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Structural Properties of Banach and Fréchet Spaces Determined by the Range of Vector Measures

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Abstract: The major theme of this paper is the interaction between structural properties of Banach and Frechet spaces and the measure-theoretic properties of measures taking values in these spaces. The emphasis shall be on the geometric/topological properties of the range of vector measures, including mainly the issue involving localization of certain (distinguished) sequences in these spaces inside the range of vector measures with or without bounded variation. Besides a brief discussion of the properties determined by the range of a vector measure, the paper concludes with a list of problems belonging to this area which are believed to be open.

Key words: Range of a vector measure, bounded variation, nuclear map, Banach space.

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1. Introduction

Ever since the discovery of Liapunov's convexity theorem in 1941, asserting that the range of a countably additive (c.a) non-atomic measure taking values in a finite dimensional is convex, the theory of vector measures has come to occupy a central position, both within and outside of functional analysis where the theory has been successfully employed in control theory and other related areas of applied mathematics. Considering that our aim in this survey shall be motivated chiefly by the role it has played in modern functional analysis involving the geometric and structural properties of Banach spaces, we shall not include any discussion regarding those aspects of vector measure theory whose scope falls outside the domain of functional analysis. In fact, we shall confine ourselves mainly to those areas of the theory that have witnessed a fruitful interplay of ideas between Banach space geometry and vector measure theory. An attempt to generalize the scalar-valued measure theory to the Banach space context typically leads to a study of Banach spaces which are already familiar objects of study in functional analysis. A case in point is the Radon-Nikodym theorem from classical measure theory which is known to hold good precisely in those Banach spaces X for which Krein-Milman theorem holds for all subsets of X which are closed and bounded. Equivalently—and more pertinently—this property is characterized by the property that X-valued 1-summing maps on $C[0,1]$ coincide with nuclear maps.

As the title of the article suggests, we shall confine ourselves mainly to the interplay between the geometry of a Banach space and the properties of the range of vector measures taking values in these spaces. Specifically, we shall see that the problem of 'localizing' certain distinguished sequences from a Banach space X inside the range of X-valued measures of a certain type leads to interesting classes of Banach spaces with nice coincidence properties in terms of operator ideals. This is indeed the case for the 'distinguished' set consisting of all null sequences in X, which can be characterized by the property that all ℓ_1-valued 1-summing maps on X are already nuclear (see Theorem 3.2). This is the subject matter of Section 3 where the 'localization problem' involving absolutely p-summable sequences has been thoroughly investigated. Section 4 deals with the problem of deciding whether a sequence which lies inside the range of a vector measure actually lies inside the range of a vector measure with better properties. Here we also address ourselves to a conjecture proposed by the author in [31], stating that a Banach space X is necessarily finite dimensional if each sequence lying inside the range of an X-valued vector measure already lies inside the range of such a measure having bounded variation.

Apart from the situation described above, there are also instances where the scalar-valued results carry over to the Banach space setting without any trade-off: the Banach-Saks property of the range of a c.a. X-valued measure holds good, regardless of the Banach space one chooses for X. However, the obverse phenomena involving the failure of an R^n-valued measure-theoretic fact in *each* infinite-dimensional Banach space are aplenty: Liapunov's theorem quoted above or the fact that the closed unit ball of the target space is the range of a vector measure of bounded variation are some of the cases in point. It turns out that these so-called finite-dimensional phenomena-at least in most of the cases of interest-when studied in the setting of Fréchet spaces lead naturally to the class of nuclear Fréchet spaces which have been extensively studied in the literature. Recalling the well-known fact that the classes of nuclear spaces and Banach spaces intersect precisely in the class of finite dimensional spaces, it is reasonable to assert that nuclear Fréchet spaces provide the ultimate infinite-dimensional setting in which these finite dimensionality phenomena hold. All these issues are discussed in Section 5 whereas the last portion of this section addresses those aspects of vector measure theory whereby the equality of (the closed convex hull of) the ranges of two vector measures determines the extent to which the two measures share certain common properties. This section concludes with a list of problems that appear to remain open in the theory of vector measures.

We conclude this introduction by admitting that the survey is by no means complete, nor is the bibliography in any way close to being comprehensive. Among the main omissions from inclusion in the text of this paper is the issue involving the (weak) Liapunov property of a Banach space X: the closure of the range of an X-valued measure (of bounded variation) is convex. The close connection of this property with the so-called compact range property is an important theme in the theory surrounding the 'range of vector measures'. Nor is included any discussion regarding non-commutative analogues of this theory in the context of Von Neumann algebras and their projection lattices. These and related issues are planned to be discussed in a subsequent work that shall appear elsewhere.

2. Notation and terminology

We shall throughout let X, Y stand for Banach spaces, unless otherwise specified, with X^* and B_X denoting, respectively, the continuous dual and the closed unit ball of X. By an X-valued (vector)-measure, we shall mean a function $\mu : \Sigma \to X$ defined on the σ-algebra Σ of a set Ω which is countably additive (c.a): For each sequence $\{A_n\} \subset \Sigma$ with $A_n \cap A_m = \phi$ we have

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n). \tag{1}$$

Taking note of the fact that a union of sets is independent of the ordering of sets involved, it is easily seen that the series on the right-hand side in (1) above converges unconditionally. We shall say that μ is of *bounded variation* if its *total variation*, $tv(\mu)$ is finite, i.e., if $tv(\mu) = |\mu|(\Omega) < \infty$ where, for $E \in \Sigma, |\mu|(E) = \sup \Sigma_{A\in\Omega} \|\mu(A)\|$, the supremum being taken over all partitions of E into a finite number of pairwise disjoint members of Σ. The space of all (bounded) sequences from X which are included inside the range of an X-valued measure (respectively of bounded variation) shall be denoted by $R(X)$

(respectively $R_{bv}(X)$) with the norm on $R(X)$ being given by

$$\|(x_n)\| = \inf \|\mu\|, \tag{2}$$

where the infimum is taken over all vector measures μ with $(x_n) \subset rg(\mu) = \{\mu(A); A \in \Sigma\}$, the range of μ and where

$$\|\mu\| = \sup\{|x^* \circ \mu|(\Omega) : x^* \in B_{X^*}\}. \tag{3}$$

For $(x_n) \in R_{bv}(X)$, we put

$$\|(x_n)\|_{bv} = \inf tv(\mu) = \inf |\mu|(\Omega), \tag{4}$$

where the infimum ranges over all vector measures μ with $(x_n) \subset rg(\mu)$. It turns out that $(R(X),\ \|\cdot\|)$ and $(R_{bv}(X),\ \|\cdot\|_{bv})$ are Banach spaces. See [28, Prop. 3.1] and [24, Prop. 2.1].

In the course of our work, we shall also make use of the spaces $R_{vbv}(X)$. The space $R_{vbv}(X)$ shall denote the collection of all bounded sequences $\overline{x} = (x_n)$ from X such that $(x_n) \subset rg(\mu)$ for some Y-valued measure μ of bounded variation where Y is a Banach space containing X (isometrically) as a subspace. As observed in [33], the formula

$$\|\overline{x}\|_{vbv} = \inf \big\{tv(\mu) : \text{there exists a vector measure of bounded variation } \mu : \Sigma \to \ell_\infty(B_{X^*}), (x_n) \subset rg(\mu)\big\} \tag{5}$$

defines a complete norm on $R_{vbv}(X)$. For $Y = X^{**}$, we shall write $R_{bbv}(X)$ for $R_{vbv}(X)$. Also, we see that $R_{bv}(X)$ coincides with $R_{vbv}(X)$ for $Y = X$. We shall also have occasion to use the space $R_b(X)$ (respectively, $R_c(X)$) consisting of all bounded sequences in X which are contained inside the range of an X^{**}-valued measure (respectively, X-valued measure having relatively compact range). We note the following chain of inclusion relations:

$$R_{bv}(X) \subset R_{bbv}(X) \subset R_{vbv}(X) \subset R(X) \subset \ell_\infty(X), \tag{6}$$

where $\ell_\infty(X)$ denotes the (Banach) space of all bounded sequences from X. Whereas all other inclusions are obvious, the proof of the inclusion $R_{vbv}(X) \subset R(x)$ is based on the following useful characterization of membership in these (vector-valued) sequence spaces.

Proposition 2.1. ([25]) *Given a bounded sequence $\overline{x} = (x_n)$ in a Banach space X and the (bounded) linear map $T = T_{\overline{x}} : \ell_1 \to X$ induced by $\overline{x}$: $T(\overline{x}) = \Sigma_n \alpha_n x_n, \overline{\alpha} = (\alpha_n) \in \ell_1$, the following statements hold:*

(i) $\overline{x} \in R_{bv}(X)$ *if and only if* T *is strictly integral.*

(ii) $\overline{x} \in R_{bbv}(X)$ *if and only if* T *is integral.*

(iii) $\overline{x} \in R_{vbv}(X)$ *if and only if* T *is absolutely summing.*

Before we explain the idea involved in the proof of this result, we pause to collect some definitions involving nuclear and (p,q)-summing maps and their basic properties that will be used in the sequel.

Definition 2.2. Given a bounded linear operator $T : X \to Y$, we shall say that T is

(a) *Nuclear* ($T \in N(X,Y)$) if there exist bounded sequences $\{f_n\}_{n=1}^{\infty} \subset B_{X^*}, \{y_n\}_{n=1}^{\infty} \subset B_Y$ and $\{\lambda_n\}_{n=1}^{\infty} \in \ell_1$ such that

$$T(x) = \sum_{n=1}^{\infty} \lambda_n \langle x, f_n\rangle\, y_n,\ x \in X.$$

(b) ∞-*nuclear* ($T \in N_\infty(X,Y)$) if there are sequences $\{f_n\}_{n=1}^{\infty} \subset X^*$, $\{y_n\}_{n=1}^{\infty} \subset Y$ with $\lim_n f_n = 0$, $\epsilon_1((y_n)) < \infty$ such that

$$T(x) = \sum_{n=1}^{\infty} \langle x, f_n\rangle\, y_n,\ \forall x \in X.$$

The norm ν_∞ on $N_\infty(X,Y)$ is defined by

$$\nu_\infty(T) = \inf\left\{\sup_n\ \|f_n\| \cdot \epsilon_1((y_n))\right\}, \tag{7}$$

where the infimum ranges over all sequences $\{f_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ admissible in the above series. $(N_\infty(X,Y), \nu_\infty)$ then becomes a Banach space. (See [9, Ch. 5]). It can be easily checked that a sequence $\overline{x} = (x_n)$ lies inside the range of an X-valued measure with relatively compact range if and only if the induced map $T_{\overline{x}} : \ell_1 \to X,\ T_{\overline{x}}(\overline{\alpha}) = \sum_{n=1}^{\infty} \alpha_n x_n$ is ∞-nuclear.

(c) (p,q)-(*absolutely*) *summing* ($p \geq q \geq 1$), if there exists $c > 0$ such that

$$\left(\sum_{i=1}^{n} \|Tx_i\|^p\right)^{1/p} \leq c \sup_{f \in B_{X^*}} \left(\sum_{i=1}^{n} |\langle x_i, f\rangle|^q\right)^{1/q} \tag{8}$$

for all $x_i \in X$, $1 \leq i \leq n$, $n \geq 1$.

Denoting the least such c by $\pi_{p,q}(T)$, it turns out that $\Pi_{p,q}(X,Y)$, the space of (p,q)-summing maps is a Banach space when equipped with the (p,q) - summing norm $\pi_{p,q}$. The special case $p=q$ corresponds to p-summing maps (= absolutely summing maps for $p=1$) which shall be denoted by $\Pi_p = \Pi_{p,p}$. For basic properties of (p,q)-summing maps, we refer to [9, Ch. 10]. Here we merely recall that p-summing maps between Hilbert spaces coincide with Hilbert-Schmidt maps and that, according to Grothendieck's theorem [9, Ch. 1], all bounded linear maps from $L_1(\mu)$ to $L_2(\nu)$ are absolutely summing (see also [30, Ch. 5].

