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The Single-Valued Extension Property and Subharmonicity

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Abstract: In this note, using subharmonicity techniques, we show stability-product of the single-valued extension property for the class of operators having spectrum without interior points. Consequently, some basic local spectral properties for the multiplication operator and the tensor product of operators are established.

Key words: local spectrum, subharmonicity, svep, tensor product.

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Throughout this note $\mathcal{B}(X)$ (resp. $\mathcal{B}(H)$) denote the algebra of bounded linear operators in a complex Banach space X (resp. a Hilbert space H). For $T \in \mathcal{B}(X)$, we denote by $\sigma(T)$ the spectrum of T. Let $x \in X$. We define $\rho_T(x)$ to be the set of complex numbers α for which there exists a neighbourhood V_α of α with u analytic on V_α having values in X such that $(zI - T)u(z) = x$ on V_α. This set is open and contains the complement of the spectrum $\sigma(T)$ of T. The function u is called a *local resolvent* of T on V_α. By definition the *local spectrum* of T at x, denoted by $\sigma(T, x)$, is the complement of $\rho_T(x)$, so it is then a compact subset of $\sigma(T)$.

In general, the local resolvent $u(z)$ is not unique. We say that T has the *single-valued extension propery* (in short SVEP) if

$$(zI - T)v(z) = 0 \qquad \text{implies} \quad v = 0$$

for any analytic function v defined on any domain D of the complex plane with values in a Banach space X. Using the principle of analytic extension, it is easy to see that an operator T having spectrum without interior points has the SVEP (for more details see [7] and [9]). Further, for operators T having the SVEP, there is a unique local resolvent which is the analytic extension of $(z-T)^{-1}x$. Using this property and Liouville's theorem, it is easy to conclude that $\sigma(T, x) \neq \emptyset$ if $x \neq 0$. Also in this case the *local spectral radius* is given by

$$r(T, x) \;=\; \max\{|z| : z \in \sigma(T, x)\} \;=\; \limsup_{k\to\infty} \|T^k x\|^{\frac{1}{k}} \,.$$

In general, these two above properties are note true for operators without the SVEP. To see that, take for example the left shift operator in l^2. See [9] for more details. Nice characterizations of SVEP can be found in [10], [14] and [16].

An operator $T \in \mathcal{B}(X)$ is said to satisfy Bishop's property (β) at a complex number λ if there exists $r > 0$ such that for every open subset Ω of the disc $D(\lambda, r)$ and any sequence $\{f_n\}$ of holomorphic function from Ω to X,

$$\lim_{n\to\infty} (T-\mu)f_n(\mu) = 0 \qquad \text{in } \ O(\Omega, X)\,,$$

implies

$$\lim_{n\to\infty} f_n(\mu) = 0 \qquad \text{in } \ O(\Omega, X)\,.$$

Where $O(\Omega, X)$ is the space of holomorphic functions from Ω to X. Let us denote by $\sigma_\beta(T)$ the set of λ for which T fails to satisfy (β) property at λ. T is said to satisfy Bishop's property (β) if T has it at every λ. If $\sigma_\beta(T) = \emptyset$, then T is said to satisfy (β) property. The right shift on l^2, for example, has Bishop's property. Obviously, T having property (β) implies that T has the SVEP. Recal that an operator T is said to have the decomposition property (δ) if, given an arbitrary open covering $\{X_1, X_2\}$ of the complex plane, every $x \in X$ admits a decomposition $x = x_1 + x_2$ where the vectors $x_1, x_2 \in X$ are such that

$$x_k \;=\; (T-\lambda)u_k(\lambda) \qquad \text{for all } \ \lambda \text{ in the complement } \ X_k^c \ \text{ of } \ X_k$$

for some analytic function $u_k : X_k^c \to X$, for $k = 1, 2$. The set of λ for which T fails to satisfy (δ) property will be denoted by $\sigma_\delta(T)$. It is clear that both $\sigma_\beta(T)$ and $\sigma_\delta(T)$ are closed subset of $\sigma(T)$. The left shift on l^2 is an example of operators having the decomposition property (δ).

