

## Topological characterization of weakly compact operators revisited <sup>†</sup>

ANTONIO M. PERALTA

*Departamento de Análisis Matemático, Facultad de Ciencias,  
Universidad de Granada, 18071 Granada, Spain,  
aperalta@ugr.es*

Received December 1, 2006

*Abstract:* In this note we revise and survey some recent results established in [8]. We shall show that for each Banach space  $X$ , there exists a locally convex topology for  $X$ , termed the “Right Topology”, such that a linear map  $T$ , from  $X$  into a Banach space  $Y$ , is weakly compact, precisely when  $T$  is a continuous map from  $X$ , equipped with the “Right” topology, into  $Y$  equipped with the norm topology. We provide here a new and shorter proof of this result. We shall also survey the results concerning sequentially Right-to-norm continuous operators.

*Key words:* Weakly compact operator, Right topology, Mackey topology, Property (V)

*AMS Subject Class. (2000):* 47B07, 47A20

### 1. INTRODUCTION

This note is thought to be a complement of the talk presented at the conference “Banach Space Theory, Cáceres 2006”. In the lecture given at the conference, the results treated in [8] were presented only tangentially. In this note we shall survey the topological characterization of weakly compact operators obtained in [8]. On the other hand, in [8] we strived for maximal clarity rather than maximal generality or maximal conciseness. Among the novelties included in this note, we present an alternative and shorter proof for this topological characterization of weakly compact operators.

In the last part of section §2, we shall review the connections between the Right topology and some previous studies, on certain locally convex topologies associated with operator ideals, due to I. Stephani [10],[11], H. Jarchow [6], and N. Ch. Wong, and Y.-Ch. Wong [14].

Throughout this note,  $X_1$  will denote the closed unit ball of a Banach space  $X$ . Whenever  $X$  and  $Y$  are Banach spaces,  $L(X, Y)$  denotes the space

---

<sup>†</sup>Partially supported by I+D MCYT project no. MTM 2005-02541, and Junta de Andalucía grants FQM 0199 and FQM 1215

of all bounded linear mappings from  $X$  to  $Y$ . The word operator will always mean bounded linear map. If  $\tau$  is a topology in  $X$  and  $A$  is a subset of  $X$ ,  $\tau|_A$  will stand for the restriction of  $\tau$  to the set  $A$ .

## 2. THE RIGHT TOPOLOGY

In this section we deal with the definition of the Right topology of a Banach space  $X$ , we will provide some characterizations of this topology.

Let  $X$  and  $Y$  be Banach spaces and let  $T : X \rightarrow Y$  be a bounded linear operator. The law

$$X \longrightarrow \mathbb{R}_0^+, \quad x \mapsto \|T(x)\|$$

defines a seminorm on  $X$ . This seminorm will be denoted by  $\|\cdot\|_T$ .

For every dual Banach space  $Y$  (with a predual denoted by  $Y_*$ ), we shall denote by  $m(Y, Y_*)$  the Mackey topology on  $Y$  relative to the duality  $(Y, Y_*)$ . According to the terminology introduced in [8], given a Banach space  $X$ , the relative topology induced on  $X$  by restricting the Mackey topology of its bidual will be termed the *Right topology of  $X$* . That is, the Right topology of  $X$  is the topology of uniform convergence on sets  $K \subset X^{**}$ , where  $K$  is absolutely convex and  $\sigma(X^*, X^{**})$  compact.

The next result was proved in [7, Proposition 2, §4].

**PROPOSITION 2.1.** *Let  $Z$  be a dual Banach space with a predual denoted by  $Z_*$ . Then the  $m(Z, Z_*)$ -topology coincides with the topology on  $Z$  generated by all the seminorms of the form  $\|\cdot\|_T$ , where  $T$  is a weak\*-weak\* continuous linear operator from  $Z$  into a reflexive Banach space.*

The following proposition is taken from [8, Proposition 2.3]. This result is the natural adaptation of the previous proposition when the Banach space  $Y$  is not assumed to be a dual space.