We shall also say that a Banach space X satisfies *Grothendieck's Theorem* (or X has (GT)) if $L(X,\ell_2) = \Pi_1(X,\ell_2)$. In view of Grothendieck's theorem quoted above, L_1 has (GT). It turns out that X^* has (GT) if and only if $L(X,\ell_1) = \Pi_2(X,\ell_1)$. For a detailed account including further examples of (GT)-spaces, see [30].

Proposition 2.3. ([16, Ch. 2], [23, Ch. 1]) *Let X and Y be Banach spaces and assume that $1 \le p,\ q < \infty$. Then the following assertions hold:*

(a) $N(X,Y) \subset \Pi_p(X,Y)\ \ni\!:\ \pi_p(T) \le \nu(T),\ \forall\, T \in N(X,Y)$.

(b) $\Pi_p(X,Y) \subset \Pi_q(X,Y)\ \ni\!:\ \pi_q(T) \le \pi_p(T),\ \forall\, T \in \Pi_p(X,Y),\ (p \le q)$.

(c) $\Pi_2^{(2)}(X,Y) \subset N(X,Y)\ \ni\!:\ \nu(TS) \le \pi_2(T)\pi_2(S),\ \forall\, S \in \Pi_2(X,Z)$, $\forall\, T \in \Pi_2(Z,Y)$.

(d) $\Pi_p(X) \subset E_r(X),\ q = \max(p,2)$.

(e) $\Pi_p \circ \Pi_q(X) \subset E_r(X),\ (\frac{1}{r} = \frac{1}{p} + \frac{1}{q})$.

Here we recall that for operator ideals $\boldsymbol{A}$ and $\boldsymbol{B}$, the symbol $\boldsymbol{A}\circ\boldsymbol{B}(X,Y)$ has been used for the component of $\boldsymbol{A}\circ\boldsymbol{B}$ on the pair (X,Y):

$$\begin{aligned}\boldsymbol{A}\circ\boldsymbol{B}(X,Y) = \{T : X \to Y :\ &\textit{there exists a Banach space } Z \textit{ and}\\ &T_1 \in \boldsymbol{A}(X,Z),\ T_2 \in \boldsymbol{B}(Z,Y) \textit{ such that } T = T_2T_1\},\end{aligned} \tag{9}$$

whereas $E_r(X)$ stands for those operators on X which have p-summable eigenvalues.

We shall also be making use of the following vector-valued sequence spaces.

Definition 2.4. For $p \ge 1$, the vector-valued sequence spaces $\ell_p[X]$ and

$\ell_p\{X\}$ are defined by:

$$\begin{aligned} \ell_p[X] &= \left\{ \bar{x} = (x_n)_{n=1}^{\infty} \subset X : \sum_{n=1}^{\infty} |\langle x_n, x^* \rangle|^p < \infty, \ \forall\, x^* \in X^* \right\}, \\ \ell_p\{X\} &= \left\{ \bar{x} = (x_n)_{n=1}^{\infty} \subset X : \sum_{n=1}^{\infty} \|x_n\|^p < \infty \right\}, \end{aligned} \tag{10}$$

which turn into Banach spaces when equipped with the norms ϵ_p and σ_p, respectively, where

$$\begin{aligned} \epsilon_p(\bar{x}) &= \sup \left\{ \left(\sum_{n=1}^{\infty} |\langle x_n, x^* \rangle|^p \right)^{1/p} : x^* \in B_{X^*} \right\}, \ \bar{x} \in \ell_p[X], \\ \sigma_p(\bar{x}) &= \left(\sum_{n=1}^{\infty} \|x\|^p \right)^{1/p}, \ \bar{x} \in \ell_p\{x\}. \end{aligned} \tag{11}$$

Clearly, $\ell_p\{X\} \subset \ell_p[X]$ with $\epsilon_p(\bar{x}) \leq \sigma_p(\bar{x})$ for all $\bar{x} \in \ell_p\{x\}$ and equality holds precisely when X is finite-dimensional. The latter statement is the famous Dvoretzky-Rogers theorem to which we shall return in Section 3. The elements of $\ell_p[X]$ shall be referred to as *weakly p-summable* sequences whereas those of $\ell_p\{X\}$ shall be called *absolutely p-summable* sequences. An easy consequence of the uniform boundedness principle shows that the space $\ell_\infty[X]$ coincides with the space of all X-valued bounded sequences and that a simple computation shows that it can be identified with $L(\ell_1, X)$, the space of all bounded linear maps via the map

$$\ell_\infty[X] \ni \bar{x} = (x_n)_{n=1}^{\infty} \to T_{\bar{x}} \in L(\ell_1, X). \tag{12}$$

We are now ready to give a

Proof of Proposition 2.1: We shall essentially make use of the 'ideal' property of absolutely summing maps and the fact that an operator is absolutely summing if and only if its composite with an isometric embedding is absolutely summing.

Let us now show how the proof works in the case of (iii) and then comment how the proof can be adapted to prove the statements (i) and (ii). To this end, suppose that $\overline{x} = (x_n) \in R_{vbv}(X)$. We can assume, without loss of generality, that there exists a c. a. measure of bounded variation $\mu : \Sigma \longrightarrow \ell_\infty(\wedge)$ for some index set $\wedge$ such that X embeds isometrically into $\ell_\infty(\wedge)$ and $(x_n) \subset rg(\mu)$.

Choose $A_n \in \Sigma$ Such that $\mu(A_n) = x_n, \quad n \geq 1$. Let $Q : L_\infty(v) \longrightarrow \ell_\infty(\wedge)$ be the associated integration map corresponding to a control measure ν of μ, defined by $Q(f) = \int f d\mu$. Since μ is of bounded variation, Theorem 3 of [8, Ch. VI] tells us that Q is absolutely summing. To see that $T_{\overline{x}} : \ell_1 \longrightarrow X$ is absolutely summing, we note that $I_X \circ T_{\overline{x}} = Q \circ P$ where $P : \ell_1 \longrightarrow L_\infty(\nu)$ is defined by $P(\overline{\alpha}) = \sum \alpha_n \chi_{A_n}$. Here I_X is the canonical embedding of X into an $\ell_\infty(\wedge)$ space. It follows that $I_X \circ T_{\overline{x}}$ and hence $T_{\overline{x}}$ is absolutely summing.

For the converse, we use the well-known fact (see [28, Prop. 1.3]) that to every absolutely summing map $Q : C(\Omega) \longrightarrow X$, there corresponds an X-valued measure μ of bounded variation such that $\overline{Q(B_{C(\Omega)})} = rg(\mu)$. Indeed, assuming that $T_{\overline{x}}$ is absolutely summing, the Pietsch factorization theorem yields that $I_X \circ T_{\overline{x}} = QP$, where $P : \ell_1 \longrightarrow L_\infty(\nu)$ is a bounded linear operator and $Q : L_\infty(\nu) \longrightarrow \ell_\infty(\wedge)$ is absolutely summing. Since $L_\infty(\nu)$ is isometric to a $C(\Omega)$ space for compact Hausdorff space Ω, the result quoted above yields the existence of an $\ell_\infty(\wedge)$-valued measure μ of bounded variation such that $\overline{Q(B_{L_\infty(\nu)})} = rg(\mu)$. This gives $(x_n) = (I_X \circ T_{\overline{x}}(e_n)) = (QP(e_n)) \subset rg(\|P\|\mu)$. Here e_n is the nth unit vector in ℓ_1 and this completes the argument.

Regarding the proofs of (i) and (ii), we follow the same line of reasoning except that the space $\ell_\infty(\wedge)$ in (iii) shall be replaced by X in (i) and by X^{**} in (ii). Combining this with the fact that Q as an absolutely summing map on a $C(K)$ space is already strictly integral, it follows that $T_{\overline{x}} = Q \circ P$ is strictly integral in case of (i) and the fact that $T_{\overline{x}}$ is integral in case of (ii) follows from the Pietsch factorization theorem applied to the map $T_{\overline{x}}$ which admits a (sub)factorization $J_X \circ T_{\overline{x}} = Q \circ P$. Here J_X is the evaluation map of X into X^{**}. ∎

With the above result at our disposal, it is now easy to justify the statement immediately preceding Proposition 2.1. Indeed, let $\overline{x} \in R_{vbv}(X)$. By (iii) of the above proposition, $T = T_{\overline{x}}$ is absolutely summing and hence 2-summing. Hence in particular, T factors over the Hilbert space H. But it is well-known (see[1], see also [30]) that the unit ball of H is the range of an H-valued vector measure ν. Thus if $T = T_2 T_1$, then $T_2 \circ \nu$ is an X-valued measure such that $\overline{x} \in rg(T_2 \circ \nu)$.

In what follows we shall, however, write $\ell_\infty\{X\}$ for the space $c_0(X)$ of all null sequences in X, rather than identify it with the space of all bounded sequences. It is also clear that for $X = K$, the scalar field, $\ell_p[X] = \ell_p\{X\} = \ell_p$, the usual sequence space of all scalar sequences which are absolutely p-summable. We shall use e_i $(i \geq 1)$ to denote the ith unit vector in ℓ_p or in ℓ_p^n. An infinite sequence shall be denoted by $(x_n)_{n=1}^\infty$ and occasionally also by

(x_n) and the symbol $\sum_n$ shall be taken to mean that n varies from 1 to ∞.

3. Banach space setting

As already mentioned in the introduction, the property involving containment of a distinguished set $S(X)$ of sequences from X inside the range of vector measures (with or without bounded variation) taking values in (a space containing) X is closed linked to the structure of the Banach space X. In this section, we shall explore the extent to which the geometry of X is determined by the indicated property when $S(X) = c_0(X)$, $\ell_p[X]$, $\ell_p\{X\}$, $2 < p < \infty$. Restricting p to the range $2 < p < \infty$ is justified by the following theorem of Anantharaman and Diestel.

Theorem 3.1. ([1, Th. 3], see also [37]) *Regardless of the Banach space X, it always holds that $\ell_2[X] \subseteq R(X)$. Indeed, given $(x_n) \in \ell_2[X]$, then $(x_n) \subset rg(\mu)$ where μ is the X-valued measure defined by*

$$\mu(E) = 2\sum_n \Big(\int_E r_n(t)dtx_n\Big), \tag{13}$$

for any Lebesgue measurable subset $E \subseteq [0,1]$.

Contrary to the case $S(X) = \ell_2[X]$ covered by the above theorem which holds good for all Banach spaces X, we shall see below that it is possible to completely describe the (sub) class of Banach spaces X that result by choosing for $S(X)$ the (sequence) space $c_0(x)$ or $\ell_p\{X\}$, for $p > 2$. We begin with

a) $S(X) = c_0(X)$.

In this subsection, we shall deal with necessary and sufficient conditions to ensure the containment of null sequences in X inside the range of measures with or without bounded variation. We begin with

Theorem 3.2. ([28]) *For a Banach space X, the following statements are equivalent:*

(i) $c_0(X) \subset R(X)$.

(ii) *X^* is isomorphic to a subspace of an L_1 space.*

(iii) $N(X,\ell_1) = \Pi_1(X,\ell_1)$.

The proof of this result is accomplished by combining the following well-known facts from the theory of vector measures and Banach space theory:

(a.1) ([27, Prop. 2]). Given $T \in \Pi_1(X, \ell_1)$, so that $T(x) = (\langle x, x_n^* \rangle)_{n=1}^{\infty}$ where $x \in X, (x_n^*) \subset X^*$, the map

$$R(X) \ \ni: \overline{x} = (x_n) \to \sum_{n=1}^{\infty} \langle x_n, x_n^* \rangle \in \mathbb{R} \tag{14}$$

defines a continuous linear functional on R(X).

