell^\infty$. If

$$T_1 = \frac{1}{2n} \sum_{k=0}^{2n-1} T^k \in \mathcal{C},$$

then, for $y = (y_n) \in c$,

$$\left(T_1(y - x_0)\right)_m = \frac{1}{2n} \left(\sum_{k=0}^{2n-1} y_{m+k} - n \right) = \frac{1}{2n} \sum_{k=0}^{2n-1} y_{m+k} - \frac{1}{2}.$$

Putting $\lim_n y_n = \alpha$, we see that

$$p\left(T_1(y - x_0)\right) = \alpha - \frac{1}{2} \quad \text{and} \quad p\left(T_1(x_0 - y)\right) = \frac{1}{2} - \alpha$$

and it follows that

$$\sup\{\ell(y) - p \circ T_1(y - x_0) : y \in Y\} = \inf\{\ell(y) - p \circ T_1(x_0 - y) : y \in Y\} = \frac{1}{2}.$$

Hence by Lemma 2.4 ((d) $\Leftrightarrow$ (a) $\Leftrightarrow$ (b)), we have uniqueness and

$$\sup\{\ell(y) - \widehat{p}(y - x_0) : y \in Y\} = \inf\{\ell(y) - \widehat{p}(y - x_0) : y \in Y\} = \frac{1}{2}.$$

The various equalities above will not allow us to conclude that $\widehat{p}(y - x_0) = \alpha - \frac{1}{2}$ but, curiously enough, this is the case as the following computation shows. Let

$$S = \sum_{k=0}^{2n-1} a_k T^k \in \mathcal{C},$$

then

$$\left(S(y - x_0)\right)_m = \left(\sum_{k=0}^{2n-1} a_k T^k(y - x_0) \right)_m$$

$$= \sum_{k=0}^{2n-1} a_k y_{m+k} - \begin{cases} \displaystyle\sum_{k=0}^{n-1} a_{2k} & \text{if } m \text{ is odd} \\ \displaystyle\sum_{k=0}^{n-1} a_{2k+1} = 1 - \sum_{k=0}^{n-1} a_{2k} & \text{if } m \text{ is even} \end{cases}$$

Hence,

$$p\big(S(y - x_0)\big) = \begin{cases} \alpha - \displaystyle\sum_{k=0}^{n-1} a_{2k} & \text{if } \displaystyle\sum_{k=0}^{n-1} a_{2k} \leq \frac{1}{2} \\[2em] \alpha - 1 + \displaystyle\sum_{k=0}^{n-1} a_{2k} & \text{if } \displaystyle\sum_{k=0}^{n-1} a_{2k} \geq \frac{1}{2} \end{cases}$$

Similarly, if $S_1 = a_0 I + a_1 T + \cdots + a_{2n} T^{2n} \in \mathcal{C}$,

$$p\big(S_1(y - x_0)\big) = \begin{cases} \alpha - \displaystyle\sum_{k=0}^{n} a_{2k} & \text{if } \displaystyle\sum_{k=0}^{n} a_{2k} \leq \frac{1}{2} \\[2em] \alpha - 1 + \displaystyle\sum_{k=0}^{n} a_{2k} & \text{if } \displaystyle\sum_{k=0}^{n} a_{2k} \geq \frac{1}{2}. \end{cases}$$

This shows that

$$\widehat{p}(y - x_0) = \alpha - \frac{1}{2}.$$

We will now show that for

$$x_0 = (1, -1, -1, \underbrace{1, \ldots, 1}_{2^2 \text{ terms}}, \underbrace{-1, \ldots, -1}_{2^3 \text{ terms}}, \ldots, \underbrace{(-1)^k, \ldots, (-1)^k}_{2^k \text{ terms}}, \ldots) \in \ell^\infty,$$

uniqueness fails. Taking $S = a_0 I + a_1 T + \cdots + a_n T^n \in \mathcal{C}$, and writing $y = (y_n) \in c$ and $x_0 = (x_n)$, we see as above that

$$\big(S(y - x_0)\big)_m = \sum_{i=0}^{n} a_i y_{m+i} - \sum_{i=0}^{n} a_i x_{m+i},$$

where the second term is some positive and negative combination of $a_0, a_1,$ $\ldots, a_n$. In particular, for $2^{2k-1} \leq n \leq 2^{2k+1}$, the 2^{2k-1}-th term of $S(y - x_0)$ is

$$\sum_{i=0}^{n} a_i y_{2^{2k-1}+i} + \sum_{i=0}^{n} a_i = \sum_{i=0}^{n} a_i y_{2^{2k-1}+i} + 1,$$

and the 2^{2k}-th term looks like

$$\sum_{i=0}^{n} a_i y_{2^{2k}+i} - \sum_{i=0}^{n} a_i = \sum_{i=0}^{n} a_i y_{2^{2k}+i} - 1.$$