**PROPOSITION 2.2.** *Let  $X$  be a Banach space. Then the Right topology of  $X$  coincides with the topology on  $X$  generated by all the seminorms  $\|\cdot\|_T$ , where  $T$  is a bounded linear operator from  $X$  into a reflexive Banach space.*

*Proof.* Let  $T : X \rightarrow R$  be a bounded linear operator from  $X$  into a reflexive Banach space  $R$ . Clearly,  $T^{**} : X^{**} \rightarrow R$  is a weak\*-weak\* continuous linear operator from  $X^{**}$  into  $R$ . Conversely, when  $T : X^{**} \rightarrow R$  is a weak\*-weak\* continuous linear operator from  $X^{**}$  into a reflexive space  $R$ , then there exists a bounded linear operator  $U : R^* \rightarrow X^*$  satisfying that  $U^* = T$ .

The reflexivity of  $R$  assures that  $U$  also is weak\*-weak\* continuous and hence  $U = V^*$  for a suitable bounded linear operator  $V : X \rightarrow R$ .

The above arguments together with Proposition 2.1 give the statement. ■

Clearly, the Right topology of a Banach space  $X$  is a locally convex topology which is compatible with the duality  $(X, X^*)$ . In particular, a bounded linear mapping  $T : X \rightarrow Z$  is (norm-)continuous if and only if it is Right-to-Right continuous.

The novelties in this note start with the next lemma. This result will allow us to give a shorter proof of the results in [8].

LEMMA 2.3. *Let  $X$  and  $Y$  be Banach spaces and  $T : X \rightarrow Y$  a linear operator. Suppose that  $T|_{X_1} : X_1 \rightarrow Y$  is Right $|_{X_1}$ -to-norm continuous, then there exists a bounded linear operator  $S$  from  $X$  into a reflexive Banach space satisfying that*

$$\|T(x)\| \leq \|x\|_S + \|x\|,$$

for all  $x \in X$ .

*Proof.* We first note that, since the convergence in the norm-topology implies convergence in the Right topology and the latter is compatible with the duality  $(X, X^*)$ ,  $T$  is bounded (by uniform boundedness principle).

From our assumptions, we know that the set

$$\mathcal{O} := \{x \in X_1 : \|T(x)\| \leq 1\}$$

is a Right $|_{X_1}$ -neighborhood of 0 in  $X_1$ . Thus, from Proposition 2.2, there exist a positive constant  $\delta$ , reflexive Banach spaces  $R_1, \dots, R_k$  and bounded linear operators  $T_i : X_i \rightarrow R_i$  ( $1 \leq i \leq k$ ), such that

$$\mathcal{O} \supseteq \mathcal{O}' := \{x \in X_1 : \|x\|_{T_i} \leq \delta, \forall 1 \leq i \leq k\}.$$

We denote

$$R := \bigoplus_{1 \leq i \leq k}^{\ell_2} R_i$$

and  $S : X \rightarrow R$  the bounded linear operator given by  $S(x) := (\delta^{-1} T_i(x))$ .

Clearly,  $R$  is reflexive and since for each  $x \in X \setminus \{0\}$ ,  $\frac{1}{\|S(x)\| + \|x\|} x$  belongs to  $\mathcal{O}' \subseteq \mathcal{O}$ , we have

$$\left\| T \left( \frac{1}{\|S(x)\| + \|x\|} x \right) \right\| \leq 1,$$

which implies that

$$\|T(x)\| \leq \|S(x)\| + \|x\| = \|x\|_S + \|x\|.$$

When  $x = 0$ , the above inequality is trivial. ■

We can now obtain the promised topological characterization of weakly compact operators proved in [8, Theorem 4 and Corolary 5]. Here, in order to get a shorter proof, we shall make use of the previous lemma together with the result on factorization of weakly compact operators through reflexive Banach spaces due to W. J. Davis, T. Figiel, W. B. Johnson, and A. Pełczyński [3].