(a.2) ([18]). X is isomorphic to a subspace of an L^1-space if and only if there exists $c > 0$ such that for all finite subsets H and G of X, we have

$$\sum_{x \in H} \|x\| \leq c \sum_{y \in G} \|y\| \tag{15}$$

whenever

$$\sum_{x \in H} |\langle x, x^* \rangle| \leq \sum_{y \in G} |\langle y, x^* \rangle|, \text{ for all } x^* \in X^*. \tag{16}$$

(a.3) ([28, Lemma 3.3]). $c_0(L^\infty(\lambda)) \subset R(L^\infty(\lambda))$ for every positive measure λ. Further, the inclusion also holds for quotient spaces of $L^\infty(\lambda)$.

(a.4) The dual of $c_0(X)$ is $\ell_1\{X^*\}$.

Regarding the containment of $c_0(X)$ in $R_{bv}(X)$ and in $R_{vbv}(X)$, it turns out that whereas there are no infinite dimensional Banach spaces X for which the inclusion: $c_0(X) \subseteq R_{bv}(X)$ holds, Hilbert spaces are (isomorphically) the only Banach spaces X such that $c_0(X) \subseteq R_{vbv}(X)$. We describe these conditions in the following theorems.

Theorem 3.3. ([24]) *For a Banach space X, the following statements are equivalent:*

(i) $c_0(X) \subseteq R_{bv}(X)$.

(ii) $c_0(X) \subseteq R_{bbv}(X)$.

(iii) $L(X, \ell_1) = N(X, \ell_1)$.

(iv) $\dim(X) < \infty$.

THEOREM 3.4. ([25]) *For a Banach space X, the following statements are equivalent:*

(i) *X is Hilbertian.*

(ii) $B_X \subseteq R_{vbv}(X)$.

(iii) $\ell_\infty[X] \subseteq R_{vbv}(X)$.

(iv) $c_0(X) \subseteq R_{vbv}(X)$.

(v) $\Pi_2(X, \ell_1) = N(X, \ell_1)$.

An alternative approach to the above theorem based on the eigenvalue theorem on nuclear operators appears in a recent work of the author [37] which also deals with other interesting results involving vector measures.

Proof of Theorem 3.4. (Sketch) To begin, we note that (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv). Thus, to complete the proof, we show that (i) $\Rightarrow$ (ii), (iv) $\Rightarrow$ (v) and (v)$\Rightarrow$(i).

(i) $\Rightarrow$ (ii). Applying Grothendieck's theorem to X which is assumed to be a Hilbert space, we have

$$L(\ell_1(\wedge), X) = \Pi_1(\ell_1(\wedge), X),$$

where the index set $\wedge$ is chosen such that it has the same cardinality as that of B_X. Combining this equality with Proposition 2.1 (iii) gives that B_X is included inside the range of a vector *bv*-measure, i.e., $B_X \subset R_{vbv}(X)$.

(iv) $\Rightarrow$ (v). Let $T \in \Pi_2(X, \ell_1)$. Then, upon combining Proposition 2.3 (c) with Proposition 2.1 (iii), it is not difficult to see that the map

$$\psi_T(S) = \sum_{n=1}^{\infty} \langle x_n, x_n^* \rangle$$

defines a continuous linear functional on $\Pi_2(\ell_1, X) = \Pi_1(\ell_1, X)$. Note that T and S can be written as

$$T(x) = \sum_{n=1}^{\infty} \langle x, x_n^* \rangle e_n, \quad S(\alpha) = \sum_{n=1}^{\infty} \alpha_n x_n,$$

where $x \in X$, $\alpha \in \ell_1$, $(x_n)_{n=1}^{\infty} \subset X$ and $(x_n^*)_{n=1}^{\infty} \subset X^*$. Further, it is easy to see that the inclusion in (iv) is already continuous, so that ψ_T is continuous when restricted to $c_0(X)$. In view of (a.4) used in the proof of Theorem 3.2, there exists $(y^*)_{n=1}^{\infty} \in (c_0(X))^* = \ell_1\{X^*\}$—the space of absolutely summable sequences in X^*—such that for each $\bar{x} = (x_n)_{n=1}^{\infty} \in c_0(X)$,

$$\psi_T(\bar{x}) = \sum_{n=1}^{\infty} \langle x_n, x_n^* \rangle = \sum_{n=1}^{\infty} \langle x_n, y_n^* \rangle.$$

This gives $(x_n^*)_{n=1}^{\infty} = (y_n^*)_{n=1}^{\infty}$, so that $\sum_{n=1}^{\infty} \|x_n^*\| = \sum_{n=1}^{\infty} \|y_n^*\| < \infty$, which yields that $T \in N(X, \ell_1)$.

(v) $\Rightarrow$ (i). Here, we make use of the 'Eigenvalue Theorem' [14] which states that "*A Banach space X is (isomorphically) a Hilbert space precisely when nuclear maps on X have absolutely summable eigenvalues*". (See also [16]). By the closed graph theorem, there exists $c > 0$ such that $\nu(S) \leq c\,\pi_2(S)$, for all $S \in \Pi_2(X, \ell_1)$. We show that $L(\ell_1, X) = \Pi_2(\ell_1 X)$. To this end, let $T \in L(\ell_1, X)$. Then for $T_n = T\,|_{\ell_1^n}$, we get, using trace duality

$$\begin{aligned}\pi_2(T_n) &= \sup\{\text{trace}(T_n S_n) : S_n \in L(X, \ell_1^n),\ \pi_2(S_n) \leq 1\} \\ &\leq c\,\sup\{\text{trace}(T_n S_n) : S_n \in L(X, \ell_1^n),\ \nu(S_n) \leq 1\} \\ &= c\,\|T_n\| \leq c\,\|T\|,\ \forall\ n \geq 1.\end{aligned}$$

In other words, we get

$$\pi_2(T) = \sup_{n \geq 1} \pi_2(T_n) \leq c\,\|T\|.$$

This gives

$$L(\ell_1, X) = \Pi_2(\ell_1, X). \tag{17}$$

Finally, let $T \in N(X)$. Then $T = T_2 D T_1$ where $D : \ell_\infty \to \ell_1$ is a diagonal (nuclear) operator and $T_1 : X \to \ell_\infty$, $T_2 : \ell_1 \to X$ are bounded linear operators. By (17), $T_2 \in \Pi_2(\ell_1, X)$, so that $T = T_2 D T_1 \in \Pi_2 \circ N(X) \subset \Pi_2^{(2)}(X)$ (by Proposition 2.3 (a)). In other words, $N(X) = \Pi_2^{(2)}(X)$, so that by virtue of Proposition 2.3 (e), we conclude that all nuclear maps on X have absolutely summable eigenvalues. By the 'Eigenvalue Theorem' stated above, this is true precisely when X is a Hilbert space!

Alternatively, one may argue as follows. It suffices to show that every separable subspace of X is Hilbertian. Thus let Y be a separable subspace of X. Then there exists a quotient map from ℓ_1 onto Y which is, by (17), absolutely 2-summing and so factors over a Hilbert space. So Y as a quotient of a Hilbert space is (isomorphically) a Hilbert space.

The author is indebted to the referee for suggesting this alternative line of argument, bypassing the use of 'Eigenvalue Theorem'. ∎

b) $S(X) = \ell_p\{X\}$, $1 \leq p < \infty$.

The description of analogous conditions guaranteeing the containment of absolutely p-summable sequences in X inside the range of vector measures, with and without bounded variation is given in the following theorem.

THEOREM 3.5. ([38]) *For a Banach space X and p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have:*

(i) $\ell_p\{X\} \subset R(X)$ *if and only if there exists $c > 0$ such that*

$$\left(\sum_{i=1}^{n} \|x_i^*\|^q\right)^{1/q} \leq c\pi_1\left(\sum_{i=1}^{n} x_i^* \otimes e_i : X \to \ell_1^n\right),$$

for all $(x_i^*)_{i=1}^n \subset X^*,\ n \geq 1.$

(ii) $\ell_p\{X\} \subset R_{vbv}(X)$ *if and only if there exists $c > 0$ such that*

$$\left(\sum_{i=1}^{n} \|x_i^*\|^q\right)^{1/q} \leq c\pi_2\left(\sum_{i=1}^{n} x_i^* \otimes e_i : X \to \ell_1^n\right),$$

for all $(x_i^*)_{i=1} \subset X^*,\ n \geq 1.$

(iii) $\ell_p\{X\} \subset R_{bbv}(X)$ *if and only if there exists $c > 0$ such that*

$$\left(\sum_{i=1}^{n} \|x_i^*\|^q\right)^{1/q} \leq c\left\|\left(\sum_{i=1}^{n} x_i^* \otimes e_i : X \to \ell_1^n\right)\right\|,$$

for all $(x_i^*)_{i=1}^n \subset X^*,\ n \geq 1.$

Moreover, it turns out that each of the equivalent conditions in (iii) above is equivalent to X^* having (q)-Orlicz property, where q is conjugate to p. This means that each unconditionally convergent series in X^* is absolutely q-convergent. Here, we recall that for $q > 2$, (q)-Orlicz property of a Banach space X is the same as cotype q which means that almost sure (a.s.) convergence in X implies q-absolute convergence. However, the equivalence breaks down for $q = 2$. Both these fundamental facts are due to M. Talagrand who proved these results in his seminal work: *Inventiones Math.* **107** (1992), 1–40 and **110** (1992), 545 – 556. Furthermore, an application of Dvoretzky-Rogers theorem yields that dim $X < \infty$ as long as $p > 2$. Finally, considering that the proofs of the statements above are more or less similar, we settle for the proof of (iii) only.

Indeed, assume that X^* has (q)-Orlicz property and fix $\bar{x} = (x_n) \in \ell_p\{X\}$. We show that $(\alpha_n x_n) \in R_{bbv}(X)$ for all $\bar{\alpha} = (\alpha_n) \in c_0$, so that by virtue of [19, Th. 1], it follows that $\bar{x} \in R_{bbv}(X)$. Now, given $T = \sum_{n=1}^{\infty} x_n^* \otimes e_n \in K(X, \ell_1)$, we see that $(x_n^*) \in \ell_1[X^*]$ so that (q)-Orlicz property of X^* combined with Holder's inequality yields $c > 0$ such that

$$\begin{aligned}\sum_{n=1}^{\infty} \|x_n\| \, \|x_n^*\| &\leq \left(\sum_{n=1}^{\infty} \|x_n\|^p\right)^{1/p} \left(\sum_{n=1}^{\infty} \|x_n^*\|^q\right)^{1/q} \\ &\leq c\, \sigma_p(\bar{x}) \epsilon_1\big((x_n^*)\big),\end{aligned}$$

which proves that the map $\psi : K(X, \ell_1) \to \ell_1\{X^*\}$ given by $\psi(T) = (\|x_n\| x_n^*)_{n=1}^{\infty}$ is well-defined and continuous. Dualising and denoting by I the class of integral operators, we get (by virtue of [13, Chap. 19])

$$\psi^* : \ell_\infty\{X^{**}\} \to I(\ell_1, X^{**})$$

where

$$\psi^*\big((x_n^{**})\big)(T) = \langle (x_n^{**}), \psi(T) \rangle = \sum_{n=1}^{\infty} x_n^{**}(x_n^*) \|x_n\| = \operatorname{trace}(ST),$$

and $S = \sum_{n=1}^{\infty} e_n^* \otimes \|x_n\| x_n^{**} \in I(\ell_1, X^{**})$. This shows that $\psi^*\big((x_n^{**})\big) = S$, so that in particular, ψ^* actually maps $c_0(X)$ into $I(\ell_1, X)$ and that

$$\psi^*(\bar{y}) = \sum_{n=1}^{\infty} e_n^* \otimes \|x_n\| y_n, \quad \bar{y} = (y_n) \in c_0(X).$$

An application of Proposition 3.1 (ii) shows that $(\|x_n\| y_n) \in R_{bbv}(X)$. In particular, $(\alpha_n x_n) \in R_{bbv}(X)$ for all $\bar{\alpha} = (\alpha_n) \in c_0$ and this completes the argument.

