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Weighted Composition Operators on Weighted Bergman Spaces

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Abstract: In this paper, we study boundedness, compactness and the essential norm of a class of weighted composition operators on weighted Bergman spaces.

Key words: Bergman spaces, Carleson measure, essential norm, weighted composition operators.

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1. Introduction

Let $H(\mathbb{D})$ denotes the space of holomorphic functions on the unit disc $\mathbb{D}$. Take $1 \le p < \infty$ and $\alpha > -1$. Then $f \in H(\mathbb{D})$ is said to be in the weighted Bergman space $A^p_\alpha(\mathbb{D})$ iff

$$\|f\|^p_{A^p_\alpha} = \int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\alpha \mathrm{d}A(z) < \infty ,$$

where $\mathrm{d}A(z)$ denote the normalised area measure on the unit disc $\mathbb{D}$.

Let φ be a holomorphic function from the unit disc $\mathbb{D}$ into itself. Then the composition operator C_φ is defined as follows

$$C_\varphi(f)(z) = f(\varphi(z)) \qquad \text{for all } f \in H(\mathbb{D}) .$$

Again, let $\psi : \mathbb{D} \to \mathbb{D}$ be a fixed holomorphic map. Then the holomorphic Toeplitz operator T_ψ is defined as follows

$$T_\psi f(z) = \psi(z) f(z) \qquad \text{for all } f \in H(\mathbb{D}) .$$

Let D denote the differential operator. Then we define the operator $DC_\varphi T_\psi$ as

$$DC_\varphi T_\psi(f) = (\psi f \circ \varphi)' \qquad \text{for all } f \in H(\mathbb{D}) .$$

Again, the operator $T_\psi DC_\varphi$ is defined for $f \in H(\mathbb{D})$ by $T_\psi DC_\varphi(f) = \psi(f\circ\varphi)^{'}$.

Fix any $a \in \mathbb{D}$ and let $\sigma_a(z)$ be the Möbius transformation of $\mathbb{D}$ which interchanges 0 and a and is defined by

$$\sigma_a(z) = \frac{a-z}{1-\overline{a}z}\,, \qquad z \in \mathbb{D}\,.$$

If $K_a(z) = \frac{1}{(1-\overline{a}z)^2}$ denotes the Bergman kernel, then

$$k_a(z) = -\sigma_a^{'}(z) = \frac{1-|a|^2}{(1-\overline{a}z)^2}$$

is the normalised kernel function for the Bergman space A^2 and $\|k_a\|_{A^2} = 1$.

We know that on a general space of analytic functions, the differential operator D is typically unbounded. On the other hand, the composition operator C_φ is bounded on various spaces of analytic functions on $\mathbb{D}$ (see [4], [13], [16]), though the products DC_φ and $C_\varphi D$ are possibly still unbounded there. Hibschweiler and Portnoy [7] defined the products DC_φ and $C_\varphi D$ and investigated boundedness and compactness of DC_φ and $C_\varphi D$ between weighted Bergman spaces using the Carleson-type measures. J.S. Choa and S. Ohno [2], J.H. Shapiro and W. Smith [14] have given some examples showing that T_ψ need not be bounded (compact) on the Bergman space A^2, but their product $T_\psi C_\varphi$ is bounded (compact) on A^2. Motivated by the work of Hibschweiler and Portnoy [7], we define new operators $DC_\varphi T_\psi$ and $T_\psi DC_\varphi$ and study their boundedness and compactness between weighted Bergman spaces using the Carleson-type conditions. Moreover, in Section 3, we also find estimates for the essenital norm of $T_\psi DC_\varphi$.

2. Bounded and compact weighted composition operators

In this section, we characterize boundedness and compactness of $DC_\varphi T_\psi$ by using Carleson measures.