**THEOREM 2.4.** *Let  $X$  and  $Y$  be Banach spaces and  $T : X \rightarrow Y$  a linear operator. The following are equivalent:*

- a)  $T$  is weakly compact;
- b)  $T$  is Right-to-norm continuous;
- c)  $T|_{X_1} : X_1 \rightarrow Y$  is Right $|_{X_1}$ -to-norm continuous;
- d) There exist a bounded linear operator  $S$  from  $X$  into a reflexive Banach space and a mapping  $N : (0, +\infty) \rightarrow (0, +\infty)$  satisfying that

$$\|T(x)\| \leq N(\varepsilon) \|x\|_S + \varepsilon \|x\|,$$

for all  $x \in X$ ,  $\varepsilon > 0$ .

*Proof.* a)  $\Rightarrow$  b) Suppose that  $T$  is weakly compact. From [3, Corollary 1], there exist a reflexive Banach space  $R$  and bounded linear operators  $S_1 : X \rightarrow R$  and  $S_2 : R \rightarrow Y$  such that  $T = S_2 \circ S_1$ . In particular

$$\|T(x)\| \leq \|S_2\| \|S_1(x)\| = \|S_2\| \|x\|_{S_2},$$

for all  $x \in X$ . The above inequality together Proposition 2.2 show that  $T$  is Right-to-norm continuous.

The implication b)  $\Rightarrow$  c) is trivial.

c)  $\Rightarrow$  d) For each natural  $n$ ,  $nT|_{X_1} : X_1 \rightarrow Y$  is Right $|_{X_1}$ -to-norm continuous. Thus, by Lemma 2.3, there exist a reflexive Banach space  $R_n$  and a bounded linear operator  $S_n : X \rightarrow R_n$  satisfying that

$$\|nT(x)\| \leq \|S_n(x)\| + \|x\|,$$

for all  $x \in X$ . We may assume  $S_n \neq 0$ , for all  $n \in \mathbb{N}$ . Let us define  $R := \bigoplus_{n \in \mathbb{N}}^{\ell_2} R_n$ ,  $S : X \rightarrow R$  a bounded operator given by

$$S(x) := \left( \frac{1}{n \|S_n\|} S_n(x) \right),$$

and  $N : (0, +\infty) \rightarrow (0, +\infty)$  by  $N(\varepsilon) := \|S_{n(\varepsilon)}\|$ , where  $n(\varepsilon) = \inf\{n \in \mathbb{N} : 1/n < \varepsilon\}$ . Finally, given  $x \in X$  we have

$$\begin{aligned} \|n(\varepsilon)T(x)\| &\leq \|S_{n(\varepsilon)}(x)\| + \|x\|; \\ \|T(x)\| &\leq \frac{1}{n(\varepsilon)} \|S_{n(\varepsilon)}(x)\| + \frac{1}{n(\varepsilon)} \|x\| \\ \|T(x)\| &\leq \|S_{n(\varepsilon)}\| \|S(x)\| + \varepsilon \|x\| = N(\varepsilon) \|x\|_S + \varepsilon \|x\|. \end{aligned}$$

Finally, the implication  $d) \Rightarrow a)$  follows from [5, Theorem 20.7.3]. ■

The previous result can be thought as a topological characterization of weak compactness in terms of Right-to-norm continuity. The class of Right-to-Right continuous linear operators is, in general, much bigger than the class of weakly compact operators. As we have seen before, since the Right topology of a Banach space  $X$  is compatible with the duality  $(X, X^*)$ , the uniform boundedness principle shows that a linear mapping between Banach spaces is Right-to-Right continuous if and only if it is bounded (see also [8]).

It is well known that a Banach space  $X$  is reflexive if and only if the identity mapping on  $X$  is weakly compact. Thus, it follows by Theorem 2.4, that a Banach space  $X$  is reflexive if and only if the identity mapping on  $X$  is Right-to-norm continuous. We therefore have the following consequence of the above.