Conversely, assume that $\ell_p\{X\} \subset R_{bbv}(X)$. By Proposition 3.1 (ii) the map $\psi : \ell_p\{X\} \to I(\ell_1, X)$ where $\psi(\bar{x}) = T_{\bar{x}}$, is well-defined and also continuous. Noting that each $\bar{x}$ in $\ell_p\{X\}$ is a limit of its 'nth-sections' in $\ell_p\{X\}$ and that $N(\ell_1, X)$ is a closed subspace of $I(\ell_1, X)$, it follows that ψ actually maps $\ell_p\{X\}$ into $N(\ell_1, X)$. Taking conjugates gives: $\psi^* : L(X, \ell_1^{**}) \to \ell_q\{X^*\}$

where $\psi^*(S)(\bar{x}) =$ trace $(T_{\bar{x}} \circ S)$, for all $S \in L(X, \ell_1^{**})$ and $\bar{x} \in \ell_p\{X\}$. Finally, let $\sum_{n=1}^{\infty} x_n^*$ be unconditionally convergent in X^*. Then for $S = \sum_{n=1}^{\infty} x_n^* \otimes e_n \in L(X, \ell_1)$, we have $\psi^*(S)(\bar{x}) = \sum_{n=1}^{\infty} \langle x_n, x_n^{\infty} \rangle$, for all $\bar{x} = (x_n) \in \ell_p\{X\}$, which yields that $\psi^*(S) = (x_n^*) \in \ell_q\{X^*\}$ and, therefore, X^* has (q)-Orlicz property.

Remark 3.6. Theorem 3.5 (iii) provides a refinement of the results of C. Piñeiro [28] and [26] pertaining to the description of Banach spaces X such that $c_0(X) \subset R_{bbv}(X)$ or $\ell_p[X] \subset R_{bbv}(X)$ for $p > 2$. The special case of our theorem corresponding to $p = 2$ was treated by Piñeiro in [25].

It is an important theme in the theory of operator ideals to know when the adjoint of a p-summing map acting between Banach spaces is q-summing for some $q \geq p$. For instance, it is well-known that the adjoint of a 2-summing map on a Banach space X is 2-summing precisely when X is a Hilbert space. The question involving the adjoint of a 1-summing (absolutely summing) map is treated below as an alternative necessary and sufficient condition for the inclusion: $\ell_p\{X\} \subset R(X)$ which is obtained by combining (i) and (iii) of Theorem 3.5.

THEOREM 3.7. ([38]) *For a Banach space X and $1 < p < \infty$, the following statements are equivalent:*

(i) $\ell_p\{X\} \subset R(X)$.

(ii) $\Pi_1(X, \ell_1) \subset \Pi_{q,1}^d(X, \ell_1)$.

Here q is conjugate to p : $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. (Sketch) We begin by noting that the proof of the of Theorem 3.5(iii) can be suitably generalized to assert the following operator-analogue of this equivalence:

(∗) *For a bounded linear map $T : X \to Y$, it holds that T maps sequences $\bar{x} = (x_n)$ in X from $\ell_p\{X\}$ into $(T(x_n)) \in R_{bbv}(Y)$ if and only if $T^* : Y^* \to X^*$ is (q,1)-summing.*

We use $(*)$ to show that (i) $\Leftrightarrow$ (ii).

(i) $\Rightarrow$ (ii). Let $T \in \Pi_1(X, Y)$ and $\bar{x} = (x_n) \in \ell_p\{X\}$ be arbitrarily chosen. In view of $(*)$, it suffices to show that $(T(x_n)) \in R_{bbv}(X)$. Note that (i) implies that $\ell_p\{X\} \subset R_c\{X\}$ and thus, it follows that $\bar{x} \in R_c(X)$ and, therefore, by [28, Prop. 1.4] applied to $\bar{x}$, there exists an unconditionally convergent series

$\sum_m y_m$ in X such that $x_n \in \sum_{m=1}^{\infty}[-y_m, y_n] = \{x \in X : x = \sum_{m=1}^{\infty} \alpha_m y_m$, for some $\bar{\alpha} = (\alpha_n) \in B_{\ell_\infty}\}$. By the definition of T, we have $\sum_{m=1}^{\infty} \|Ty_m\| < \infty$, which yields that the map $T_z : \ell_1 \longrightarrow Y$ induced by $z = (z_n) = (Tx_n)$ is nuclear and, a priori, strictly integral, so that by Proposition 2.1 (i), $(Tx_n) \in R_{bv}(Y)$.

(ii) $\Rightarrow$ (i). Here we invoke Theorem 3.5 to prove our assertion. To this end, fix $n \geq 1$ and $(x_i^*)_{i=1}^n \subset X^*$. Then for $S = \sum_{i=1}^n x_i^* \otimes e_i \in \Pi_1(X, \ell_1^n)$, we have $S \in \Pi_{q,1}^d(X, \ell_1^n)$. Now (ii) yields that there exists $c > 0$ such that

$$\pi_{q,1}(T^*) = \pi_{q,1}^d(T) \leq c\,\pi_1(T),\ \forall\, T \in \Pi_1(X, \ell_1^n),\ n \geq 1. \tag{18}$$

By the given hypothesis, $S^* \in \Pi_{q,1}(\ell_\infty^n, X^*)$ which translates into the estimate

$$\left(\sum_{i=1}^m \|S^*(\bar{\alpha}_i)\|^q\right)^{1/q} \leq \pi_{q,1}(S^*) \sup\left\{\sum_{i=1}^m |\langle \bar{\alpha}_i, \bar{\beta}\rangle| : \bar{\beta} \in B_{\ell_\infty^n}\right\} \tag{19}$$

for all $(\bar{\alpha}_i)^m \subset \ell_\infty^n$ and $m \geq 1$.

Combining (18) and (19) and noting that $S^*(e_i) = x_i^*,\ 1 \leq i \leq n$, we get

$$\left(\sum_{i=1}^n \|x_i^*\|^q\right)^{1/q} \leq c\,\pi_1\left(\sum_{i=1}^n x_i^* \otimes e_i : X \to \ell_1^n\right)$$

which was required to be proved. ∎

Proceeding on similar lines gives,

Theorem 3.8. ([38]) *For a Banach space X, we have*

(i) *$\ell_p\{X\} \subset R_{bbv}(X)$ if and only if $L(X, \ell_1) = \Pi_{q,1}^d(X, \ell_1)$.*

(ii) *$\ell_p\{X\} \subset R_{vbv}(X)$ if and only if $\Pi_2(X, \ell_1) \subseteq \Pi_{q,1}^d(X, \ell_1)$.*

Remark 3.9. (i) From the proof of Theorem 3.7, it is clear that there is nothing sacrosanct about ℓ_1 in the above two theorems. In fact, it is easily seen that these statements hold with ℓ_1 replaced by any Banach space and consequently, ℓ_1 serves as a 'test space' for the stated inclusions.

(ii) It is interesting to note that Theorems 3.7 and 3.8 when applied to $p = \infty$ yield the case of $\lambda = c_0(X)$ already encountered in Theorems 3.2, 3.3 and 3.4. This follows because, as per our notation, $\ell_\infty\{X\}$ denotes the space $c_0(X)$ of all null sequences in X, in which case the dual of $\ell_\infty\{X\}$

gets identified with $\ell_1\{X^*\}$. Combining this with the easily checked fact that $\prod_1^d(X, \ell_1)$ coincides with nuclear maps $N(X, \ell_1)$, the desired conclusions follow as given in the following corollary.

COROLLARY 3.10. *For a Banach space X, we have*

(i) $c_0(X) \subset R(X)$ *if and only if* $\Pi_1(X, \ell_1) = N(X, \ell_1)$. *Further, this is true if and only if X^* is a subspace of an $L^1(\mu)$-space.*

(ii) $c_0(X) \subset R_{bv}(X)$ *if and only if* $L(X, \ell_1) = N(X, \ell_1)$. *Further, this holds exactly when dim $X < \infty$.*

(iii) $c_0(X) \subset R_{vbv}(X)$ *if and only if* $\Pi_2(X, \ell_1) = N(X, \ell_1)$. *Further, this holds precisely when X is Hilbertian.*

Remark 3.11. The above discussion leaves open the question of describing those Banach spaces X with bounded sequences in X being contained inside the range of X-valued measures (without bounded variation)—the case of bounded variation having already been treated in Theorems 3.3 and 3.4. It is interesting to note that the classes of Banach spaces X described by these theorems are exactly those in which the stated property holds for all bounded sequences precisely when it holds for all null sequences in X. However, that is far from being the case in the absence of bounded variation: a Banach space X in which bounded sequences can be localized inside the range of an X-valued measure is necessarily reflexive (in fact, even super-reflexive) whereas there are non-reflexive spaces, e.g., c_0, ℓ_∞, in which all null sequences are included inside the range of vector-measures. In other words –in the absence of bounded variation– the property of null sequences in X being contained inside the range of an X-valued measure does not imply the same property for all bounded sequences in X. To the best of our knowledge, the problem explicitly stated below is open:

(P) Characterize Banach spaces X such that each bounded sequence in X (respectively; the unit ball of X) is contained inside the range of an X-valued measure. (See [7] for some partial results of J.M.F. Castillo and F. Sánchez.) Closely related to the above problem is the one where it is demanded that the unit ball B_X be (equal to) the range of an X-valued measure. Fortunately, there is a complete description of those spaces X for which the stated equality holds. This was achieved by Anantharaman and Diestel already in 1991.

THEOREM 3.12. ([1]) *The closed unit ball B_X of X is the range of an X-valued measure if and only if X^* is isometrically isomorphic to a reflexive subspace of $L^1(\mu)$ for some probability measure μ.*

As a consequence of the well known fact that L_p embeds in L_1 isometrically, if $1 \leq p \leq 2$, we see that for $2 \leq p < \infty$, the closed unit ball of ℓ_p^n or of ℓ_p is the range of a vector measure. The same holds for $L_p(\mu)$-spaces. Combining Theorem 3.12 with an old and important theorem of L. E. Dor (*Israel J. Math.* **24**(3-4) (1976), 260 – 268) that ℓ_p^3 does not embed isometrically in L_1 for $p > 2$, it follows that the closed unit ball of ℓ_p^n is not equal to the range of a measure for $1 < p < 2$. As another useful consequence, we recover a well-known fact that the closed unit ball of every 2-dimensional normed space is the range of a vector measure. This follows in view of the '2-universality' of $L_1[0,1]$—a fact due originally to Lindenstrauss—which means that $L_1[0,1]$ contains an isometric copy of each 2-dimensional space.

