Invariant Metric f-Structures on Specific Homogeneous Reductive Spaces

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Abstract: For homogeneous reductive spaces G/H with reductive complements decomposable into an orthogonal sum $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$ of three $Ad(H)$-invariant irreducible mutually inequivalent submodules we establish simple conditions under which an invariant metric f-structure (f, g) belongs to the classes $\mathbf{G_1 f}$, $\mathbf{NKf}$, and $\mathbf{Kill\,f}$ of generalized Hermitian geometry. The statements obtained are then illustrated with four examples. Namely we consider invariant metric f-structures on the manifolds of oriented flags $SO(n)/SO(2) \times SO(n-3)$ $(n \geq 4)$, the Stiefel manifold $SO(4)/SO(2)$, the complex flag manifold $SU(3)/T_{max}$, and the quaternionic flag manifold $Sp(3)/SU(2) \times SU(2) \times SU(2)$.

Key words: Homogeneous reductive space, f-structure, invariant structure, nearly Kähler structure, flag manifold.

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Introduction

The concept of generalized Hermitian geometry (see, for example, [16]) was created in the 1980s as a natural consequence of the development of Hermitian geometry and the theory of almost contact structures. One of the central objects in this concept is the metric f-structure (f, g), that is, an f-structure [21] f compatible with an invariant Riemannian metric g.

An interesting problem that arises in this context is to determine whether a given metric f-structure belongs to the main classes of generalized Hermitian geometry, for example, to the classes $\mathbf{G_1 f}$ (see [16]), $\mathbf{NKf}$ (see [6] and [7]), and $\mathbf{Kill\,f}$ (see [14] and [15]). It should be emphasized that in the case of naturally reductive manifolds [18] there exist a number of results that transform this problem into an easy computational task ([6], [8], [4], [5]). However, in the case of an arbitrary Riemannian metric this problem is not an easy one, at least because it involves the calculation of the implicitly defined Levi-Civita connection.

In this paper we consider invariant metric f-structures (f, g) on specific

homogeneous reductive spaces G/H, namely on homogeneous reductive spaces that satisfy the following set of conditions:

1) G is a compact semisimple Lie group (hence the Killing form B of G is negative definite).

2) The reductive complement $\mathfrak{m}$ admits the decomposition

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$$

into the direct sum of $Ad(H)$-invariant irreducible mutually non-equivalent submodules and this decomposition is B-orthogonal.

3)
$$0 \neq [\mathfrak{m}_i, \mathfrak{m}_{i+1}] \subset \mathfrak{m}_{i+2} \pmod 3, \ i = 1, 2, 3.$$

4)
$$[\mathfrak{m}_i, \mathfrak{m}_i] \subset \mathfrak{h}, \ i = 1, 2, 3,$$

where $\mathfrak{h}$ is the Lie algebra of H.

In this case it is not difficult to obtain an explicit formula for the Levi-Civita connection of a Riemannian manifold $(G/H, g)$. At the same time, for any nontrivial invariant f-structure which is not an almost complex structure [18] there exists such $i \in \{1, 2, 3\}$ that either $\operatorname{Im} f = \mathfrak{m}_i$ or $\operatorname{Ker} f = \mathfrak{m}_i$. This, in its turn, has enabled us to obtain easy-to-check characteristic conditions (Theorem 2 and Theorem 3) for metric f-structures (f, g) under which they belong to the aforementioned classes of generalized Hermitian geometry.

Note that this paper was initiated by the study of the manifolds of oriented flags $SO(n)/SO(2) \times SO(n-3)$ ($n \geq 4$). In [10] it was shown that these homogeneous spaces satisfy the conditions 1) – 4). In the last section of this paper we provide other examples of such spaces. Namely, by making use of Theorem 2 and Theorem 3, we consider invariant metric f-structures on the Stiefel manifold $SO(4)/SO(2)$, the complex flag manifold $SU(3)/T_{max}$, and the quaternionic flag manifold $Sp(3)/SU(2) \times SU(2) \times SU(2)$.

1. Preliminaries

1.1. Invariant f-structures on homogeneous reductive spaces Homogeneous reductive spaces make up the main subject of our further considerations. Therefore we begin with recollecting some basic facts related to them.

DEFINITION 1. [19] Let G be a connected Lie group, H its closed subgroup, $\mathfrak{g}$ and $\mathfrak{h}$ the corresponding Lie algebras. G/H is called a *homogeneous reductive space* if there exists $\mathfrak{m} \subset \mathfrak{g}$ such that

1) $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$.

2) For any $h \in H$ $Ad(h)\mathfrak{m} \subset \mathfrak{m}$.