Definition 1. Take $0 < p < \infty$. A positive measure μ on $\mathbb{D}$ is called a p-Carleson measure in $\mathbb{D}$ if

$$\sup_{I\subset\partial\mathbb{D}} \frac{\mu(S(I))}{|I|^p} < \infty\,, \tag{2.1}$$

where $|I|$ denotes the arc length of I and $S(I)$ denotes the Carleson square based on I,

$$S(I) = \left\{ z \in \mathbb{D} \,:\, 1-|I| \le |z| < 1\,,\ \frac{z}{|z|} \in I \right\}.$$

Again, μ is called a compact p-Carleson measure if

$$\lim_{|I|\to 0} \frac{\mu(S(I))}{|I|^p} = 0\,. \tag{2.2}$$

DEFINITION 2. Let φ be a holomorphic mapping defined on $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. Take $p \geq 1$ and let $\alpha > -1$. Then the counting function for the weighted Bergman spaces A^p_α is

$$N_{\varphi,\alpha,p}(\omega) = \sum_{\varphi(z)=\omega} |\varphi'(z)|^{p-2}(1-|z|^2)^\alpha$$

for $0 \neq \omega \in \mathbb{D}$.

Recall that the pseudohyperbolic metric ρ is defined by

$$\rho(z,\omega) = \left|\frac{z-\omega}{1-\bar{z}\omega}\right|, \qquad z,\omega \in \mathbb{D}\,.$$

Let $D(a)$ denotes the pseudohyperbolic disc $\{z : \rho(a,z) < 1/8\}$. The following results are well known.

THEOREM 2.1. ([5, Theorem A]) *Take* $1 < p \leq q < \infty$. *Let* μ *be a finite positive Borel measure on* $\mathbb{D}$. *Then the following statements are equivalent:*

(1) *The inclusion map* $i : A^p_\alpha \to L^q(\mathbb{D}, \mathrm{d}\mu)$ *is bounded.*

(2) *The measure* μ *is an* $(\alpha+2)q/p$*-Carleson measure.*

(3) *For all* $a \in \mathbb{D}$ *we have*

$$\int_{\mathbb{D}} |k_{a,\alpha}(z)|^q \mathrm{d}\mu(z) \leq C\,,$$

where $k_{a,\alpha}(z) = (1-|a|^2)^{(\alpha+2)/p}(1-\bar{a}z)^{-2(\alpha+2)/p}$.

THEOREM 2.2. ([16, Theorem 8.2.5]) *Take* $1 < p \leq q < \infty$. *Let* μ *be a finite positive Borel measure on* $\mathbb{D}$. *Then the following statements are equivalent:*

(1) *The inclusion map* $i : A^p_\alpha \to L^q(\mathbb{D}, \mu)$ *is compact.*

(2) *The measure* μ *is a vanishing* $(\alpha+2)q/p$*-Carleson measure.*

(3) *For all* $a \in \mathbb{D}$ *we have*

$$\lim_{|a|\to 1} \int_{\mathbb{D}} |k_{a,\alpha}(z)|^q \mathrm{d}\mu(z) = 0\,.$$

The proof of the following lemma follows on similar lines as in [4, Proposition 3.11].

Lemma 2.3. *Given* $1 \le p, q < \infty$. *Take* $T = DC_\varphi T_\psi$ *or* $T_\psi DC_\varphi$. *Let* φ *be a holomorphic mapping defined on* $\mathbb{D}$ *with* $\varphi(\mathbb{D}) \subseteq \mathbb{D}$ *and* $\psi \in H(\mathbb{D})$ *be such that* $T : A^p_\alpha \to A^q_\alpha$ *is bounded. Then* $T : A^p_\alpha \to A^q_\alpha$ *is compact (respectively, weakly compact) if and only if whenever* $\{f_n\}$ *is a bounded sequence in* A^p_α *converging to zero uniformly on compact subsets of* $\mathbb{D}$, *then* $\|T(f_n)\|_{A^q_\alpha} \to 0$ *(respectively,* $\{T(f_n)\}$ *is a weak null sequence in* A^q_α*).*

We state a result of Luecking [10, Theorem 2.2] for the case $n = 1$ and $1 \le p \le q$.

Theorem 2.4. *Take* $1 \le p \le q$ *and let* $\alpha > -1$. *Let* $\mu \ge 0$ *be a finite measure on* $\mathbb{D}$. *Then the followings are equivalent:*

(1) $\|f'\|_{L^q(\mu)} \le C\|f\|_{A^p_\alpha}$ *for all* $f \in A^p_\alpha$.