Using Gaussian random variables, L. R. Piazza [32] was able to explicitly construct an ℓ_2-valued measure whose range coincides with the closed unit ball of ℓ_2. His construction exploits the rotational invariance of the Gaussian distribution which yields the remarkable property of Gaussian variables that if $g_1, g_2, \ldots, g_n$ are independent standard Gaussian variables on a probability space (Ω, Σ, P) and if $a_1, a_2, \ldots, a_n$ are real scalars, then $\sum_{i=1}^n a_i g_i$ is Gaussian with variance $\sigma^2 = \sum_{i=1}^n a_i^2$. Combining this property with the fact that the equality

$$\left(\int_\Omega |g(w)|^p \, dP(w)\right)^{\frac{1}{p}} = \left(\frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} |t|^p \, e^{\frac{-t^2}{2}} \, dt\right)^{\frac{1}{p}}$$

holds for all standard Gaussian variables g and for all $0 < p < \infty$, we see that

$$\left(\int_\Omega \Big| \sum_{i=1}^n a_i g_i(w) \Big|^p \, dP(w)\right)^{\frac{1}{p}} = \left(\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} |t|^p \, e^{\frac{-t^2}{2\sigma^2}} \, dt\right)^{\frac{1}{p}}$$

for any $0 < p < \infty$ and $\sigma = \sum_{i=1}^n a_i^2$. To show that B_{ℓ_2} is indeed the range of a measure, he defines $\mu : \Sigma \longrightarrow \ell_2$ by

$$\mu(A) = \left(\sqrt{2\pi} \int_A g_n dP\right)_{n=1}^{\infty}.$$

The first equality noted above shows that μ is a well-defined (c.a.) measure with $\|\mu(A)\|_2 \leq 1$. Finally, given $a = (a_n) \in B_{\ell_2}$, the above-noted properties of Gaussian variables combined with the orthonormality of the sequence $\{g_n\}$ in $L_2(P)$ yields that for all $t \geq 0$,

$$\mu\left(\sum_{n=1}^{\infty} a_n g_n > t\right) = \exp\left(\frac{-t^2}{2}\right) a.$$

Taking $t = 0$, we get range $(\mu) = B_{\ell_2}$.

The idea of the proof employed above actually provides an isometric embedding of ℓ_2 into $L_p(\Omega, \Sigma, P)$ for any $0 < p < \infty$ via the map

$$(a_n) \longrightarrow m_p^{-1} \cdot \sum_{n=1}^{\infty} a_n g_n$$

where m_p denotes the p-th moment of the Gaussian variable

$$m_p = \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |t|^p \, e^{\frac{-t^2}{2}} \, dt \right)^{\frac{1}{p}}.$$

On the other hand, the original proof of Banach (see S. Kaczmarz and H. 'Steinhauss Theorie der Orthogonalreihen', Chelsea, 1951) that the closed unit ball of ℓ_2 is the range of a measure instead makes use of Rademacher averages and shows essentailly how to construct an isometric embedding of ℓ_2 into $L_1[0,1]$ and consequently a quotient map of norm one from $L_\infty[0,1]$ onto ℓ_2 which in turn yields that the unit ball of ℓ_2 can be realized as the range of a measure.

Remark 3.13. We have not included the discussion of the case $S(X) = \ell_p[X]$, $2 \leq p < \infty$, in our description of Banach spaces X for which the inclusions $\ell_p[X] \subseteq R(X)$, $R_{bbv}(X)$, $R_{vbv}(X)$ hold. Whereas the first two inclusions are a subject of discussion in C. Piñeiro's works [26] and [27], the issue related to the inclusion $\ell_p[X] \subset R_{vbv}(X)$ is discussed in detail in a recent work of the author [36].

We give a sample of these results in

THEOREM 3.14. *Given a Banach space X, $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, we have*

(i) ([26]) $\ell_p[X] \subset R(X)$ *if and only if there exists $c > 0$ such that for all $(x_i)_{i=1}^n \subset X$, $(x_i^*) \subset X^*$ and $n \geq 1$,*

$$\sum_{i=1}^{n} |\langle x_i, x_i^* \rangle| \leq c\pi_1 \left(\sum_{i=1}^{n} x_i^* \otimes x_i : X \to \ell_1^n \right) \epsilon_p\big((x_i)_{i=1}^n\big).$$

(ii) ([27]) $\ell_p[X] \subseteq R_{bbv}(X)$ *if and only if there exists* $c > 0$ *such that for all* $(x_i)_{i=1}^n \subset X$, $(x_i^*)_{i=1}^n \subset X^*$ *and* $n \geq 1$,

$$\sum_{i=1}^n |\langle x_i, x_i^* \rangle| \leq c \left\| \left(\sum_{i=1}^n x_i^* \otimes x_i : X \to \ell_1^n \right) \right\| \epsilon_p\big((x_i)_{i=1}^n\big).$$

(iii) ([36]) $\ell_p[X] \subseteq R_{bvb}(X)$ *if and only if there exists* $c > 0$ *such that for all* $(x_i)_{i=1}^n \subset X$, $(x_i^*)_{i=1}^n \subset X^*$ *and* $n \geq 1$,

$$\sum_{i=1}^n |\langle x_i, x_i^* \rangle| \leq c\pi_2 \left(\sum_{i=1}^n x_i^* \otimes x_i : X \to \ell_1^n \right) \epsilon_p\big((x_i)_{i=1}^n\big).$$

Before we proceed further, we digress a little on the question involving the possibility of replacing ℓ_1 by ℓ_2 in Corollary 3.10. It turns out that the situation that results in the process in respect of Corollary 3.10 (i) as given below, besides being interesting in its own right, shall also be seen to have implications in the theory of vector measures (see the next section). Further, the analogous situation resulting from ℓ_2 replacing ℓ_1 in Corollary 3.10 (ii) and (iii) characterizes X as finite-dimensional! See [36] for details of proof and applications.

Theorem 3.15. ([36]) *For a Banach space* X, *the following statements are equivalent:*

(i) $\Pi_1(X, \ell_2) = N(X, \ell_2)$.

(ii) X^* *has* (GT) *and* (GL).

(GL) *in* (ii) *above refers to the Gordon-Lewis property which means that for each Banach space* Y *and* $T \in \Pi_1(X, Y)$, *the composed map* $J_Y \circ T$ *factorizes over an* L_1*-space. Here,* $J_Y : Y \to Y^{**}$ *denotes the natural injection.*

4. Reverse implications

In this section, we address ourselves to the problem of describing (necessary and) sufficient conditions that would ensure that a sequence sitting inside the range of a vector measure already lies inside the range of a vector measure with bounded variation. Thus, for example, given $\overline{x} \in R(X)$, it makes sense to know if, under suitable extra conditions, $x \in R_{vbv}(X)$ or better still, $x \in R_{bv}(X)$. In what follows, we describe these conditions. We begin with

Theorem 4.1. ([20]) *For a Banach space X, the following statements are equivalent:*

(i) $R(X) = R_{vbv}(X)$.

(ii) $\Pi_2(X, \ell_1) = \Pi_1(X, \ell_1)$.

(iii) $\Pi_2(X, \ell_2) = \Pi_1(X, \ell_2)$.

(iv) $\Pi_2(X, Y) = \Pi_1(X, Y)$, *for some infinite-dimensional Banach space Y.*

Proof. (Sketch) A detailed study of Banach spaces satisfying the property $\Pi_2(L_\infty, X) = L(L_\infty, X)$ was carried out by Dubinsky, Pelcynski and Rosenthal in [10] where such spaces are characterized by each of the equivalent properties (ii), (iii) and (iv) given above. However, the equivalence (i) $\Leftrightarrow$ (ii) can be proved by using the following results of Piñeiro [20]:

For a Banach space X and a bounded sequence $\overline{x} = (x_n) \subset X$, the following statements are equivalent:

(A)

(a) $\overline{x} \in R_{vbv}(X)$.

(b) $\sum\limits_{n=1}^{\infty} |\langle x_n, x_n^* \rangle| < \infty$, $\forall$ *2-summing maps:* $x \in X \to (\langle x, x_n^* \rangle) \in \ell_1$.

(B)

(a) $(\alpha_n x_n) \in R(X)$, $\forall\, \overline{\alpha} = (\alpha_n) \in c_0$.

(b) $\sum\limits_{n=1}^{\infty} |\langle x_n, x_n^* \rangle| < \infty$, $\forall$ *1-summing maps:* $x \in X \to (\langle x, x_n^* \rangle) \in \ell_1$.

Indeed, assuming (i), it follows from Proposition 2.1 (iii) that $T : \ell_1 \to X$ where $T(\alpha) = \sum_{n=1}^{\infty} \alpha_n x_n$, $\overline{\alpha} = (\alpha_n) \in \ell_1$ is 1-summing for all sequence (x_n) lying inside the range of some X-valued measure (with relatively compact range), which can be identified with $N_\infty(\ell_1, X)$, the class of ∞-nuclear mappings. This gives rise to the inclusion

$$N_\infty(\ell_1, X) \subset \Pi_1(\ell_1, X) = \Pi_2(\ell_1, X),$$

which is easily seen to be continuous, by the closed graph theorem. Dualising the above inclusion yields

$$\Pi_2(X, \ell_1^{**}) \subset \Pi_1(X, \ell_1^{**}).$$

In particular, we get

$$\Pi_2(X, \ell_1) = \Pi_1(X, \ell_1).$$

Conversely, let $\bar{x} \in R(X)$. By (A), it suffices to prove A (b). Now, consider the map $T_{\overline{x}} : \ell_1 \to X$. Clearly, $T_{\overline{x}}$ maps B_{ℓ_1} into the closed convex hull of the range of an X-valued measure which, by virtue of [8, p. 274], is itself the range of an X-valued measure. Thus $(\alpha_n x_n) = (\alpha_n T(e_n))_n$ is included inside the range of an X-valued measure, so that by virtue of B (b),

$$\sum_{n=1}^{\infty} |\langle x_n, x_n^* \rangle| < \infty,$$

for all $S = (\langle \cdot, x_n^* \rangle) \in \Pi_1(X, \ell_1) = \Pi_2(X, \ell_1)$, which is A(b). This completes the proof. ∎

Next we treat the case involving the equality $R_{vbv}(X) = R_{bbv}(X)$.

Theorem 4.2. ([25]) *For a Banach space, it holds that* $R_{vbv}(X) = R_{bbv}(X)$ *if and only if* X^* *has* (GT).

Proof. Assume that $R_{vbv}(X) = R_{bb}(X)$. Then Theorem 2.1 combined with cotype 2 property of ℓ_1 yields

$$\Pi_2(\ell_1, X) = \Pi_1(\ell_1, X) = I(\ell_1, X).$$

By the open mapping theorem, there exists $c > 0$ such that

$$i(T) \leq c\, \pi_2(T), \text{ for all } T \in L(X, \ell_1)$$

which gives $\Pi_2(X, \ell_1) = L(X, \ell_1)$. In other words, X^* is a (GT)-space.

Conversely, assume that X^* has (GT) and let $\overline{x} \in R_{vbv}(X)$. By Theorem 2.1, $T = T_{\overline{x}} \in \Pi_1(\ell_1, X) = \Pi_2(\ell_1, X)$. Thus, we can write $T = T_1 T_2$ where $T_1 : \ell_2 \to X$ and $T_2 : \ell_1 \to \ell_2$ are bounded linear maps. This gives $T^* = T_1^* T_1^*$ with $T_1^* : X^* \to \ell_2$ and $T_2^* : \ell_2 \to \ell_\infty$. By the given hypothesis, $T_1^* \in \Pi_1(X^*, \ell_2)$. But then $T^* \in \Pi_1(X^*, \ell_\infty) = I(X^*, \ell_\infty)$, by virtue of [9, Cor. 6.24]. Finally T itself is integral by [9, Th. 5.15]. Equivalently, $\overline{x} \in R_{bbv}(X)$, again by Theorem 2.1. ∎

An intriguing problem in the theory of vector measures that has been around for quite some time now asks whether a subset of a Banach space X lying inside the range of an X^{**}-valued measure is already contained inside the range of an X-valued measure. Restricting to sequences in place of subsets of X, the problem amounts to asking whether the equality $R_b(X) = R(X)$ always holds! The analogous problem involving the equality $R_{bbv}(X) = R_{bv}(X)$

—in the presence of bounded variation—was answered in the negative by B. Marchena and C. Piñeiro [19] who used the $\mathcal{L}_\infty$-space constructed by Bourgain and Delbaen [6] which has the Radon-Nikodym property and also lacks copies of c_0. Piñeiro's construction depends upon the following theorem which is interesting in its own right.