Recall that a bounded operator $T \in \mathcal{B}(X)$ is decomposable in the sense of Foiaş if, for each finite open cover $\sigma(T) \subset U_1 \cup \ldots \cup U_n$ of the spectrum of T, there are closed $T-$invariant subspaces $X_1, \ldots, X_n$ of X with

$$X = X_1 + \ldots + X_n \qquad \text{and} \qquad \sigma(T|X_i) \subset U_i\,.$$

It has been shown in [1] that the properties (β) and (δ) are dual to each other. The union of the two classes of operators (β) and (δ) is the class of the decomposable operators which is one of the most important class of operators in local spectral theory. All these classes, except the decomposition property (δ), are subclass of the class SVEP(X) of operators satisfying the

SVEP. Another interesting subclass is the class DC(X) of operator satisfying Dunford's property (C). An operator is said to have Dunford's property (C) if, for each closed subset F of the complex plane, the corresponding local spectral subspace

$$X_T(F) = \{x \in X : \sigma(T, x) \subset F\}$$

is closed. This notion was introduced by Dunford in his characterization of spectral operators. It is known that Bishops' property (β) implies Dundord's property (C), but the reverse implication is not true (see[17]). The significance of Bishop's property stems from the fact it characterizes, up to similarity, the restriction of decomposable operators to closed invariant subspaces. This nice result was obtained by Albrecht and Eschmeier [1]. The importance of (β) property in localizing the analytic functional calculus of an operator goes back to the nice work of M. Putinar in 1980s.

A well known open problem in local spectral theory asks whether the sum and the product of two commuting decomposable operators are decomposable. Following the above Albrecht-Eschmeier's result, to give a positive answer it would be sufficient to show that the sum and product satisfy the much weaker Bishop's property. This problem was initiated by Foiaş in 1960s in the particular case of generalized scalar operators. Partial results on the SVEP for the sum and the generalized commutator were obtained by Foiaş-Vasilescu [11], Laursen-Neumann [14], Aupetit-Drissi [3], and for the sum and product by Sun [21] and Miller-Neumann [17].

Let $T, S \in \mathcal{B}(X)$, the multiplication operator $D(T, S)$ defined by T, S is by definition the operator defined on $\mathcal{B}(X)$ by $D(T, S)(A) = TAS$. A natural question arises: Under which conditions on T and S, can we assert the single-valued extension property for the multiplication operator $D(T, S)$?

1. Subharmonicity and the single-valued extension property of $D(T, S)$

In this section, we shall use some analytic property of the local spectrum in order to obtain a result on the single-valued extension property of the multiplication operators. We refer to [2] and [12] for all standard definitions and properties concerning subharmonic functions.

Let D be a non-empty open subset of the complex plane and $g : D \to R \cup \{-\infty\}$ which is locally upper bounded on D. The upper regularization g^* is by definition

$$g^*(\alpha) = \limsup_{z \to \alpha} g(z)\,;$$

g^* is an upper semicontinuous function on D such that $g^*(z) \geq g(z)$.

Given an X-analytic function u and an analytic function f with values in $\mathcal{B}(X)$. It is well-known that the multifunction $z \to \sigma(f(z))$ is upper semi-continuous. It is also continuous if the spectrum is totally disconnected. Unfortunetely, the multifunction $z \to \sigma(T, u(z))$ is not upper semi-continuous, even in the case of matrices. For that we introduce the so-called upper semicontinuous regularization of the spectrum as

$$\sigma^*(T, u(z)) = \cap_{r>0} \overline{\cup_{\lambda \in D, |\lambda - z| < r} \sigma(T, u(z))} \,.$$

In [3], using deep results from classical analysis, the following two basic results on the subharmonicity of the local spectrum has been established.

Lemma 1.1. *Let $T \in \mathcal{B}(X)$ having the single-valued extension property and let u be an analytic function on an open set D, with values in X. Then $g(z) = \log(r(T, u(z)))$ satisfy the mean inequality and therefore $g*$ is subharmonic in D. Moreover the set of z for which $g^*(z) > g(z)$ is a polar subset of D.*

Lemma 1.2. *Let $T \in \mathcal{B}(X)$ and let u be an analytic function on an open set D, with values in X. Suppose that $\sigma(T)$ has no interior points. Then $\sigma^*(T, u(z))$ is constant on D.*

Using the above property of subharmonicity of the local spectrum, we obtain the following stability-product for the single valued extension property.