(2) $\mu(D(a)) = O(1-|a|^2)^{q(\alpha+2+p)/p}$ *as* $|a| \to 1$.

Theorem 2.5. *Take* $1 \le p < \infty$ *and* $\alpha > -1$. *Let* φ *be a holomorphic self-map of* $\mathbb{D}$ *with* $\varphi' \in A^p_\alpha$ *and* $\psi \in A^p_\alpha$ *such that* $\psi' \in A^p_\alpha$. *Let* $\mathrm{d}\mu(\omega) = |\psi(\omega)|^p N_{\varphi,\alpha,p}(\omega)\mathrm{d}A(\omega)$. *Suppose* $\mu(D(a)) = O(1-|a|^2)^{(\alpha+2+p)}$ *as* $|a| \to 1$. *Then* $DC_\varphi T_\psi : A^p_\alpha \to A^p_\alpha$ *is bounded if and only if* $|\psi'|^p N_{\varphi,\alpha,p}\mathrm{d}A$ *is a Carleson measure on* A^p_α.

Proof. First suppose that $|\psi'|^p N_{\varphi,\alpha,p}\mathrm{d}A$ is a Carleson measure on A^p_α. Then for $f \in A^p_\alpha$

$$\begin{aligned}\|DC_\varphi T_\psi(f)\|^p_{A^p_\alpha} &= \int_{\mathbb{D}} |(\psi f \circ \varphi)'(z)|^p (1-|z|^2)^\alpha \mathrm{d}A(z) \\ &= \int_{\mathbb{D}} |(\psi f)'(\varphi(z))|^p |\varphi'(z)|^p (1-|z|^2)^\alpha \mathrm{d}A(z)\,.\end{aligned}$$

By making a non-univalent change of variables as done in [13, p. 86], we see that

$$\begin{aligned}\|DC_\varphi T_\psi(f)\|^p_{A^p_\alpha} &= \int_{\mathbb{D}} |(\psi f)'(\omega)|^p N_{\varphi,\alpha,p}(\omega)\mathrm{d}A(\omega)\\ &\leq \int_{\mathbb{D}} |f(\omega)|^p|\psi'(\omega)|^p N_{\varphi,\alpha,p}(\omega)\mathrm{d}A(\omega)\\ &\quad + \int_{\mathbb{D}} |f'(\omega)|^p|\psi(\omega)|^p N_{\varphi,\alpha,p}(\omega)\mathrm{d}A(\omega)\,.\end{aligned}$$

Since $|\psi'|^p N_{\varphi,\alpha,p}\mathrm{d}A$ is a Carleson measure on A^p_α, the first term in the above inequality is bounded by some constant times $\|f\|^p_{A^p_\alpha}$. Again, by Theorem 2.4, we get that the second term is bounded by some constant times $\|f\|^p_{A^p_\alpha}$. Therefore, $DC_\varphi T_\psi : A^p_\alpha \to A^p_\alpha$ is bounded.

For the converse, assume $DC_\varphi T_\psi$ is bounded. Then there exists a constant $C > 0$ such that

$$\|DC_\varphi T_\psi(f)\|^p_{A^p_\alpha} \leq C\|f\|^p_{A^p_\alpha} \qquad \text{for all } f \in A^p_\alpha\,.$$

Also, there exists a constant $M > 0$ such that for $f \in A^p_\alpha$

$$\begin{aligned}\|DC_\varphi T_\psi(f)\|^p_{A^p_\alpha} &\geq M\int_{\mathbb{D}} |(\psi f)'(\omega)|^p N_{\varphi,\alpha,p}(\omega)\mathrm{d}A(\omega)\\ &\geq M\bigg\{\int_{\mathbb{D}} |f(\omega)|^p|\psi'(\omega)|^p N_{\varphi,\alpha,p}(\omega)\mathrm{d}A(\omega)\\ &\qquad - \int_{\mathbb{D}} |f'(\omega)|^p|\psi(\omega)|^p N_{\varphi,\alpha,p}(\omega)\mathrm{d}A(\omega)\bigg\}\\ &\geq M\bigg\{\int_{\mathbb{D}} |f(\omega)|^p\mathrm{d}\nu(\omega) - \int_{\mathbb{D}} |f'(\omega)|^p|\psi(\omega)|^p N_{\varphi,\alpha,p}(\omega)\mathrm{d}A(\omega)\bigg\}\,,\end{aligned}$$