THEOREM 4.3. ([19]) *For a Banach space X, the following statements are equivalent for a bounded sequence $(x_n) \subset X$:*

(i) $(x_n) \in R_{bbv}(X)$.

(ii) $(\alpha_n x_n) \in R_{bv}(X),\ \forall\ (\alpha_n) \in c_0$.

(iii) $(\alpha_n x_n) \in R_{bbv}(X),\ \forall\ (\alpha_n) \in c_0$.

Remark 4.4. As already seen in the proof of Theorem 4.1, given $(x_n) \in R(X)$, it follows that $(\alpha_n x_n) \in R(X)$, for all $(\alpha_n) \in c_0$. However, converse is not true. Indeed, if X is a non-reflexive $\mathcal{L}_\infty$-space, then $(\alpha_n x_n) \in R(X)$, for all bounded sequences $(x_n) \subset X$ and for all $(\alpha_n) \in c_0$. This follows from Theorem 3.2 because X^* is an L_1-space. However, by non-reflexivity of X, there exists a bounded sequence which is not in $R(X)$. The above theorem shows that the converse holds as long as the vector measures under reference are assumed to have bounded variation.

We are now ready to present Piñeiro's counter example to the equality $R_{bbv}(X) = R_{bv}(X)$.

Let X be the $\mathcal{L}_\infty$-space of Bourgain and Delbaen of the first class which has (RNP) and the Schur property, and so contains an isomorph of ℓ_1. In particular, X contains an unconditional basic sequence (x_n), say.

CLAIM. $(x_n) \in R_{bbv}(X)$ but $(x_n) \notin R_{bv}(X)$.

To show that $(x_n) \in R_{bbv}(X)$, by Theorem 4.3, it suffices to show that $(\alpha_n x_n) \in R_{bbv}(X)$, for all $(\alpha_n) \in c_0$. Now X^* is an L_1-space, so by Theorem 3.2, $(\alpha_n x_n) \in R(X)$, for all $(\alpha_n) \in c_0$. Since $(\alpha_n x_n)$ is also an unconditional basis for X, Theorem 5 of [1] tells us that $(\alpha_n x_n) \in \ell_2[X]$. Further, it was shown in [36, Cor. 3.2] that $\ell_2[X] \subset R_{vbv}(X)$ which yields, by virtue of Proposition 2.1, that the map $S : \ell_1 \to X$ given by

$$S(\beta) = \sum_n \beta_n \alpha_n x_n, \quad \beta = (\beta_n) \in \ell_1$$

is absolutely summing. Since X is an L_∞-space, Corollary 6.24 of [9] gives that S is, in fact, integral, which means that $(\alpha_n x_n) \in R_{bbv}(X)$.

In particular, $(x_n) \in R(X)$ and, by the same argument as used above, $(x_n) \in \ell_2[X]$. Finally, assume that $(x_n) \in R_{bv}(X)$. Then the induced map $T_{\overline{x}} : \ell_1 \to X$ is strictly integral. Since X has (RNP), invoking [9, Ch. VII, 6.9] yields that $T_{\overline{x}}$ is nuclear and, a priori, compact. But then (x_n) has a subsequence which is convergent to zero in view of $(x_n) \in \ell_2[X]$. This contradicts the fact that (x_n) is normalized and the claim is established.

We conclude this section with another open problem which was explicitly posed in [36]. This problem is motivated by the now well-known fact that a Banach space X is finite-dimensional exactly when all X-valued measures are of bounded variation. Before we put our problem in perspective, we collect together a list of equivalent conditions for the equality $R(X) = R_{bbv}(X)$.

Theorem 4.5. ([36], see also[19]) *Given a Banach space X, the following statements are equivalent:*

(i) $R(X) = R_{bbv}(X)$.

(ii) $R_c(X) = R_{bv}(X)$.

(iii) $L(X, \ell_1) = \Pi_1(X, \ell_1)$.

(iv) $L(X^*, \ell_1) = \Pi_1(X^*, \ell_1)$.

(v) *X and X^* have* (GT).

Proof. (Sketch) It is shown in [11] that (i) $\Leftrightarrow$ (ii) and (iii) $\Leftrightarrow$ (iv). We show that (i) $\Rightarrow$ (iii) $\Rightarrow$ (v) $\Rightarrow$ (i). To show that (i) $\Rightarrow$ (iii), we have $R(X) = R_{vbv}(X)$ and $R_{vbv}(X) = R_{bbv}(X)$. By virtue of Theorems 4.1 and 4.2, these equalities translate into $\Pi_2(X, \ell_1) = \Pi_1(X, \ell_1)$ and $\Pi_2(X, \ell_1) = L(X, \ell_1)$, respectively. Combining these two equalities gives (iii). Now, assuming that (iii) holds, we easily see that X^* has (GT). Using the equivalence (iii) $\Leftrightarrow$ (iv), we get that X^{**} and hence X has (GT). Finally, the implication (v) $\Rightarrow$ (i) follows by observing that (GT)* translates into $R_{vbv}(X) = R_{bbv}(X)$ by virtue of Theorem 4.2 whereas (GT)-property yields $\Pi_1(X, \ell_2) = L(X, \ell_2)$. In particular, $\Pi_1(X, \ell_2) = \Pi_2(X, \ell_2)$. Equivalently, $R(X) = R_{vbv}(X)$, by Theorem 4.1. Combining the two conclusions gives (i). ∎

Combining the above theorem with Theorem 3.1 yields the following interesting corollary.

Corollary 4.6. ([36]) *A Banach space X with* (GL)*-property such that $R(X) = R_{bbv}(X)$ is finite dimensional.*

Proof. By the above theorem X has (GT) and (GT)*. Now (GT)-property means $L(X, \ell_2) = \Pi_1(X, \ell_2)$. Combining with the (GL)-property, (GT)* gives, by virtue of Theorem 3.10 $N(X, \ell_2) = \Pi_1(X, \ell_2)$. These two equalities together yield $L(X, \ell_2) = N(X, \ell_2)$ which is possible only if dim $X < \infty$. Indeed, a nuclear operator into ℓ_2 is a composite of two 2-summings maps (see [9, Th. 5.31]). Thus a nuclear map on X is a composite of two 2-summing maps and, therefore, is a Hilbert space by virtue of the 'Eigenvalue Theorem' mentioned earlier combined with Proposition 2.3 (e). However, every infinite-dimensional Hilbert space admits non-nuclear operators! ∎

Remark 4.7. (i) The above corollary does not hold in the absence of the (GL)-property. In fact, Pisier [29] constructed an infinite-dimensional Banach space verifying the equivalent properties listed in Theorem 4.5. The above corollary yields that Pisier space lacks (GL)-property.

(ii) It is possible to use Theorem 3.15 to provide what appears at best to be a slightly different (though by no means a simpler) argument used in the proof of Pelczynski's famous theorem that the disc algebra $A(D)$ does not have the (GL)-property. After all, our argument uses the highly non-trivial fact discovered by Bourgain that $A(D)$ has (GT)*. For details of this argument, see [36].

We conclude this section with the following conjecture which was proposed in [36].

Conjecture. For a Banach space X, the equality $R(X) = R_{bv}(X)$ holds exactly when dim $X < \infty$.

5. Fréchet space setting

Having gained a fair amount of understanding as to the richness of vector measure theory in the setting of Banach spaces where we have seen how the geometrical properties of a Banach space influence the nature of range of certain kinds of vector measures taking values in these spaces, it is time to explore the extent to which it is possible to extend these ideas beyond the class of Banach spaces. In this section, we treat the case of Fréchet-valued measures and show that the theory retains its richness, at least in respect of those aspects of the theory which, in the Banach space setting, characterize finite dimensionality. Remarkably, it turns out that the appropriate property of Fréchet spaces verifying the Fréchet-analogue of these properties of vector measures is nuclearity!

We begin with the Fréchet-analogue of Theorem 3.4.

THEOREM 5.1. ([2]) *The following statements are equivalent for a Fréchet space X:*

(i) $c_0(X) \subseteq R_{vbv}(X)$.

(ii) X *is Hilbertizable.*

Here, 'Hilbertizability' is meant in the sense that the given Fréchet topology of X is determined by a sequence of seminorms which arise from semi-inner products. Further, the symbols $R_{vbv}(X)$, $R_{bbv}(X)$ and $R_{bv}(X)$ etc. are defined for Fréchet spaces X in a manner similar to that already defined in Section 2 for Banach spaces. The case of Hilbert spaces as covered under Theorem 3.4 follows as a special case of this theorem.

The above theorem motivates the question of describing those Fréchet spaces X for which the above inclusion holds with $R_{vbv}(X)$ replaced by $R_{bv}(X)$ or say by $R(X)$. Whereas the case of $R(X)$ seems to be unknown for Fréchet spaces X, in the Banach space case treated in Theorem 3.2, the inclusion $c_0(X) \subseteq R(X)$ translates into the statement that X^* is a subspace of an L_1-space.

Before we treat the Fréchet analogue of the inclusions $c_0(X) \subseteq R_{bv}(X)$ and $c_0(X) \subseteq R_{bbv}(X)$, we note that, by virtue of Theorem 3.3, the only Banach spaces that result in the process are the finite dimensional ones. We also recall [17] that the only Banach spaces X for which all X-valued measures are of bounded variation are those which are finite dimensional. The Fréchet-space version of this latter statement is an old result of C. Duchon which showed that such spaces are necessarily nuclear.

THEOREM 5.2. ([11]) *A Fréchet space X is nuclear if and only if each X-valued (c.a.) measure has bounded variation.*

Regarding the inclusion $c_0(X) \subseteq R_{bv}(X)$ for Fréchet spaces, we have the following theorem of Bonet and Madrigal.

THEOREM 5.3. ([2]) *For a Fréchet space X, the inclusion $c_0(X) \subseteq R_{bv}(X)$ holds if and only if X is nuclear.*

Before discussing the Fréchet analogues of the statement (iii) of Theorem 3.5, let us recall that a Banach space X is said to have the *Radon-Nikodym property* (RNP) if each X-valued measure of bounded variation which is absolutely continuous w.r.t. a scalar measure has a Radon-Nikodym derivative. In other words, if $\mu : \Sigma \to X$ is a measure of bounded variation and $\nu : \Sigma \to \mathbb{R}$ is a scalar measure such that $\nu(A) = 0$ implies $\mu(A) = 0$, then there exists a Bochner-integrable function $f \in L^1(\Omega, X)$ such that

$$\mu(E) = \int_E f \, d\nu, \ \forall \ E \in \Sigma.$$

Surprisingly, it turns out that if μ is chosen to be an arbitrary measure-not necessarily of bounded variation-then the above property characterizes X has finite-dimensional. The Fréchet space analogue of this result is a famous theorem of G.E.F. Thomas. See also [17].