Theorem 1.3. *Let T, S be two commuting operators in $\mathcal{B}(X)$. Suppose that T, S have spectrum without interior points. Then ST has the SVEP.*

Proof. Let $u(\lambda)$ be an $X-$analytic function on an open set U of the complex plane, such that

$$(ST - \lambda I)u(\lambda) = 0 \,.$$

Suppose, without loss of generality, that u is never zero on U. Let us consider (for $\mu \neq 0$) the function

$$g_\lambda(\mu) = \frac{1}{\mu} \left(\frac{\lambda}{\mu} I - T \right)^{-1} T u(\lambda)$$

which is analytic for $\frac{\lambda}{\mu} \notin p(T)$. Then

$$(S - \mu I) g_\lambda(\mu) = u(\lambda) \qquad \text{for every} \quad \mu \in V_\lambda \subset U \,.$$

So, if $\frac{\lambda}{\mu} \notin \sigma(T)$, then we can find an analytic function g_λ such that

$$(S - \mu I)g_\lambda(\mu) = u(\lambda) \qquad \text{for every} \ \ \mu \in V_\lambda \,.$$

Thus, $\mu \notin \sigma(S, u(\lambda))$. Hence, $\sigma(S, u(\lambda)) \subset \left\{\frac{\lambda}{\alpha}, \alpha \in \sigma(T)\right\}$ for every $\lambda \in U$. This implies $\sigma^*(S, u(\lambda)) \subset \left\{\frac{\lambda}{\alpha}, \alpha \in \sigma(T)\right\} \cup \{0\}$. Since S and T commute, the case $\sigma^*(S, u(\lambda)) = \{0\}$ leads to the trivial case ST quasi-nilpotent. By lemma 1.2, $\sigma^*(S, u(\lambda))$ is constant and non-empty on U. Let $\alpha \in \sigma^*(S, u(\lambda))$, then $\frac{\lambda}{\alpha} \in \sigma(T)$ for every $\lambda \in U$. Therefore, $\sigma(T)$ contains the open set $(\frac{1}{\alpha})U$, which contradict the hypothesis $\sigma(T)$ has empty interior. Hence, $u(\lambda) = 0$, for any $\lambda \in U$. ∎

Corollary 1.4. *Let T and S be in $\mathcal{B}(X)$. Suppose that S and T have spectrum without interior points. Then the multiplication operator $D_{T,S}$ has the SVEP.*

Proof. Since $D_{T,S} = L_S R_T$ with L_S and R_T commuting operators satisfying

$$\sigma(R_S) = \sigma(S) \qquad \text{and} \qquad \sigma(L_T) = \sigma(T)\,.$$

On the other hand, S having the SVEP implies that the left multiplication operator L_S has the SVEP. By applying Theorem 1.3, we obtain the result. ∎

Remark 1. It is well-knwon (see [14, p. 278]) that if S and T satisfy respectively Dunford's property and decomposition property (δ), then both L_S and R_T have Dunford's property. Combining this result with Sun's result [21], we obtain that $D_{T,S}$ has the SVEP if S has Dunford's property and T has decomposition propery (δ).

Remark 2. It is well-known that $\sigma(D_{T,S}) = \{\lambda_1\lambda_2 : \lambda_1 \in \sigma(T), \lambda_2 \in \sigma(S)\}$. Suppose that T is invertible. In the particular case where $S = T^{-1}$, we have $D_{T,T^{-1}}(P) = TPT^{-1}$ which we will denote by D_T. It is clear that D_T is invertible and $D_T^{-1} = D_{T^{-1}}$. By the spectral mapping theorem

$$\sigma(D_T)^{-1} = \sigma(D_{T^{-1}})\,.$$

Consequently, if the spectrum of T is in the unit disc, then the spectrum of D_T is in the unit circle. So, D_T has the SVEP for all operator T having the spectrum in the unit disc.