where $\mathrm{d}\nu(\omega) = |\psi'(\omega)|^p N_{\varphi,\alpha,p}(\omega)\mathrm{d}A(\omega)$. Since $DC_\varphi T_\psi$ is bounded, by using Theorem 2.4, we obtain

$$\int_{\mathbb{D}} |f(\omega)|^p\mathrm{d}\nu(\omega) \leq K\|f\|^p_{A^p_\alpha}$$

for some constant $K > 0$. Thus by Theorem 2.1, $|\psi'(\omega)|^p N_{\varphi,\alpha,p}(\omega)\mathrm{d}A(\omega) = \mathrm{d}\nu(\omega)$ is a Carleson measure on A^p_α. ∎

THEOREM 2.6. *Take $1 \leq p < \infty$ and $\alpha > -1$. Let φ be a holomorphic self-map of $\mathbb{D}$ with $\varphi' \in A^p_\alpha$ and $\psi \in A^p_\alpha$ such that $\psi' \in A^p_\alpha$. Suppose $|\psi(\omega)|^p N_{\varphi,\alpha,p}(\omega) = o(1-|\omega|^2)^\alpha$ as $|\omega| \to 1$. Then $DC_\varphi T_\psi : A^p_\alpha \to A^p_\alpha$ is compact if and only if $|\psi'|^p N_{\varphi,\alpha,p}\mathrm{d}A$ is a vanishing Carleson measure on A^p_α.*

Proof. First suppose that $DC_\varphi T_\psi : A^p_\alpha \to A^p_\alpha$ is compact. Then by using the similar argument as in [13, p. 86], there exists a positive constant $C > 0$ such that for $f \in A^p_\alpha$

$$\|DC_\varphi T_\psi(f)\|^p_{A^p_\alpha} \geq C \int_{\mathbb{D}} |(\psi f)'(\omega)|^p N_{\varphi,\alpha,p}(\omega)\mathrm{d}A(\omega)\,.$$

So, we have

$$\int_{\mathbb{D}} |f(\omega)|^p |\psi'(\omega)|^p N_{\varphi,\alpha,p}(\omega)\mathrm{d}A(\omega) \\ \leq C\left\{\|DC_\varphi T_\psi(f)\|^p_{A^p_\alpha} + \int_{\mathbb{D}} |f'(\omega)|^p |\psi(\omega)|^p N_{\varphi,\alpha,p}(\omega)\mathrm{d}A(\omega)\right\}.$$

In the above inequality, if we take $f = k_{a,\alpha} \in A^p_\alpha$, then

$$\int_{\mathbb{D}} |k_{a,\alpha}(\omega)|^p |\psi'(\omega)|^p N_{\varphi,\alpha,p}(\omega)\mathrm{d}A(\omega) \\ \leq C\left\{\|DC_\varphi T_\psi(k_a)\|^p_{A^p_\alpha} + \int_{\mathbb{D}} |k'_{a,\alpha}(\omega)|^p |\psi(\omega)|^p N_{\varphi,\alpha}(\omega)\mathrm{d}A(\omega)\right\}. \tag{2.3}$$

Since $DC_\varphi T_\psi$ is compact and the unit vectors $k_{a,\alpha}$ tends to zero uniformly on compact subsets of $\mathbb{D}$ as $|a| \to 1$, by Lemma 2.3, we have $\|DC_\varphi T_\psi(k_{a,\alpha})|^p_{A^p_\alpha} \to 0$ as $|a| \to 1$.