THEOREM 5.4. ([39]) *For a Fréchet space X, the following statements are equivalent:*

(i) *X is nuclear.*

(ii) (RNP) *holds for each X-valued (c.a.) measure.*

Another important finite-dimensionality result involving vector measures was obtained by C. Piñeiro [26]. He showed that for $p > 2$, each weakly p-summable sequence in a Banach space X lies inside the range of an X-valued measure of bounded variation precisely when X is finite dimensional. A strengthening of this result was obtained by the author with the weakly p-summable sequences in X being replaced by absolutely p-summable sequences for $p > 2$. This follows from Theorem 3.5 (iii) and the discussion following it. A simultaneous generalization of this result and the theorem of Bonet and Madrigal (Theorem 5.3) was achieved by the author in a recent work [37] which makes use of some ideas from Banach space theory in the proof of this theorem:

THEOREM 5.5. ([37]) *A Fréchet space is nuclear if and only if $\ell_p\{X\} \subseteq R_{bv}(X)$ for some (all) $p > 2$.*

Proof. (Sketch) The proof makes use of the following facts (proved in [35]) which are also of independent interest.

(A) $(R_{bv}(X), \tau_{bv})$ is a Fréchet space where τ_{bv} is the (l.c.) topology on $R_{bv}(X)$ generated by the sequence of seminorms $\{\|\cdot\|_n^{bv}; n \geq 1\}$, defined by

$$\|\overline{x}\|_n^{bv} = \inf\{|\mu|_n : (x_n) \subset rg(\mu) \text{ for some } X\text{-valued measure } \mu \text{ of bounded variation.}\}$$

Here, $|\mu|_n = |\mu|_n(\Omega)$ and $|\mu|_n(A) = \sup\left\{\sum_{A\in P} \|\mu(A_i)\|_n\right\}$, $A \in \sum$, where the supremum ranges over all (finite) partitions P of A into pairwise disjoint members of $\sum$.

(B) Given $\overline{u} = (x_n^*) \in \ell_1[X_\beta^*]$, the map

$$\psi_{\overline{u}}(\overline{x}) = \sum_{n=1}^{\infty} \langle x_n, x_n^* \rangle, \quad \overline{x} = (x_n) \in R_{bv}(X)$$

defines a continuous linear functional on $R_{bv}(X)$.

In view of Theorem 5.3, it suffices to assume that $\ell_p\{X\} \subset R_{bv}(X)$ and show that X is nuclear. We begin by noting that the σ_p-topology on $\ell_p\{X\}$ and τ_{bv} on $R_{bv}(X)$ are both stronger than the uniform topology τ_∞. It follows that the inclusion map: $\ell_p\{X\} \to R_{bv}(X)$ has a closed graph and is, therefore, continuous by the closed graph theorem combined with (A). This yields that $\ell^1[X_\beta^*] \subset \ell_q\{X_\beta^*\}$ where $\frac{1}{p} + \frac{1}{q} = 1$. Indeed, given $u = (x_n^*) \in \ell^1[X_\beta^*]$, (B) yields that $\psi = \psi_u$ is continuous on $R_{bv}(X)$ and hence its restriction on $\ell_p\{X\}$ is also continuous. But $(\ell_p\{X\})^* = \ell_q\{X_\beta^*\}$. Thus there exists $\bar{v} = (y_n^*) \in \ell_q\{X^*\}$ such that

$$\sum_{n=1}^{\infty} \langle x_n, y_n^* \rangle = \psi(\bar{x}) = \psi_{\bar{u}}(\bar{x}) = \sum_{n=1}^{\infty} \langle x_n, x_n^* \rangle, \text{ for each } \bar{x} = (x_n) \in \ell_p\{X\}.$$

In other words, $\bar{u} = (x_n^*) = (y_n^*) = \bar{v}$ and the desired inclusion is established. ∎

We now show that for each $n \geq 1$, there exists $m > n$ such that the canonical map: $i_{mn} : x_n \to x_m$ is $(q, 1)$-summing where $X_n = X^*(U_n^0)$ is the linear span of U_n^0 in X^* equipped with the norm

$$\|x^*\|_n^* = \sup\{|\langle x^*, x \rangle : x \in U_n\}, \quad x_n^* \in X_n.$$

To this end, fix $n \geq 1$ and note that $\ell_1[X_n]$ embeds continuously into $\ell^1[X_\beta^*]$. As shown above, we have $\ell_1[X_\beta^*] \subset \ell_q\{X_\beta^*\}$, which gives that the (inclusion) map $\psi : \ell_1[X_n] \to \ell_q\{X_\beta^*\}$ has a closed graph. Now being a complete

(DF) space, X^*_β is fundamentally ℓ_q-bounded by [35]. In other words, bounded subsets of $\ell_q\{X^*_\beta\}$ are contained and bounded in some $\ell_q\{X_n\}$. Taking the lead from Bonet-Madrigal [2], we can show that for $m \geq n$, the canonical mapping $i_{mn} : X_n \to X_m$ is nuclear. Indeed, we can write

$$\ell_q\{X^*_\beta\} = \bigcup_{n=1}^{\infty} \ell_q\{X_n\}.$$

Now let Z denote $\ell_q\{X^*_\beta\}$ equipped with the inductive topology induced by the inclusions: $\ell_q\{X_k\} \subset \ell_q\{X^*_\beta\}$. Let $I : \ell_q\{X^*_\beta\} \to Z$ denote the identity map and let $\psi_k : \ell_q\{X_k\} \to Z$ be the canonical inclusions. Clearly, the composite map $Io\psi : \ell^1[X_n] \to Z$, having a closed graph is continuous by the closed graph theorem applied to the Banach space $\ell^1[X_n]$ and the (LB)-space Z [21, Th. 24.31 and 24.36]. Finally, Grothendick's factorization theorem [10, Th. 24.33] applied to the continuous linear maps $Io\psi$ and $\{\psi_k\}$, yields $m \geq 1$ such that

$$\psi(\ell_1[X_n]) = Io\psi(\ell_1[X_n]) \subseteq \psi_m(\ell_q\{X_m\}) = \ell_q\{X_m\}.$$

Equivalently, the map $\psi : \ell_1[X_n] \to \ell_q\{X_m\}$ is well-defined and also continuous or, in other words, the canonical map $i_{mn} : X_n \to X_m$ is absolutely $(q, 1)$-summing. Finally noting that $q < 2$, so that by virtue of [15, Cor. 5.7], a composite of sufficiently many $(q, 1)$-summing maps produces a nuclear map, we conclude that X is a nuclear space in view of [22, Ch. 4].

We conclude this section with a sample of recent results on (spectral) measures taking values in (the space of continuous linear operators on) a Köthe echelon space [3]. Let us recall that a *Köthe matrix* is an infinite matrix $(a_n(i))_{i,n=1}^{\infty}$ of positive numbers such that

$$0 < a_n(i) \leq a_{a+1}(i), \quad i, n \in \mathbb{N}.$$

The *Köthe (echelon) space* associated to the Köthe matrix $A = (a_n(i))$ and $p \in [1, \infty)$ is the linear space

$$\lambda_p(A) = \left\{ \overline{x} = (x_n) \in w : q_n^{(p)}(\overline{x}) = \left(\sum_i a_n(i)|x_i|^p \right)^{1/p} < \infty, \ \forall\, n \in \mathbb{N} \right\},$$

equipped with the Fréchet space topology generated by the sequence of (semi) norms $\{q_n^{(p)} : n \in \mathbb{N}\}$. Here w stands for the space of all scalar-valued sequences. For a detailed account concerning these spaces, see [13] and [21].

The first result is the Köthe space analogue of a well known fact about vector measures in ℓ_p spaces: *Every ℓ_p-valued measure has a relatively compact range if and only $1 \leq p < 2$.*

THEOREM 5.6. ([3])

(i) *Let $1 \leq p < 2$. Then every $\lambda_p(A)$-valued measure has relatively compact range.*

(ii) *Let $2 \leq p < \infty$. Then every $\lambda_p(A)$-valued measure has relatively compact range if and only if $\lambda_p(A)$ is Montel.*

The connection between the structural properties of a (Köthe) sequence space and the measure-theoretic properties of the (canonical) spectral measure is treated in the next theorem. Recall that a vector measure $P : \sum \to L_s(X)$ taking values in the space of continuous linear operators on a Hausdorff locally convex space X is a *spectral measure* if $P(E \cap F) = P(E)\,P(F)$ for all $E, F \in \sum$ with $E \cap F = \emptyset$. Here $L_s(X)$ means that $L(X)$ is equipped with the topology of pointwise convergence whereas $L_b(X)$ shall be understood to mean that $L(X)$ is endowed with strong topology (of uniform convergence on bounded subsets of X). We shall also say that P is *boundedly* σ-additive if it is c.a. additive as an $L_b(X)$-valued measure. We can now state

THEOREM 5.7. ([3]) *A Fréchet normal sequence space λ is Montel if and only every $L_s(\lambda)$-valued measure is boundedly σ-additive.*

It turns out that it suffices to check the condition of boundedly σ-additivity in the above theorem for a special spectral measure, the so-called *canonical spectral measure* which is defined by

$$P : 2^{\mathbb{N}} \to L(\lambda)$$
$$P(E)(X) = x\chi_E, \quad x \in \lambda$$

and $E \in 2^{\mathbb{N}}$. For details, see [3, Prop. 3.1].

The next result characterizes the situation when the measure P has finite variation.

THEOREM 5.8. ([3]) *Let $p \in (1, \infty)$. The canonical spectral measure $P : 2^{\mathbb{N}} \to L_s(\lambda_p(A))$ has bounded variation if and only if $\lambda_p(A)$ is nuclear.*

Remark 5.9. The case $p = 1$ has been excluded in Theorem 5.8 because, regardless of the nuclearity of $\lambda_1(P)$, the canonical spectral measure $P : 2^{\mathbb{N}} \to L_s(\lambda_1(A))$ always has bounded variation. Indeed, for a given fixed $x \in \lambda_1(A)$ and an arbitrary finite partition $\{E_j\}_{j=1}^r$ of $\mathbb{N}$, we have for each $n \geq 1$

$$\sum_{j=1}^{r} q_n^{(1)}\big(P(E_j)x\big) = \sum_{j=1}^{r} \sum_{i \in E_j} a_n(i)|x_i| = \sum_{i=1}^{\infty} a_n(i)|x_i| = q_n^{(1)}(x).$$

Thus, it follows that P has bounded variation.

Remarkably, it is still possible to characterize nuclearity of $\lambda_1(P)$ in terms of the canonical spectral measure provided it is assumed to have bounded variation in $L_b(\lambda_1(P))$ instead of in $L_s(\lambda_1(P))$. More precisely, we have the following partial converse.

THEOREM 5.10. ([3]) *Let the Köthe space $\lambda_1(A)$ be Montel. Then $\lambda_1(A)$ is nuclear if and only if the canonical spectral measure $P : 2^{\mathbb{N}} \to L_b(\lambda_1(A))$ has bounded variation.*

Remark 5.11. A study of the function-space theoretic analogues of the above results in the context of canonical spectral measures has also been carried out in the setting of Köthe function spaces in a recent work by J. Bonet, W. J. Ricker and S. Okada [4].

6. RANGE-DETERMINED PROPERTIES

In this final section, we briefly touch upon an important aspect of the general theme of this paper which was not discussed in the previous sections. The section then concludes with some open problems involving the range of vector measures discussed earlier. Most of the results treated in this section are due to L. Rodriguez Piazza and his collaborators which deal with the properties of a vector measure determined by its range.

Given a pair of vector measure μ, ν taking values in a Banach space X, it makes sense to ask if μ and ν share some common properties in the event that their ranges coincide: $rg(\mu) = rg(\nu)$. For the property of bounded variation, it turns out that the indicated equality completely determines if μ and ν are simultaneously of bounded/unbounded variation. In this direction, we have the following remarkable theorem of L. R. Piazza [32].