$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is the *reductive decomposition* corresponding to G/H and $\mathfrak{m}$ is the *reductive complement*.

For any homogeneous reductive space G/H its reductive complement $\mathfrak{m}$ can be identified with the tangent space to G/H at the point $o = H$ in the following sense:

for any $h \in H$, $d\tau(h)_o = Ad(h)$, where $\tau(g) : G/H \to G/H$, $xH \to (gx)H$.

Since all homogeneous spaces to be discussed in this paper are reductive, we agree to identify their reductive complements and their tangent spaces at the point o.

An *affinor structure* on a smooth manifold is known to be a tensor field of type (1,1) realized as a field of endomorphisms acting on its tangent bundle. In this paper we will be primarily interested in the *almost complex structure* [18] (such an affinor structure J that $J^2 = -\,\mathrm{id}$) and the *f-structure* [21] (an affinor structure f satisfying $f^3 + f = 0$).

DEFINITION 2. [1] Let G/H be a homogeneous manifold, F an affinor structure. F is called *invariant* with respect to G if for any $g \in G$

$$d\tau(g) \circ F = F \circ d\tau(g).$$

It is known that any invariant affinor structure F on a reductive homogeneous space G/H is completely determined by its value F_o at the point $o = H$, where F_o is a linear operator on the reductive complement $\mathfrak{m}$ such that

$$F_o \circ Ad(h) = Ad(h) \circ F_o \text{ for any } h \in H.$$

For this reason, further we will not distinguish an invariant structure F on G/H and its value F_o at the point $o = H$.

1.2. Some important classes in generalized Hermitian geome-
try   The concept of generalized Hermitian geometry appeared in the 1980s
and is mostly associated with the works of V. F. Kiritchenko (see, for example,
[16] and [17]). It should be mentioned that this theory is a natural consequence
of the development of Hermitian geometry and the theory of almost contact
structures with many applications.

In the sequel by $\mathfrak{X}(M)$ we will denote the set of all smooth vector fields
on a manifold M.

One of the central objects in generalized Hermitian geometry is a *metric
f-structure* [16] (f, g), where f is an f-structure compatible with a (pseudo)
Riemannian metric $g = \langle \cdot, \cdot \rangle$ in the following sense:

$$\langle fX, Y \rangle + \langle X, fY \rangle = 0 \text{ for any } X, Y \in \mathfrak{X}(M).$$

Evidently, this definition generalizes the notion of an almost Hermitian struc-
ture J in Hermitian geometry. A manifold M equipped with a metric f-
structure is called a *metric f-manifold*.

It is worth noticing that the main classes of generalized Hermitian geom-
etry (see [16], [7], [8], [14], and [15]) in the special case $f = J$, where J is
an almost complex structure, coincide with those of Hermitian geometry (see
[13]). In this section we will mainly concentrate on the classes **Kill f**, **NKf**,
and **G₁f** of metric f-structures.

A fundamental role in generalized Hermitian geometry is played by the
tensor T of type $(2, 1)$ which is called a *composition tensor* [16]. In [16] it was
shown that such a tensor exists on any metric f-manifold and it is possible to
evaluate it explicitly:

$$T(X, Y) = \frac{1}{4} f \big(\nabla_{fX}(f) fY - \nabla_{f^2X}(f) f^2 Y \big),$$

where ∇ is the Levi-Civita connection of a (pseudo) Riemannian manifold
(M, g), $X, Y \in \mathfrak{X}(M)$.

With the help of this tensor one can define the structure of a so-called
adjoint Q-algebra (see [16]) on $\mathfrak{X}(M)$ by the formula $X * Y = T(X, Y)$. It
gives the opportunity to introduce some classes of metric f-structures in terms
of natural properties of the adjoint Q-algebra.

For example, if

$$T(X, X) = 0 \text{ for any } X \in \mathfrak{X}(M) \tag{1}$$

(that is, if $\mathfrak{X}(M)$ is an anticommutative Q-algebra) then f is referred to as a G_1f-*structure.* $\mathbf{G_1f}$ denotes the class of G_1f-structures, which was first introduced (in a more general situation) in [16].

A metric f-structure on (M, g) is said to be a *Killing* f-*structure* [14, 15] if

$$\nabla_X(f)X = 0 \text{ for any } X \in \mathfrak{X}(M) \tag{2}$$

(that is, if f is a Killing tensor). The class of Killing f-structures is denoted by $\mathbf{Kill\,f}$.