Also, for a given $\epsilon > 0$, we can find $0 < r < 1$ such that

$$|\psi(\omega)|^p N_{\varphi,\alpha,p}(\omega)\mathrm{d}A(\omega) \leq \epsilon(1-|\omega|^2)^\alpha \qquad \text{on } \ |\omega| \geq r\,. \tag{2.4}$$

Now take the integral

$$\int_{\mathbb{D}} |k'_{a,\alpha}(\omega)|^p |\psi(\omega)|^p N_{\varphi,\alpha}(\omega)\mathrm{d}A(\omega) = \int_{r\mathbb{D}} |k'_{a,\alpha}|^p |\psi(\omega)|^p N_{\varphi,\alpha}(\omega)\mathrm{d}A(\omega) \\ + \int_{\mathbb{D}\setminus r\mathbb{D}} |k'_{a,\alpha}(\omega)|^p |\psi(\omega)|^p N_{\varphi,\alpha}(\omega)\mathrm{d}A(\omega)\,.$$

Also,

$$\begin{aligned}\int_{r\mathbb{D}} &|k'_{a,\alpha}(\omega)|^p|\psi(\omega)|^p N_{\varphi,\alpha}(\omega)\mathrm{d}A(\omega)\\ &\le \left(\max_{|\omega|\le r}|k'_{a,\alpha}(\omega)||\psi(\omega)|\right)^p \int_{\mathbb{D}} N_{\varphi,\alpha}(\omega)\mathrm{d}A(\omega) \qquad (2.5)\\ &\le M\left(\max_{|\omega|\le r}|k'_{a,\alpha}(\omega)|\right)^p\left(\max_{|\omega|\le r}|\psi(\omega)|\right)^p \quad \to 0 \quad \text{as} \quad |a|\to 1\end{aligned}$$

because $k'_{a,\alpha} \to 0$ uniformly on compact subsets of $\mathbb{D}$.

Again, by condition (2.4), we have

$$\begin{aligned}\int_{\mathbb{D}\setminus r\mathbb{D}} |k'_{a,\alpha}(\omega)|^p|\psi(\omega)|^p N_{\varphi,\alpha}(\omega)\mathrm{d}A(\omega) &\le \epsilon\int_{\mathbb{D}\setminus r\mathbb{D}}(1-|\omega|^2)^\alpha|k'_{a,\alpha}(\omega)|^p\mathrm{d}A(\omega)\\ &\le M\epsilon\|k'_{a,\alpha}\|^p_{A^p_\alpha} \le M\epsilon\,. \qquad (2.6)\end{aligned}$$

From (2.5) and (2.6), we have

$$\limsup_{|a|\to 1}\int_{\mathbb{D}} |k'_{a,\alpha}(\omega)|^p|\psi(\omega)|^p N_{\varphi,\alpha}(\omega)\mathrm{d}A(\omega) \le M\epsilon\,.$$

Since $\epsilon > 0$ was arbitrary, we get

$$\lim_{|a|\to 1}\int_{\mathbb{D}} |k'_{a,\alpha}(\omega)|^p|\psi(\omega)|^p N_{\varphi,\alpha}(\omega)\mathrm{d}A(\omega) = 0\,.$$

From condition (2.3), we have

$$\lim_{|a|\to 1}\int_{\mathbb{D}} |k_{a,\alpha}(\omega)|^p|\psi'(\omega)|^p N_{\varphi,\alpha}(\omega)\mathrm{d}A(\omega) = 0\,.$$

Therefore, by Theorem 2.2, we get that $|\psi'|^p N_{\varphi,\alpha,p}\mathrm{d}A$ is a vanishing Carleson measure on A^p_α.