THEOREM 6.1. ([32], see also [33]) *Given vector measure μ and ν taking values in a Banach space, then $rg(\mu) = rg(\nu)$ implies that $tv(\mu) = tv(\nu)$. More generally, it suffices to assume that $rg(\mu)$ is a translate of $rg(\nu)$.*

Regarding the property of *finite σ-variation* (which means that there exists a sequence $\{A_n\} \subset \sum$ with $\Omega = \bigcup_{n=1}^{\infty} A_n$ such that $|\mu|(A_n) < \infty$, for all $n \geq 1$), we have the following analogous result.

THEOREM 6.2. ([33], see also [34]) *For X-valued vector measures μ and ν such that $rg(\mu) = rg(\nu)$, it holds that ν has σ-finite variation if and only if ν does.*

Notwithstanding these positive results in the presence of the equality of ranges, the situation is seen to change dramatically if $rg(\mu)$ is assumed only to be a subset of $rg(\nu)$. In fact, Anantharaman and Diestel [1] showed that there exist c_0-valued measures μ and ν with $rg(\mu) \subseteq rg(\nu)$ such that ν is of bounded variation but μ is not! The next theorem characterizes such Banach spaces where 'monotonicity' of total variation holds under inclusion of vector measure ranges.

THEOREM 6.3. ([32]) *Let X be a Banach space. Then for all X- valued measures μ and ν such that $rg(\mu) \subset rg(\nu)$ with ν being of bounded variation, it follows that μ is also of bounded variation if and only if X is a subspace of L_1.*

More precisely, there exists $c > 0$ such that $tv(\mu) \leq ctv(\nu)$ if and only if X is c-isomorphic to a subspace of an L^1-space.

However, as opposed to total variation, monotonicity of σ-finite variation is possible only in trivial situations.

THEOREM 6.4. ([33]) *Let X be a Banach space. Then for all X- valued measures μ and ν such that $rg(\mu) \subset rg(\nu)$ with ν having σ-finite variation, it follows that μ also has σ-finite variation precisely when X is finite dimensional.*

Continuing in the same vein, we have

THEOREM 6.5. ([33]) *For X-valued measures μ and ν such that $\overline{co}(rg\mu)$ is a translate of $\overline{co}(rg\nu)$, it follows that μ is Bochner differentiable w.r.t. $|\mu|$ if and only if ν is Bochner differentiable w.r.t. $|\nu|$.*

Here, differentiability is understood in the sense in which it is meant in Theorem 5.4 where the conclusion can be equivalently stated by saying that μ is differentiable w.r.t. to ν. Unfortunately, the above theorem does not hold for Pettis differentiability. Indeed, there are ℓ_∞-valued measures μ and ν satisfying the assumptions of Theorem 6.5 with only one of them being Pettis differentiable. For an example of this phenomenon, see [34].

An alternative approach to Theorems 6.1 and 6.5 and based on the idea of conical measures is included in another important contribution by L. Rodriguez Piazza and C. Romero Moreno [34] which also provides a far reaching generalization of Theorem 6.1 in the language of (p, q)-summing norms of the integration maps associated with μ and ν, respectively. On the other hand, the p-nuclear analogue of Theorem 6.5 for $p \geq 1$ which states that under the assumptions of Theorem 6.5, p- nuclear norms of the associated integration maps coincide, yields this theorem as a special case by taking $p = 1$. This follows from the fact that a vector measure is Bochner differentiable if and only if the associated integration map is nuclear. Here, we recall that corresponding to a vector measure $\mu : \sum \to X$, there exists the 'integration map'

$$L^1(\mu) \to X,$$

defined by

$$I_\mu(f) = \int_\Omega f d\mu, \quad f \in L^1(\mu),$$

where $L^1(\mu)$ denotes the space of those $\sum$-measurable functions $f : \Omega \to \mathbb{R}$ such that $f \in L^1(\Omega, \langle \mu, x^* \rangle)$, for each $x^* \in X^*$ and that for each $E \in \sum$, there exists a (unique) vector $\int\limits_E f d\mu \in X$ such that

$$\left\langle \int_E f\, d\mu,\, x^* \right\rangle = \int_E f d\langle \mu,\, x^* \rangle, \quad \forall\, x^* \in X^*.$$

THEOREM 6.6. ([31]) *Let $\mu : \sum_1 \to X$, $\nu : \sum_2 \to X$ be vector measures such that $\overline{co}(rg\mu) = \overline{co}(rg\nu)$. Then*

$$\overline{I_\mu(L^1(\mu))} = \overline{I_\nu(L^1(\nu))}.$$

In other words, the range of a vector measure determines (closure of the range of) its integration map.

We conclude this paper with a list of open problems.

7. Open problems

We saw in Section 3 that the situation is reasonably satisfactory in respect of sequences from a 'distinguished' set $S(X)$ of sequences from a Banach space X being localized inside the range of a vector measure of a certain type. This was seen to be the case particularly for $S(X) = c_0(X)$, $\ell_p\{X\}$ and also for $\ell_p[X]$, $1 \le p < \infty$. The case $S(X) = c_0(X)$ also characterizes the situation when compact subsets of X can be located inside the range of vector measures of certain types. This is based upon the well-known fact that a compact set is contained inside the closed convex hull of a (norm) null sequence combined with another fundamental fact from the vector measure theory which states that the closed convex hull of a set K is the range of a vector measure whenever K is such. This motivates the problem involving weakly compact sets (resp., weakly null sequences) being contained inside the range of vector measures:

Problem 1. Describe those Banach spaces such that weakly compact subsets (resp. weakly null sequences) of X are contained inside the range of vector measures taking values in (a) X; (b) X^{**}; (c) a superspace containing X, with or without bounded variation.

Comments. Theorem 3.12 shows that L_p-spaces for $2 \le p < \infty$ enjoy the property that weakly compact subsets of these spaces lie inside the range of a vector measure. However, there also exist non-reflexive Banach spaces for which this property holds. In fact, the separable L_∞ space of Bourgain and Delbaen which was discussed in Section 4 has this property as was noted in [1]. Moreover, it appears to be unknown if weakly compact subsets of c_0 lie inside the range of c_0-valued vector measures, even when all compact subsets of c_0 are known to be contained inside the range of c_0-valued vector measures (see Theorem 3.2).

Problem 2. Does there exist a weakly compact subset of c_0 which is not included inside the range of a c_0-valued measure?

Our next problem is based on Theorem 3.2 and the trivial observation that if X has the property that each null sequence in X is included inside the range of an X-valued measure, then so does each absolutely p-summable sequence in X for each $p \ge 1$. Thus one can ask:

Problem 3. Does there exist a Banach space X such that for each $p \ge 1$, each absolutely p-summable sequence in X lies inside the range of an X-valued measure but $X^* \subsetneq L^1$?

Comments. Corollary 3.10 shows how Theorem 3.2 is subsumed in the framework provided by Theorem 3.5 (i) which deals with the case of $S(X) = \ell_p\{X\}$. In some sense, one can say that the inclusion $c_0(X) \subset R(X)$ is the 'limiting case' of the inclusions $\ell_p\{X\} \subset R(X)$ for $p \geq 1$ as $p \to \infty$. In other words, the above problem asks whether the inclusion $\ell_p\{X\} \subset R(X)$ for all $p \geq 1$ implies that $c_0(X) \subset R(X)$. In a similar way, Corollary 3.10 combined with Theorem 3.4 and Theorem 3.5 (ii) motivate the next problem.

Problem 4. Given a Banach space X such that $\ell_p\{X\} \subset R_{vbv}(X)$ for each $p \geq 1$, prove or disprove that X is a Hilbert space.

Our next problem pertains to the chain of inclusions (6) already encountered in Section 2:

$$R_{bv}(X) \subset R_{bbv}(X) \subset R_{vbv}(X) \subset R(X) \subset \ell_\infty(X).$$

Whereas easy examples show that the last three inclusions are proper, it was unknown for a long time if it is also the case for the inclusion $R_{bv}(X) \subset R_{bbv}(X)$. In Section 4, we saw how the Bourgain-Delbaen space could be used to show that the indicated inclusion is indeed proper. This leads to the investigation of those Banach spaces X for which the inclusion is an equality.

Problem 5. Characterize Banach spaces X such that $R_{bv}(X) = R_{bbv}(X)$.

We can also formulate

Problem 6. Characterize Banach spaces X such that $R(X) = R_b(X)$.

Comment. As already commented upon in the paragraphs preceding Theorem 4.3, it is still unknown if the equality in Problem 6 holds for all Banach spaces.

The following two problems pertain to the 'preservation' of differentiability of a vector measure when its range coincides with the range of another measure which is differentiable.

Problem 7. Describe Banach spaces X such that whenever μ and ν are X-valued measures with $\overline{co}(rg(\mu)) = \overline{co}(rg(\nu))$, then μ is Pettis-differentiable if and only if ν is.

Comments. For Bochner differentiability, there are no issues involved: Theorem 6.5 tells us that regardless of the Banach space X, μ and ν are simultaneously Bochner differentiable or Bochner non-differentiable. However, that is not exactly the case for Pettis differentiability as was noted in the discussion following Theorem 6.5. Further, there is nothing sacrosanct about Bochner/Pettis differentiability: one may as well pose the following problems.

PROBLEM 8. Does the range determine McShane differentiability? In other words, does Theorem 6.5 hold with Bochner differentiability replaced by McShane differentiability?

Here 'McShane differentiability' is understood in the obvious sense that the density function of the vector measure is McShane integrable.

On similar lines, we have

PROBLEM 9. Characterize Banach spaces X such that

(i) $c_0(X) \subset R_D(X)$.

(ii) $\ell_p\{X\} \subset R_D(X)$, $1 \leq p < \infty$.

(iii) $\ell_p[X] \subset R_D(X)$, $1 \leq p < \infty$.

Here, $R_D(X)$ is used to denote the space of all X-valued (bounded) sequences which lie inside the range of a vector measure which is differentiable in the sense of 'D' where 'D' stands for an integration theory like Bochner, Pettis, McShane or Birkhoff integration.

Comments. Bochner differentiability of the measures involved need not be included in the above problem because there are no (infinite-dimensional) Banach spaces verifying inclusion (i), and also (ii) and (iii) at least for $p > 2$.

We conclude with the following Fréchet analogues of Theorems 3.1, 3.2 and 6.6.

PROBLEM 10. Characterize Fréchet spaces X such that

(i) $\ell_2[X] \subset R(X)$.

(ii) $c_0(X) \subset R(X)$.

PROBLEM 11. Is it true that for a given Fréchet space X it holds that whenever μ, ν are X-valued measures such that $\overline{co}(rg(\mu)) = \overline{co}(rg(\nu))$, it follows that $\overline{I_\mu(L^1(\mu))} = \overline{I_\nu(L^1(\nu))}$, where $I_\mu,\ I_\nu$ are the associated integration maps?

The following problem is motivated by Theorem 6.4.

Problem 12. Is it true that monotonicity of σ-finite variation in a Fréchet space X holds exactly when X is nuclear?

In other words, is it true that nuclearity of X is equivalent to the following condition:

Whenever μ and ν are X-valued measures such that $rg(\mu) \subset rg(\nu)$, then μ has finite σ-variation if ν has it.

Here, finite σ-variation of μ is understood in the same sense as for Banach spaces, namely that there exists a sequence $\{A_n\} \subset \sum$ with $\Omega = \bigcup_{n=1}^{\infty} A_n$ such that $|\mu|_n(A_n) < \infty$ for all $n \geq 1$, where $|\mu|_n$ is defined as in the proof of Theorem 5.5.

The last problem pertains to the circle of ideas involving spectral measures and is motivated by the desire to realize certain distinguished subsets of operators on a (Köthe) sequence space as included inside the range of a spectral measure.

Problem 13. Describe linear topological properties of a (Köthe) sequence space λ such that all compact sets of (compact operators on) λ are contained inside the range of a (boundedly σ-additive) spectral measure in $L_s(\lambda)$.

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