The defining property of *nearly Kähler* f-*structures* (or NKf-*structures*) is

$$\nabla_{fX}(f)fX = 0 \text{ for any } X \in \mathfrak{X}(M). \tag{3}$$

This class of metric f-structures, which is denoted by $\mathbf{NKf}$, was first determined in [3] (see also [7, 6]). It is not difficult to see that for $f = J$ the classes $\mathbf{Kill\,f}$ and $\mathbf{NKf}$ coincide with the well-known class $\mathbf{NK}$ of *nearly Kähler structures* [12].

The following relations between the classes mentioned are evident:

$$\mathbf{Kill\,f} \subset \mathbf{NKf} \subset \mathbf{G_1f}. \tag{4}$$

The classical result bellow will be used to rewrite formulas (1), (2) and (3) in a form more suitable for further considerations.

THEOREM 1. ([18]) *Let (M, g) be a Riemannian manifold, $M = G/H$ a homogeneous reductive space with the reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Then the Levi-Civita connection with respect to g can be expressed in the form*

$$\nabla_X Y = \frac{1}{2}[X, Y]_{\mathfrak{m}} + U(X, Y), \tag{5}$$

where U is the symmetric bilinear mapping $\mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ defined by the formula

$$2g(U(X, Y), Z) = g(X, [Z, Y]_{\mathfrak{m}}) + g([Z, X]_{\mathfrak{m}}, Y) \text{ for any } X, Y, Z \in \mathfrak{m}. \tag{6}$$

It can be shown in the standard way that the application of (5) to (1), (2) and (3) produces the following result.

LEMMA 1. *Let (M, g) be a Riemannian manifold, $M = G/H$ a reductive homogeneous space with the reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Then for an invariant metric f-structure (f, g) on M the following holds.*

1) $f \in \mathbf{G_1f}$ if and only if

$$f(2U(fX, f^2X) - f(U(fX, fX)) + f(U(f^2X, f^2X))) = 0 \tag{7}$$
$$\text{for any } X \in \mathfrak{m};$$

2) $f \in \mathbf{NKf}$ if and only if

$$\frac{1}{2}[fX, f^2X]_\mathfrak{m} + U(fX, f^2X) - f(U(fX, fX)) = 0 \quad \text{for any } X \in \mathfrak{m}; \tag{8}$$

3) $f \in \mathbf{Kill\,f}$ if and only if

$$\frac{1}{2}[X, fX]_\mathfrak{m} + U(X, fX) - f(U(X, X)) = 0 \quad \text{for any } X \in \mathfrak{m}. \tag{9}$$

2. Main results

Assumption 1. Suppose that for a homogeneous reductive space G/H with the reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ the following is true.

A_1) G is a compact semisimple Lie group (hence the Killing form B on $\mathfrak{g}$ is negative definite).

A_2) The reductive complement $\mathfrak{m}$ admits the decomposition

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \tag{10}$$

into the direct sum of $Ad(H)$-invariant irreducible mutually non-equivalent submodules and this decomposition is B-orthogonal.

A_3)
$$0 \neq [\mathfrak{m}_i, \mathfrak{m}_{i+1}] \subset \mathfrak{m}_{i+2} \pmod 3, \ i = 1, 2, 3. \tag{11}$$

A_4)
$$[\mathfrak{m}_i, \mathfrak{m}_i] \subset \mathfrak{h}, \ i = 1, 2, 3. \tag{12}$$

In the view of A_1) and A_2) any invariant Riemannian metric g on G/H is uniquely determined by the triple of positive real numbers (a_1, a_2, a_3) which implies that

$$g = a_1 g_0 \mid_{\mathfrak{m}_1 \times \mathfrak{m}_1} + a_2 g_0 \mid_{\mathfrak{m}_2 \times \mathfrak{m}_2} + a_3 g_0 \mid_{\mathfrak{m}_3 \times \mathfrak{m}_3}, \tag{13}$$

where g_0 is an invariant inner product generated by the negative of the Killing form B. Further we will refer to (a_1, a_2, a_3) as to the *characteristic numbers of g*. We will also denote the projection of X onto $\mathfrak{m}_i$ by X_i for any $X \in \mathfrak{m}$.

Assumption 1 makes it possible to calculate the symmetric bilinear mapping $U(X, Y)$ defined in the previous section. The proof of the following result, which is nothing but the simplification of (6) in the view of Assumption 1, can be found in [20].

LEMMA 2. *Suppose that G/H satisfies Assumption 1. Then the symmetric bilinear mapping U is defined by the formula*

$$U(X,Y) = \frac{a_3 - a_2}{2a_1}([X_2, Y_3] + [Y_2, X_3]) + \frac{a_3 - a_1}{2a_2}([X_1, Y_3] + [Y_1, X_3]) + \frac{a_2 - a_1}{2a_3}([X_1, Y_2] + [Y_1, X_2]). \tag{14}$$

Here and below we assume that G/H satisfies Assumption 1.