Conversely, suppose that $|\psi'|^p N_{\varphi,\alpha,p}\mathrm{d}A$ is a vanishing Carleson measure on A^p_α. Let $\{f_n\}$ be a norm bounded sequence in A^p_α such that $\|f_n\|_{A^p_\alpha} \le 1$ and $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. Our aim is to prove that $DC_\varphi T_\psi$ is compact. By Lemma 2.3, it is enough to show that $\|DC_\varphi T_\psi(f_n)\|_{A^p_\alpha} \to 0$ as $n\to\infty$. Using the similar argument as in [13, p. 86], we have

$$\begin{aligned}\|DC_\varphi T_\psi(f_n)\|^p_{A^p_\alpha} \le C\Bigg\{&\int_{\mathbb{D}} |f_n(\omega)|^p|\psi'(\omega)|^p N_{\varphi,\alpha}(\omega)\mathrm{d}A(\omega)\\ &+\int_{\mathbb{D}} |f'_n(\omega)|^p|\psi(\omega)|^p N_{\varphi,\alpha}(\omega)\mathrm{d}A(\omega)\Bigg\}\,. \qquad (2.7)\end{aligned}$$

Since $|\psi^{'}|^p N_{\varphi,\alpha,p}\mathrm{d}A$ is a vanishing Carleson measure on A^p_α, so by Theorem 2.2, the first integral tends to zero as $n \to \infty$. By using the same arguments as in the direct part, we can prove that the second integral also tends to zero. ∎

3. Essential norm

In this section, we find estimates for the essential norm of operator $T_\psi DC_\varphi$.

Suppose φ is a holomorphic mapping defined on $\mathbb{D}$. Let $\varphi(\mathbb{D}) \subseteq \mathbb{D}$ and $\psi \in H(\mathbb{D})$ be such that $\psi\varphi^{'} \in A^p_\alpha$. We define the measure $\mu_{\varphi,\psi,p}$ on $\mathbb{D}$ by

$$\mu_{\varphi,\psi,p}(E) = \int_{\varphi^{-1}(E)} |\psi(z)\varphi^{'}(z)|^p (1-|z|^2)^\alpha \mathrm{d}A(z)\,, \tag{3.1}$$

where E is a measurable subset of the unit disc $\mathbb{D}$.

Using [3, Lemma 2.1], we can easily prove the following lemma.

Lemma 3.1. *Let φ be a holomorphic mapping defined on $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. Take $\psi \in H(\mathbb{D})$ such that $\psi\varphi^{'} \in A^p_\alpha$. Then*

$$\int_{\mathbb{D}} g \mathrm{d}\mu_{\varphi,\psi,p} = \int_{\mathbb{D}} |\psi(z)\varphi^{'}(z)|^p (g\circ\varphi)(z)(1-|z|^2)^\alpha \mathrm{d}A(z)\,,$$

where g is an arbitrary measurable positive function in $\mathbb{D}$.

The following two lemmas are proved in [5].

Lemma 3.2. *Take $0 < r < 1$ and denote $\mathbb{D}_r = \{z \in \mathbb{D} : |z| < r\}$. Let μ be a positive Borel measure on $\mathbb{D}$. Take*

$$\|\mu\|_r = \sup_{|I|\le 1-r} \frac{\mu(S(I))}{|I|^p}\,,$$

$$\|\mu\| = \sup_{I\subset\partial\mathbb{D}} \frac{\mu(S(I))}{|I|^p}\,,$$

where I run through arcs on the unit circle. Let μ_r denotes the restriction of measure μ to the set $\mathbb{D}\setminus\mathbb{D}_r$. Further, if μ is a Carleson measure on A^p_α, so is μ_r and $\|\mu_r\| \le C\|\mu\|_r$, where $C > 0$ is a constant.

Lemma 3.3. *For $0 < r < 1$ and $1 < p < \infty$ and let*

$$\|\mu\|^*_r = \sup_{|a|\ge r} \int_{\mathbb{D}} |k_{a,\alpha}(z)|^p \mathrm{d}\mu(z)\,.$$

Moreover, if μ is a Carleson measure on A^p_α, then $\|\mu_r\| \le C\|\mu\|^*_r$.

Take $f(z) = \sum_{s=0}^{\infty} a_s z^s$ holomorphic on $\mathbb{D}$. For a positive integer n, define the operators $R_n f(z) = \sum_{s=n+1}^{\infty} a_s z^s$ and $Q_n = Id - R_n$, where Id is the identity map.