Recall that $z \to \sigma(T, u(z))$ is said to be *lower semi-continuous* in an open set D, if given a sequence $\{z_n\}$ in D, for which $z_n \to z$, we have

$$\sigma(T, u(z)) \subset \underline{\lim}\sigma(T, u(z_n)) .$$

It is easy to see that if $z \to \sigma(T, u(z))$ is lower-semicontinuous for every analytic function on every open subset D of the complex plane, then T satisfy Dunford's property (C). Surprisingly, Dunford's property characterizes the lower semi-continuity of the local spectrum as it is showing in the next lemma.

LEMMA 1.5. *Let $T \in \mathcal{B}(X)$ satisfying Dunford property and let $u(z)$ be an X-analytic function. Then the local spectrum is lower semi-continuous.*

Proof. Suppose the contrary. Let $\{z_n\}$ be in D such that $z_n \to z$. Then there exists $\lambda \in \sigma(T, u(z))$ and $\lambda \notin \underline{\lim}\sigma(T, u(z_n))$. Hence, by definition of liminf of sequence of compact sets, there exists a neighberhood U_λ of λ and a sub-sequence $\{z_{n_k}\}$ of $\{z_n\}$ for which

$$U_\lambda \cap \sigma(T, u(\lambda_{n_k})) = \emptyset \qquad \text{for all} \quad k . \tag{1}$$

Let $F = \overline{\cup_{k\geq 1}\sigma(T, u(\lambda_{n_k}}))$. Then $\sigma(T, u(z_{n_k})) \subset F$ for every k. That is, $u(z_{n_k}) \in X_T(F)$. Since $z_{n_k} \to z$, we obtain using Dunford's property and the analyticity of u that

$$\sigma(T, u(z)) \subset F = \overline{\cup_{k\geq 1}\sigma(T, u(z_{n_k}))} .$$

So, $U_\lambda \cap (\cup_{k\geq 1}\sigma(T, u(z_{n_k}))) \neq \emptyset$, which leads to a contradiction with (1). ∎

2. Some local properties for tensor product of operators

Given two bounded linear operators A, B defined in a Hilbert space H, we consider their tensor product $A \otimes B$ of A and B, defined in the tensor product $H \otimes H$ of the space H by itself equipped with a suitable norm. In the particular case where H is of finite dimensional, that is A and B are represented by matrices, C. Stéphanos in 1900 showed that the set of eigenvalues (point spectrum) of $A \otimes B$ coincides with the set of the product of the eigenvalues (point spectrum) of A and B.

In 1965, Brown-Pearcy [6], have obtained an infinite dimensional generalization of Stephanos' result by showing that for A, B bounded linear operators in a Hilbert space $\mathcal{H}$, we have

$$\sigma(A \otimes B) = \sigma(A)\sigma(B) . \tag{2}$$

Further results were obtained by Schechter [20]. See [22] for more details. The aim of this section is to study some basic local spectral properties related to the tensor product of two operators. Among other results we obtain conditions under which $A \otimes B$ has the single-valued extension property, the Bishop's (β) property and at last the decomposability of $A \otimes B$.

Given A, B in $\mathcal{B}(H)$, it is known that if B is diagonalizable operator with eigenvalues $\{\lambda_n\}$, then $A \otimes B = \oplus_n \lambda_n A$ a direct sum of multiple of A. Suppose that A has the SVEP, and B is diagonalizable. Then, by [7, Proposition 1.1.3, p. 3],

$$\begin{gathered} A \otimes B \quad \text{has the SVEP and} \\ \sigma(A \otimes B, x) = \cup_n |\lambda_n| \sigma(A, x) \quad \text{for any} \quad x \in H\,. \end{gathered} \tag{3}$$

Moreover, $H_{A \otimes B}(F) = \oplus_n H_{\lambda_n A}(F)$ for any closed set F.

Recall that two operators A and B are said to be *approximately equivalent*, denoted by $A \sim_a B$, if there exist unitaries operators U_n such that $U_n^* A U_n \to B$ in the norm. Using the theorem of Weyl-von Neumann-Berg (see[4]) on approximation of normal operators by diagonalizable one, we obtain

PROPOSITION 2.1. *Suppose that A has the SVEP and N is a normal operator. Then $N \otimes A$ is in the closure of* SVEP(H).