**COROLLARY 2.5.** [8] *A Banach space  $X$  is reflexive if, and only if, the Right topology for  $X$  coincides with the norm topology.*

The above results are strongly related with certain locally convex topologies on Banach spaces associated with ideals of operators. Quite recently, while this note was being written, we were told about the significant papers [6], [12, 13, 14] and [10, 11], which are directly connected with the results we obtained in [8]. In a personal communication, Professor N.-Ch. Wong told us about the connections appearing among our note and the just quoted papers. We are also in debt with the referee of the paper for pointing out the connections with reference [6]. Although the approaches of Wong-Wong in [14],

(respectively, Stephani in [10, 11] and Jarchow in [6]) and Peralta-Villanueva-Wright-Ylinen in [8] differ considerably, they have some connections which are worth enough to be reviewed and surveyed here.

Let  $\mathcal{L}$  denote the class of all bounded linear operator between Banach spaces. According to the notation in [14], an *operator ideal* on the class of Banach spaces over  $\mathbb{K}$  ( $\mathbb{R}$  or  $\mathbb{C}$ ) is a subclass  $\mathcal{U} \subset \mathcal{L}$  satisfying the following properties:

- ( $OI_1$ )  $Id_K \in \mathcal{U}(\mathbb{K}, \mathbb{K})$ ;
- ( $OI_2$ ) For each couple of Banach spaces  $X, Y$ ,  $\mathcal{U}(X, Y)$  is a subspace of  $L(X, Y)$ ;
- ( $OI_3$ )  $STR \in \mathcal{U}(Z, W)$ , whenever  $Z, X, Y, W$  are Banach spaces,  $R \in L(Z, X)$ ,  $S \in L(Y, W)$ , and  $T \in \mathcal{U}(X, Y)$ ;

where, for each couple of Banach spaces  $X$  and  $Y$ ,  $\mathcal{U}(X, Y) = \mathcal{U} \cap L(X, Y)$ .

Given an operator ideal  $\mathcal{U}$ , on the class of Banach spaces and a fixed Banach space  $X$ , the symbol  $\tau(\mathcal{U})(X)$  will denote the projective topology on  $X$  generated by the family

$$\{T \in \mathcal{U}(X, Y) : Y \text{ is a Banach space}\}.$$

More concretely,  $\tau(\mathcal{U})(X)$  is the topology on  $X$  generated by all the seminorms of the form  $\|\cdot\|_T$ , where  $T \in \mathcal{U}(X, Y)$  and  $Y$  runs in the set of all Banach spaces. Let  $\mathcal{U}^I$  denote the operator ideal defined by  $\mathcal{U}^I(X, Y) = L(X_{\tau(\mathcal{U})(X)}, Y)$ , where  $L(X_{\tau(\mathcal{U})(X)}, Y)$  stands for the space of all linear mappings from  $X$  to  $Y$  which are  $\tau(\mathcal{U})$ -to-norm continuous. Then it is known that  $\mathcal{U}^I$  coincides with the injective hull,  $\mathcal{U}^{inj}$ , of  $\mathcal{U}$  (compare [10], [6]).

Let  $\mathcal{W}$  denote the operator ideal of all weakly compact operators between Banach spaces. A consequence of Proposition 2.2 establishes that for each Banach space  $X$ , the Right topology of  $X$  and the  $\tau(\mathcal{W})(X)$  topology coincide.

Let us recall that the operator ideal  $\mathcal{W}$  is injective. Thus, when particularized to our setting, Lemmas 2.1 and 3.2 in [14] (see also [11]) give an alternative proof of our main theorem.

**PROPOSITION 2.6.** [14] *Let  $T : X \rightarrow Y$  be a linear mapping between two Banach spaces. Then  $T$  is weakly compact if and only if  $T$  is Right-to-norm continuous.*

Similar considerations can be made to deduce Corollary 2.5, and the comments preceding it, from [14, Example 5.2 (iv)] and [14, Theorem 3.9], respectively.