Recall that the essential norm of an operator T is defined as:

$$\|T\|_e = \inf\{\|T - K\| \,:\, K \text{ is compact operator}\}.$$

By using [5, Proposition 3], we get the following genealization of [4, Lemma 3.16, p. 134] for A^p_α.

Lemma 3.4. *If T is a bounded linear operator on A^p_α, then*

$$C \limsup_{n\to\infty} \|TR_n\| \le \|T\|_e \le \liminf_{n\to\infty} \|TR_n\|$$

for some positive constant C independent of T.

In the following theorem we give the upper and lower estimates for the essential norm of the operator $T_\psi DC_\varphi$.

Theorem 3.5. *Let φ be a holomorphic mapping defined on $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. Take $\psi \in H(\mathbb{D})$ such that $\psi\varphi' \in A^p_\alpha$. Suppose that the induced measure $\mu_{\varphi,\psi,p}$ is a Carleson measure on A^p_α. Further, suppose $T_\psi DC_\varphi$ is bounded on A^p_α. Then there is a absolute constant $M \ge 1$ such that*

$$\limsup_{|a|\to 1} \int_{\mathbb{D}} \frac{(1-|a|^2)^{2+\alpha}}{|1-\overline{a}\omega|^{2(2+\alpha)}} \mathrm{d}\mu_{\varphi,\psi,p}(\omega) \le \|T_\psi DC_\varphi\|^p_e$$
$$\le M \limsup_{|a|\to 1} \int_{\mathbb{D}} \frac{(1-|a|^2)^{2+\alpha}}{|1-\overline{a}\omega|^{2(2+\alpha)}} \mathrm{d}\mu_{\varphi,\psi,p}(\omega)\,.$$

Proof. First we prove the upper estimate. By Lemma 3.4, we have

$$\|T_\psi DC_\varphi\|^p_e \le \lim_{n\to\infty} \|T_\psi DC_\varphi R_n\|^p_{A^p_\alpha} = \lim_{n\to\infty} \sup_{\|f\|_{A^p_\alpha}\le 1} \|(T_\psi DC_\varphi R_n)f\|^p_{A^p_\alpha}\,.$$

So, by using Lemma 3.1, we have

$$\begin{aligned}\|(T_\psi DC_\varphi R_n)f\|^p_{A^p_\alpha} &= \int_{\mathbb{D}} |\psi(z)(R_nf(\varphi(z)))^{'}|^p(1-|z|^2)^\alpha \mathrm{d}A(z)\\ &= \int_{\mathbb{D}} |\psi\varphi^{'}(z)|^p|(R_nf)^{'}(\varphi(z))|^p(1-|z|^2)^\alpha \mathrm{d}A(z)\\ &= \int_{\mathbb{D}} |(R_nf)^{'}(\omega)|^p \; di\mu_{\varphi,\psi,p}(\omega)\\ &= \int_{\mathbb{D}\setminus\mathbb{D}_r} |(R_nf)^{'}(\omega)|^p \mathrm{d}\mu_{\varphi,\psi,p}(\omega)\\ &\quad + \int_{\mathbb{D}_r} |(R_nf)^{'}(\omega)|^p \mathrm{d}\mu_{\varphi,\psi,p}(\omega)\,.\end{aligned}$$

Using [4, p. 133], we have

$$|R_nf(\omega)| = |\langle R_nf, K_\omega\rangle| = |\langle f, R_nK_\omega\rangle| \le \|f\|_{A^p_\alpha}\|R_nK_\omega\|_{A^q_\alpha}\,.$$

Again, we have

$$|(R_nf)^{'}(\omega)| = |\langle R_nf^{'}, K_\omega\rangle| = |\langle f^{'}, R_nK_\omega\rangle| \le \|f^{'}\|_{A^p_\alpha}\|R_nK_\omega\|_{A^q_\alpha}\,.$$