LEMMA 3. *For any invariant affinor structure f on G/H $f(\mathfrak{m}_i)$ $(i = 1, 2, 3)$ is $Ad(H)$-invariant.*

Proof. A_2) yields that $Ad(h)\mathfrak{m}_i \subset \mathfrak{m}_i$ for any $h \in H$. Hence

$$f(Ad(H)\mathfrak{m}_i) \subset f(\mathfrak{m}_i).$$

f is an invariant affinor structure, therefore

$$Ad(H)(f(\mathfrak{m}_i)) \subset f(\mathfrak{m}_i).$$

∎

PROPOSITION 1. *Let f be an invariant affinor f-structure on G/H with $\operatorname{Ker} f \neq \{0\}$ and $\operatorname{Im} f \neq \{0\}$, and assume that G/H satisfies Assumption 1. Then there exists $i \in \{1, 2, 3\}$ such that either $\operatorname{Im} f = \mathfrak{m}_i$ or $\operatorname{Ker} f = \mathfrak{m}_i$.*

Proof. As $\mathfrak{m} = \operatorname{Ker} f \oplus \operatorname{Im} f$, for any $i \in \{1, 2, 3\}$ we have

$$\mathfrak{m}_i = (\operatorname{Ker} f)_i \oplus (\operatorname{Im} f)_i,$$

where

$$(\operatorname{Ker} f)_i = \mathfrak{m}_i \cap \operatorname{Ker} f, \ (\operatorname{Im} f)_i = \mathfrak{m}_i \cap \operatorname{Im} f.$$

Suppose that there exists $k \in \{1, 2, 3\}$ such that $(\mathrm{Ker}\, f)_k \neq 0$ and $(\mathrm{Im}\, f)_k \neq$ 0. Obviously, for any $X \in (\mathrm{Ker}\, f)_k$ and $h \in H$

$$f(Ad(h)X) = Ad(h)(f(X)) = 0,$$

which implies that $(\mathrm{Ker}\, f)_k$ is $Ad(H)$-invariant.

The same is true for $(\mathrm{Im}\, f)_k$. Indeed, for any $X \in (\mathrm{Im}\, f)_k$ we have $Ad(h)X \in \mathrm{Im}\, f$ (by Lemma 3) and $Ad(h)X \in \mathfrak{m}_k$ (by A_2)).

In this way we have obtained that $\mathfrak{m}_k$ is decomposed into the sum of the two non-trivial $Ad(H)$-invariant subspaces, which contradicts Assumption 1. ∎

Proposition 1 yields that for any non-trivial invariant affinor f-structure f which is not an almost complex structure the following is true:

1) either $f\,|_{\mathfrak{m}_i} = J$, $f\,|_{\mathfrak{m}_j \oplus \mathfrak{m}_k} = 0$,

2) or $f\,|_{\mathfrak{m}_i} = 0$, $f\,|_{\mathfrak{m}_j \oplus \mathfrak{m}_k} = J$,

where $\{i, j, k\} = \{1, 2, 3\}$, and J is an almost complex structure.

Let us consider the first of these two cases. The following statement is valid.

THEOREM 2. *Suppose that G/H satisfies Assumption 1, and g is an arbitrary invariant Riemannian metric on G/H. Let (f, g) be an invariant metric f-structure such that $f\,|_{\mathfrak{m}_i} = J$, $f\,|_{\mathfrak{m}_j \oplus \mathfrak{m}_k} = 0$, where $\{i, j, k\} = \{1, 2, 3\}$, and J is an almost complex structure. Then*

1) *(f, g) is not a Killing f-structure;*

2) *(f, g) belongs to the class* **NKf** *(and, consequently, to the class* **G₁f***).*

Proof. We assume that

$$f\,|_{\mathfrak{m}_1} = J, \ f\,|_{\mathfrak{m}_2 \oplus \mathfrak{m}_3} = 0 \tag{15}$$

(the results for the other cases are obtained via cyclic rearrangement of indices).