A consequence of [6, Proposition 4.1] is the following: for a Banach space  $X$ , the Right topology,  $\tau(\mathcal{W})(X)$ , of  $X$  is complete if and only if  $X$  is reflexive. This can be also proved applying Corollary 2.5 and the following direct argument: If  $X$  is not reflexive then we can find a norm-one element  $z \in X^{**} \setminus X$ . By Goldstine's theorem  $X_1$  is  $\sigma(X^{**}, X^*)$ -dense in  $X_1^{**}$ . Since the Mackey topology of  $X^{**}$  is compatible with the duality  $(X^{**}, X^*)$  it follows that  $X_1$  is  $m(X^{**}, X^*)$ -dense in  $X_1^{**}$ , hence there exists a net  $(x_\lambda) \subset X_1$  converging to  $z$  in the  $m(X^{**}, X^*)$ -topology. Finally, since the Right topology of  $X$  coincides with the restriction to  $X$  of the Mackey topology of its bidual, we deduce that  $(x_\lambda)$  is Right-Cauchy in  $X_1$ . However,  $(x_\lambda)$  cannot be Right-convergent in  $X$  because  $z$  has been taken in  $X^{**} \setminus X$  and the Mackey topology is Hausdorff.

It should be also noted here that the topics surveyed in this note have been also revisited by Professor Wright in [15]. In the just quoted note, the results concerning the Right topology are surveyed from an independent point of view. Among the novelties introduced by Professor Wright we can find an interesting generalization of a classical result due to Nikodym.

### 3. SEQUENTIALLY RIGHT-TO-NORM CONTINUITY

Our motivation to study sequentially Right-to-norm continuous operators was the Eberlein-Šmulian Theorem. The latter affirms that weak compactness is, in some sense, property determined by sequences instead of nets. The first question that one can ask clearly is whether a sequentially Right-to-norm continuous operator between Banach spaces is Right-to-norm continuous. The answer was shown to be negative, in general. For example, the identity on  $\ell_1$  is not Right-to-norm continuous. However, thanks to the Schur property on  $\ell_1$ , one can easily check that every Right-null sequence in  $\ell_1$  is automatically weakly-null and hence norm-null. This shows that the identity on  $\ell_1$  is sequentially Right-to-norm continuous.

In [8], we define *pseudo weakly compact* operators between Banach spaces as those linear mapping which are sequentially Right-to-norm continuous. A Banach space  $X$  is said to be *Sequentially Right* if every pseudo weakly compact operator  $T : X \rightarrow Y$  is weakly compact.

We have already seen that  $\ell_1$  is not sequentially Right. In order to have a wide class of Banach spaces which are sequentially Right, we refer to a previous paper by Wright and Ylinen [16]. In the just quoted paper, the author showed that every  $C^*$ -algebra is sequentially Right (see [16, Proposition 2.2]). We shall see now how sequentially Right spaces are related to those Banach

spaces satisfying property (V) of Pelczynski.

First we recall that a series  $\sum_n x_n$  in a Banach space  $X$  is called *weakly unconditionally Cauchy (w.u.C.)* if there exists  $C > 0$  such that for any finite subset  $F \subset \mathbb{N}$  and  $\varepsilon_n = \pm 1$ , we have  $\|\sum_{n \in F} \varepsilon_n x_n\| \leq C$ . A linear mapping between Banach spaces,  $T : X \rightarrow Y$  is *unconditionally converging* if, for every w.u.C. series  $\sum_n x_n$  in  $X$ , the series  $\sum_n T(x_n)$  is unconditionally convergent. Finally, a Banach space  $X$  is said to have Pelczynski's *property (V)* if, for every Banach space  $Y$ , every unconditionally converging operator is weakly compact.