Proof. Following the theorem of Weyl-von Neumann-Berg, there exists a diagonalizable operator D such that $D \sim_a N$. That is, there exist unitaries operators U_n for which $U_n^* D U_n \to N$. So, $V_n^*(D \otimes A)V_n \to N \otimes A$, with $V_n = I \otimes U_n$. Since A has the SVEP, we obtain from (3) that $D \otimes A$ has the SVEP. Since SVEP is preserved by similarity, we can conclude that $N \otimes A$ is in the closure of SVEP(H). ∎

Remark 3. Using [19, Theorem 9], the above result can be strengthned to $N \otimes A$ is in the closure of the class $\beta(H)$ of operators satisfying Bishop's property.

A natural question that arises is if given two operators A, B in $\mathcal{B}(H)$ with the SVEP, does $A \otimes B$ has the SVEP? As an application of Theorem 1.3, we obtain the following partial positive answer.

PROPOSITION 2.2. *Let A, B be in $\mathcal{B}(H)$. Suppose that $\sigma(A)$ and $\sigma(B)$ have empty interior. Then $A \otimes B$ has the SVEP.*

Proof. Following J. Dixmier [8, p. 25], we have

$$A \otimes B = (A \otimes I)(I \otimes B)\,,$$

with $(A \otimes I)(I \otimes B) = (I \otimes B)(A \otimes I)$, where I is the identity operator and $A \otimes I$ can be seen as an infinite diagonal matrix with $A's$ in the main diagonal. So,

$$\sigma(A \otimes I) = \sigma(A) \qquad \text{and} \qquad \sigma(I \otimes B) = \sigma(B)\,.$$

Applying Theorem 1.3, we obtain the result. ∎

Remark 4. It is easy to see that if S has Dunford's property (C) then $S \otimes I$ has it too. This follows from [7, Proposition 1.4] and the fact that $S \otimes I = \oplus_n S$. If T has the decomposability property δ, then $I \otimes S$ has Dunford's property (C). In [21] (see also [16] for a nice corrected version), Sun showed that a product of commuting operators with Dunford's property (C) has the SVEP. Using these two results and the duality between property (β) and property (δ), we obtain the second result on the SVEP of $A \otimes B$.

PROPOSITION 2.3. *Let A, B be in $\mathcal{B}(H)$. Suppose that A has Dunford's property (C) and B has Bishop's property (β). Then $A \otimes B$ has the SVEP.*

For similar question on Bishop's property of $A \otimes B$, we have the following result

LEMMA 2.4. *Let A, B be in $\mathcal{B}(H)$. Suppose A and B are without Bishop's (β) property at λ. Then $A \otimes B$ is without Bishop's (β) property at λ.*

Proof. Suppose that $\sigma_\beta(A) \neq \emptyset$ and $\sigma_\beta(B) \neq \emptyset$. Let $\alpha \in \sigma_\beta(A)$ and $\xi \in \sigma_\beta(B)$. Then there exist two non-zero sequences $\{u_n\}, \{v_n\}$ of $H-$valued analytic functions in the neighborhood of λ, such that

$$\lim_{n\to\infty} (A - \gamma I) u_n(\gamma) = 0 \qquad \text{in } \; O(V_\alpha, H) \tag{4}$$

and

$$\lim_{n\to\infty} (B - \mu I) v_n(\mu) = 0 \qquad \text{in } \; O(V_\xi, H)\,. \tag{5}$$

Noticing that

$$\begin{aligned}
&\Big(A \otimes B - \gamma\mu(1 \otimes 1)\Big)\Big(u_n(\gamma) \otimes v_n(\mu)\Big) \\
&\qquad = \Big((A - \gamma) \otimes B + \gamma \otimes (B - \mu)\Big)\Big(u_n(\gamma) \otimes v_n(\mu)\Big) \\
&\qquad = (A - \gamma) u_n(\gamma) \otimes B v_n(\mu) + \gamma u_n(\gamma) \otimes (B - \mu) v_n(\mu)\,.
\end{aligned}$$

By (4) and (5), we obtain

$$\big(A \otimes B - \gamma\mu\big)\big(u_n(\gamma) \otimes v_n(\mu)\big) \to 0 \qquad \text{in } \ O(V_\alpha, H) \otimes O(V_\xi, H)\,.$$