These are repeated an infinite number of times as k gets larger and consequently, putting $\lim_n y_n = \alpha$, we see that

$$p\big(S(y - x_0)\big) = \limsup_n \Big(\sum_{k=0}^n a_k y_{m+k} - \sum_{k=0}^n a_k x_{m+k} \Big) = \alpha + 1.$$

This implies $\widehat{p}(y - x_0) = \alpha + 1$. Similarly, $p\big(S(x_0 - y)\big) = 1 - \alpha$ and $\widehat{p}(x_0 - y) = 1 - \alpha$. Thus,

$$-1 = \sup \big\{ \ell(y) - p\big(S(y - x_0)\big) : y \in c \big\} < \inf \big\{ \ell(y) + p\big(S(x_0 - y)\big) : y \in c \big\} = 1,$$

showing non-uniqueness of Banach limit.

To see the connection with nested sequences of balls, note that, $Z = \ell^{-1}(0) = c_0 \subseteq c = Y \subseteq \ell^\infty = X$. Let $x_0 \in \ell^\infty$ as in the previous paragraph and let $z_n \in c_0$ and define $y_n = \alpha_n(1, 1, \ldots, 1, \ldots) + z_n \in c$, $y = \alpha(1, 1, \ldots, 1, \ldots) + z \in c$, where $\alpha_n \geq 0$, $\alpha_1 < 1$ and $\alpha_n \uparrow \infty$. Let $r_n = \alpha_n + 1$ for $n \geq 1$. Then $\{B_{\widetilde{p}}(y_n + Z, r_n)\}$ is a nested sequence of $\widetilde{p}$-balls in Y/Z. We check for $S = a_0 I + a_1 T + \cdots + a_n T^n \in \mathcal{C}$ (vide Theorem 2.6 (b)) that

$$\inf_n \frac{d_{S,n}}{r_n} \geq 2.$$

Recall that $\widetilde{p \circ S} \geq \widetilde{p} \circ \widetilde{S}$ (Lemma 2.1 (b)). Explicitly writing down the expressions for $S(y_n - x_0 - y + z)$ and $S(y_n + x_0 + y + z)$ and arguing as before, we see that (with $z' \in Z$),

$$p\big(S(y_n - x_0 - y + z) + z'\big) = \alpha_n - \alpha + 1,$$

and

$$p\big(S(y_n + x_0 + y + z) + z'\big) = \alpha_n + \alpha + 1.$$

Hence,

$$d_{S,n} \geq 2\alpha_n + 2 = 2r_n \quad \Rightarrow \quad \inf_n \frac{d_{S,n}}{r_n} \geq 2.$$

Similarly,

$$\widetilde{p}(y_n - x_0 - y + c_0) = \alpha_n - \alpha + 1 = r_n - \alpha \geq r_n \quad \text{if } \alpha \leq 0,$$

$$\widetilde{p}(y_n + x_0 + y + c_0) = \alpha_n + \alpha + 1 = r_n + \alpha \geq r_n \quad \text{if } \alpha \geq 0.$$

So, there are no $y \in c$ and $n_0 \in \mathbb{N}$ such that

$$\widetilde{p}(y_{n_0} \pm (x_0 + y) + c_0) < r_{n_0},$$

thus Theorem 2.6 (b) and (c) both fail for the Banach limit.

EXAMPLE 3.5. Let X be Banach space and $Y \subseteq X$ be a closed subspace such that there is a norm 1 projection on X with range Y. Let $\mathcal{G} = \{I, P\}$ and $p(x) = \|x\|$. Let $\ell \in Y^*$ with $\|\ell\| = 1$. Clearly, Y and ℓ are $\mathcal{G}$-invariant. Thus, there exists a norm-preserving invariant extension $\widetilde{\ell} : X \to X$ of ℓ. Note that by invariance, for any $x \in X$,

$$\widetilde{\ell}(x) = \widetilde{\ell}(Px) = \ell(Px),$$

and hence, $\widetilde{\ell} = \ell \circ P$, showing uniqueness of the invariant extension.

Remark 3.6. We thus see that uniqueness of invariant extension implies uniqueness of norm one projection from X onto Y. Indeed, if P_1 and P_2 are two distinct norm 1 projections from X onto Y, then there exists $x \in X$ and $\ell \in Y^*$ such that $\ell(P_1 x) \neq \ell(P_2 x)$ and by the above, the P_1-invariant extension of ℓ differs from the P_2-invariant extension of ℓ.