Take $0 < r < 1$ and $|\omega| \le r, \omega \in \mathbb{D}$. Also, take the Taylor expansion of $K_\omega(z) = \sum_{k=1}^{\infty}(k+1)\overline{\omega}^k z^k$. Using this Taylor expansion, we get the estimate $|R_nK_\omega(z)| \le \sum_{k=n+1}^{\infty} r^k(k+1)$ and so $|(R_nK_\omega)'(z)| \le \sum_{k=n+1}^{\infty} kr^k(k+1)$. Thus for any $\epsilon > 0$, we can find n large enough such that

$$\int_{\mathbb{D}} |R_nK_\omega(z)|^q(1-|z|^2)^\alpha \mathrm{d}A(z) < \epsilon^q\,.$$

Therefore, for a fixed r, we have

$$\sup_{\|f\|_{A^p_\alpha}\le 1}\int_{\mathbb{D}_r} |(R_nf)^{'}(\omega)|^p \mathrm{d}\mu_{\varphi,\psi,p}(\omega) \;\to\; 0 \quad \text{as} \quad n\to\infty\,.$$

Let $\mu_{\varphi,\psi,p,r}$ denotes the restriction of measure $\mu_{\varphi,\psi,p}$ to the set $\mathbb{D}\setminus\mathbb{D}_r$. So by using Lemma 3.3 and Theorem 2.1, we have

$$\begin{aligned}\int_{\mathbb{D}\setminus\mathbb{D}_r} |(R_nf)^{'}(\omega)|^p \mathrm{d}\mu_{\varphi,\psi,p,r}(\omega) &\le M\|\mu_{\varphi,\psi,p,r}\|\|(R_nf)^{'}\|^p_{A^p_\alpha}\\ &\le M\|\mu_{\varphi,\psi,p}\|^*_r\|f^{'}\|^p_{A^p_\alpha} \;\le\; M\|\mu_{\varphi,\psi,p}\|^*_r\,,\end{aligned}$$

where M is an absolute constant and $\|\mu_{\varphi,\psi,p}\|_r^*$ is defined as in Lemma 3.3. Therefore,

$$\lim_{n\to\infty} \sup_{\|f\|_{A^p_\alpha}\le 1} \|(T_\psi DC_\varphi R_n)f\|^p_{A^p_\alpha} \le \lim_{n\to\infty} M\|\mu_{\varphi,\psi,p}\|_r^* \,.$$

Thus, $\|T_\psi DC_\varphi\|_e^p \le M\|\mu_{\varphi,\psi,p}\|_r^*$. Taking $r\to 1$, we have

$$\begin{aligned}\|T_\psi DC_\varphi\|_e^p &\le M\lim_{r\to 1}\|\mu_{\varphi,\psi,p}\|_r^*\\ &= M\limsup_{|a|\to 1}\int_{\mathbb{D}} |k_{a,\alpha}(\omega)|^p \mathrm{d}\mu_{\varphi,\psi,p}(\omega)\\ &= M\limsup_{|a|\to 1}\int_{\mathbb{D}} \frac{(1-|a|^2)^{2+\alpha}}{|1-\overline{a}\omega|^{2(2+\alpha)}}\mathrm{d}\mu_{\varphi,\psi,p}(\omega)\,,\end{aligned}$$

which is the desired upper bound.

As for the lower bound, consider the function $k_{a,\alpha}$. Then $k_{a,\alpha}$ is a unit vector and $k_{a,\alpha}\to 0$ uniformly on compact subsets of $\mathbb{D}$. Also fix a compact operator K on A^p_α. Then $\|K(k_{a,\alpha})\|_{A^p_\alpha}\to 0$ as $|a|\to 1$.

Therefore,

$$\begin{aligned}\|T_\psi DC_\varphi\|_e^p \ge \|T_\psi DC_\varphi - K\|^p_{A^p_\alpha} &\ge \limsup_{|a|\to 1}\|(T_\psi DC_\varphi)k_{a,\alpha}\|^p_{A^p_\alpha}\\ &= \limsup_{|a|\to 1}\int_{\mathbb{D}} \frac{(1-|a|^2)^{2+\alpha}}{|1-\overline{a}\omega|^{2(2+\alpha)}}\mathrm{d}\mu_{\varphi,\psi,p}(\omega)\,.\end{aligned}$$

Thus we get the result. ∎

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