For every w.u.C. series  $\sum x_n$  in a Banach space  $X$ , there is a bounded linear operator  $U : c_0 \rightarrow X$  satisfying that  $U(e_n) = x_n$ , where  $(e_n)$  is the canonical basis of  $c_0$ . It is well known that  $e_n$  converges to zero in the Strong\* topology of the commutative C\*-algebra  $c_0$ . Since in every C\*-algebra, the Strong\*-topology and the Right topology coincide on bounded sets (compare [1, Theorem II.7]), it follows that  $(e_n)$  is Right-null. Since a linear mapping between Banach spaces is bounded if and only if it is Right-to-Right continuous, we deduce that  $T(e_n) = x_n$  is Right-null in  $X$ . This result was established in [8, Lemma 13]. Since, by [4, Exercise 8, page 54], an operator between Banach spaces fails to be unconditionally converging if and only if it fixes a copy of  $c_0$ , we clearly have:

PROPOSITION 3.1. [8] *Every pseudo weakly compact operator between two Banach spaces is unconditionally converging.*

COROLLARY 3.2. [8] *Every Banach space satisfying property (V) is sequentially Right.*

Since every JB\*-triple also satisfies property (V) (compare [2]), it follows that every JB\*-triple is sequentially Right.

The questions posed at the end of [8] remain open at this moment. It still being an open question if every sequentially Right Banach space satisfies property (V).

#### REFERENCES

- [1] AKEMANN, C.A., The dual space of an operator algebra, *Trans. Amer. Math. Soc.* **126** (1967), 286–302.
- [2] CHU, CH.-H., MELLON, P., JB\*-triples have Pelczynski's Property V, *Manuscripta Math.* **93** no. 3 (1997), 337–347.



- [3] DAVIS, W.J., FIGIEL, T., JOHNSON, W.B., PELCZYNSKI, A., Factoring weakly compact operators, *J. Funct. Anal.* **17** (1974), 311-327.
- [4] DIESTEL, J., "Sequences and Series in Banach Spaces", Graduate Texts in Mathematics, 92. Springer-Verlag, New York, 1984.
- [5] JARCHOW, H., "Locally Convex Spaces", Teubner, Stuttgart, 1981.
- [6] JARCHOW, H., On certain locally convex topologies on Banach spaces, in "Functional Analysis: Surveys and Recent Results, III", North-Holland Math. Stud. 90, 1984, 79–93.
- [7] PERALTA, A.M., RODRÍGUEZ PALACIOS, A., Grothendieck's inequalities for real and complex JBW\*-triples, *Proc. London Math. Soc.* (3) **83** no. 3 (2001), 605–625.
- [8] PERALTA, A.M., VILLANUEVA, I., WRIGHT, J.D.M., YLINEN, K., Topological characterisation of weakly compact operators, *J. Math. Anal. Appl.* **325** no. 2 (2007), 968–974.
- [9] SAKAI, S., "C\*-algebras and W\*-algebras", Springer-Verlag, Berlin, 1971.
- [10] STEPHANI, I., Injektive Operatorenideale über der Gesamtheit aller Banachräume und ihre topologische Erzeugung, *Studia Math.* **38** (1970), 105–124.
- [11] STEPHANI, I., Generating topologies and quotients of injective operator ideals in "Banach space theory and its applications" (Bucharest, 1981), Lecture Notes in Math. 991, Springer, 1983, 239–258.
- [12] WONG, N.-CH., Topologies and bornologies determined by operator ideals. II, *Studia Math.* **111** no. 2 (1994), 153–162.
- [13] WONG, N.CH., The triangle of operators, topologies, bornologies, to appear in the Proceedings of International Congress of Chinese Mathematicians 2004, Hong Kong, by AMS-IP.
- [14] WONG, N.CH., WONG, Y.-CH., TOPOLOGIES AND BORNOLOGIES DETERMINED BY OPERATOR IDEALS, *Math. Ann.* **282** NO. 4 (1988), 587–614.
- [15] WRIGHT, J.D.M., Right topology for Banach spaces and weak compactness, *preprint* 2006.
- [16] WRIGHT, J.D.M., YLINEN, K., Multilinear maps on products of operator algebras, *J. Math. Anal. Appl.* **292** no. 2 (2004), 558–570.