EXAMPLE 3.7. INVARIANT EXTENSIONS OF POSITIVE FUNCTIONALS. We recall the setting of [3, Example 4.6] : Let $Y \subseteq X$ be a subspace of an ordered linear space $(X, \geq)$. Assume that Y is cofinal in X, that is, given any $x \in X$, there exists $y \in Y$ such that $x \leq y$. Then we know that any $f \in Y^\#$, $f \geq 0$ has an extension $\widehat{f} \in X^\#$, $\widehat{f} \geq 0$ if and only if $\widehat{f} \leq q$ on X where

$$q(x) = \inf\{f(y) : x \leq y, \ y \in Y\}.$$

and $\widehat{f}$ is unique if and only if q is linear on X. Let a semigroup $\mathcal{G}$ (with $I \in \mathcal{G}$) act on X in such a way that $T(Y) \subseteq Y$, $Tx \geq 0$ if $x \geq 0$ and $f(Ty) = f(y)$ for all $y \in Y$, i.e. f is invariant under $\mathcal{G}$ on Y. Note that

$$x \leq y \quad \Rightarrow \quad Tx \leq Ty \in Y \quad \Rightarrow \quad q(Tx) \leq f(Ty) = f(y) \text{ for all } y \in Y$$

Therefore, $q(Tx) \leq q(x)$ for all $x \in X$. We also know that $f \leq q$ has an $\mathcal{G}$-invariant extension $\hat{f} \leq \widehat{q}$ on X. Consequently, $\hat{f}$ is also a positive extension by the remarks made above. We reiterate that the necessary and sufficient condition for f to have a unique positive extension (i.e. an extension which is unique with respect to positivity) is that q is linear on X.

EXAMPLE 3.8. We conclude this paper with another application. In [8, Theorem 4, page 33], it is proved that there exists a non-negative finitely additive set function $m(P)$ defined on *all* subsets P of the unit circle $\mathbb{T}$ in $\mathbb{C}$ that is invariant under rotations. This is obtained as follows. Let B be the space of *all* bounded real valued functions on $\mathbb{T}$ and Y the space of all

bounded Lebesgue measurable function on $\mathbb{T}$. Let $\ell(f) = \int_{\mathbb{T}} f(\theta)d\theta$, ($f \in Y$, $d\theta$ normalized Haar measure on $\mathbb{T}$). Y obviously has an order unit 1 and therefore the function

$$q(g) = \inf\{\ell(f) : g \le f, \ f \in Y\}$$

is well-defined (as we have seen earlier).

If $A_\rho(f)(\theta) = f(\theta + \rho)$, ρ being a rotation of the circle, one verifies easily that q is rotation invariant: $q(A_\rho g) = q(g)$ for all $g \in B$. Rotations of the circle commute and hence ℓ can be extended to all of B such that $\widetilde{\ell}$ is linear, $\widetilde{\ell}(A_\rho g) = \widetilde{\ell}(g)$ for all $g \in B$, i.e. $\widetilde{\ell}$ is invariant under rotations and $\widetilde{\ell} \le q$.

We investigate whether the positive extension is unique. For this, the necessary and sufficient condition, as observed above, is that q is linear. We will show that q is non-linear. If $g = \chi_P$ is the *indicator* function of P, where P is chosen to be non-measurable, then

$$q(\chi_P) = \inf\{\ell(f) : \chi_P \le f, \ f \in Y\}.$$

f can be approximately uniformly by simple functions and therefore it suffices to take a simple function in the definition of $q(\chi_P)$, from which it will be clear that f must be of the form χ_E, E a Lebesgue measurable set in $\mathbb{T}$ with $P \subseteq E$ and hence $q(\chi_P) = \inf\{m(E) : P \subseteq E, \ E \text{ Lebesgue measurable}\} = m^*(P)$, m^* being the *outer measure* which provides the extension of Lebesgue measure on $\mathbb{T}$. From [2, Theorem 1.3.5, page 17], we must have

$$m^*(P) + m^*(P^c) > 1$$

as P is non-measurable. But if q is linear, then

$$q(\chi_P) + q(\chi_{P^c}) = q(\chi_P) + q(1 - \chi_P) = 1,$$

i.e. $m^*(P) + m^*(P^c) = 1$, a contradiction.

Thus, the extension is non-unique. The question whether there is a unique invariant extension, i.e. whether $\widehat{q}$ is linear on X, appears more difficult and we leave it as an open question for that elusive creature known as the interested reader.

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