This implies that $\sigma_\beta(A \otimes B) \neq \emptyset$. ∎

Remark 5. The following questions are naturally raised :

(1) Given A, B in $\mathcal{B}(H)$. Is

$$\sigma_\beta(A)\sigma_\beta(B) \subset \sigma_\beta(A \otimes B)?$$

By duality, if the answer to question (1) is positive, then one can obtain the following inclusions

$$\sigma_\delta(A)\sigma_\delta(B) \subset \sigma_\delta(A \otimes B).$$

(2) Given A, B in $\mathcal{B}(H)$. Let $x \in H$. Following the lines of proof of Lemma 2.4, it is easy to see that $\sigma_{A\otimes I}(x \otimes 1) = \sigma_A(x)$. Suppose that A and B are commuting. What can be said about $\sigma_{A\otimes B}(x \otimes y)$.

References

[1] Albrecht, E., Eschmeier, J., Analytic functional models and local spectral theory, *Proc. London Math. Soc. (3)* **75** (1997), 323–348.

[2] Aupetit, B., "A Primer on Spectral Theory", Springer-Verlag, New York, 1991.

[3] Aupetit, B., Drissi, D., Local spectrum and subharmonicity, *Proc. Edinburgh Math. Soc. (2)* **39** (1996), 571–579.

[4] Berg, I., An extension of the Weyl-von Neumann theorem to normal operators, *Trans. Amer. Math. Soc.* **160** (1971), 365–371.

[5] Brelot, M., "Éléments de la Thèorie du Potential", Centre de Documentation Universitaire, Paris, 1965.

[6] Brown, A., Pearcy, C., Spectra of tensor products of operators, *Proc. Amer. Math. Soc.* **17** (1966), 162–166.

[7] Colojoară, I., Foiaş, C., "The Theory of Generalized Spectral Operators", Gordon and Breach, Science Publishers, New York, 1968.

[8] Dixmier, J., "Les Algèbres d'Opérateurs dans l'Espace Hilbertien,", Gauthier-Villars, Paris, 1957.

[9] Drissi, D., Local spectrum and Kaplansky's theorem on algebraic operators, *Colloq. Math.* **75** (1998), 159–165.

[10] Finch, J.K., The single valued extension property on a Banach space, *Pacific J. Math.* **58** (1975), 61–69.

[11] Foiaş, C., Vasilescu, F.-H., On spectral theory of commutator, *J. Math. Anal. Appl.* **31** (1970), 473–486.

[12] Hayman, W.K., Kennedy, P.B., "Subharmonic Functions", Academic Press, London, 1976.

[13] Hille, E., On roots and logarithms of elements of a complex Banach algebra, *Math. Ann.* **136** (1958), 46 – 57.

[14] Laursen, K.B., Neumann, M.M., "An Introduction to Local Spectral Theory", The Clarendon Press, Oxford University Press, New York, 2000.

[15] Laursen, K.B., Miller, V.G., Neumann, M.M., Local spectral properies of commutators, *Proc. Edinburgh Math. Soc. (2)* **38** (1995), 313 – 329.

[16] Miller, T.L., Neumann, M.M., The single-valued extension property for sums and products of commuting operators, *Czechoslovak Math. J.* **52** (2002), 635 – 642.

[17] Miller, T.L., Miller, V.G., An operator satisfying Dunford condition (C) but without Bishop's property (β), *Glasg. Math. J.* **40** (1998), 427 – 430.

[18] Mbekhta, M., Sur la theorie spectrale locale et limite des nilpotents, *Proc. Amer. Math. Soc.* **110** (1990), 621 – 631.

[19] Prăjitură, G.T., The classes of local spectral theory, *J. Math. Anal. Appl.* **305** (2005), 175 – 182.

[20] Schechter, M., On the spectra of operators on tensor products, *J. Funct. Anal.* **4** (1969), 95 – 99.

[21] Sun, S.L., The sum and product of decomposable operators (chinese), *Northeast. Math. J.* **5** (1989), 105 – 117.

[22] "Elementary Operators and Applications", edited by M. Mathieu (Proceedings of the International Workshop, Blaubeuren, June 9 – 12, 1991), World Scientific, River Edge, NJ, 1992.