1) **Kill f** is defined by the formula (9). Taking (14), (15) and Assumption 1 into account we obtain

$$U(X, fX) = \frac{a_3 - a_1}{2a_2}[(fX)_1, X_3] + \frac{a_2 - a_1}{2a_3}[(fX)_1, X_2],$$

$$f(U(X,X)) = \frac{a_3 - a_2}{a_1} f([X_2, X_3]).$$

Besides,

$$\frac{1}{2}[X, fX]_{\mathfrak{m}} = \frac{1}{2}[X_2, (fX)_1] + \frac{1}{2}[X_3, (fX)_1].$$

Hence, (9) is equivalent to the following relation:

$$\frac{a_3 - a_2 - a_1}{2a_2}[(fX)_1, X_3] + \frac{a_2 - a_1 - a_3}{2a_3}[(fX)_1, X_2]$$

$$- \frac{a_3 - a_2}{a_1} f([X_2, X_3]) = 0$$

for any $X \in \mathfrak{m}$.

By A_3), $[\mathfrak{m}_i, \mathfrak{m}_j] \neq 0$ $(i, j \in \{1, 2, 3\}, i \neq j)$. Therefore (f, g) belongs to **Kill f** if and only if the characteristic numbers of g satisfy the following set of conditions:

$$\begin{cases} \dfrac{a_3 - a_2 - a_1}{2a_2} = 0, \\[2mm] \dfrac{a_2 - a_1 - a_3}{2a_3} = 0, \\[2mm] \dfrac{a_3 - a_2}{a_1} = 0. \end{cases}$$

Evidently, this system is inconsistent.

2) The defining property of **NKf** is (8). As (15) holds, (14) yields that

$$U(fX, fX) = U(fX, f^2 X) = 0.$$

Moreover, by Assumption 1,

$$\frac{1}{2}[fX, f^2 X]_{\mathfrak{m}} = \frac{1}{2}[(fX)_1, (f^2 X)_1]_{\mathfrak{m}} = 0.$$

Thus (8) holds for any Riemannian metric. As a particular case, any f satisfying (15) is a $G_1 f$-structure. ∎

Now let us consider the second group of f-structures.

THEOREM 3. *Suppose that G/H satisfies Assumption 1, and g is an arbitrary invariant Riemannian metric on G/H with the characteristic numbers (a_1, a_2, a_3). Let (f, g) be an invariant metric f-structure such that $f \mid_{\mathfrak{m}_i} = 0$, $f \mid_{\mathfrak{m}_j \oplus \mathfrak{m}_k} = J$, where $\{i, j, k\} = \{1, 2, 3\}$, and J is an almost complex structure. Then*

1) (f, g) is a $G_1 f$-structure;

2) (f, g) is a nearly Kähler f-structure if and only if $a_j = a_k$ and

$$[fX, f^2 X]_\mathfrak{m} = 0 \text{ for any } X \in \mathfrak{m}; \tag{16}$$

3) (f, g) is a Killing f-structure if and only if $a_j = a_k = \frac{4}{3} a_i$ and

$$\begin{cases} [Z, fZ]_\mathfrak{m} = 0, \\ [Y, fZ] + f([Y, Z]) = 0 \end{cases} \quad \text{for any } Y \in \mathfrak{m}_i, \ Z \in \mathfrak{m}_j \oplus \mathfrak{m}_k.$$

Proof. Without loss of generality it can be assumed that

$$f \mid_{\mathfrak{m}_1} = 0, \ f \mid_{\mathfrak{m}_2 \oplus \mathfrak{m}_3} = J. \tag{17}$$

1) It is evident that both

$$U(fX, f^2 X) = \frac{a_3 - a_2}{2a_1} ([(fX)_2, (f^2 X)_3] + [(f^2 X)_2, (fX)_3]) \tag{18}$$

and

$$U(fX, fX) = \frac{a_3 - a_2}{a_1} [(fX)_2, (fX)_3] \tag{19}$$

belong to Ker f for any $X \in \mathfrak{m}$. Therefore (7) holds regardless of the choice of (a_1, a_2, a_3).

2) Clearly,

$$\frac{1}{2}[fX, f^2 X]_\mathfrak{m} = \frac{1}{2}[(fX)_2, (f^2 X)_3] + \frac{1}{2}[(fX)_3, (f^2 X)_2]. \tag{20}$$

Using (18), (19) and (20) we can rewrite (8) as follows:

$$\frac{a_3 - a_2 + a_1}{2a_1} [(fX)_2, (f^2 X)_3] + \frac{a_3 - a_2 - a_1}{2a_1} [(f^2 X)_2, (fX)_3] = 0$$
$$\text{for any } X \in \mathfrak{m}.$$

$a_3 - a_2 + a_1 \neq 0$ (otherwise $a_3 - a_2 - a_1 = 0$ and hence $a_1 = 0$). Thus $f \in \mathbf{NKf}$ with respect to (a_1, a_2, a_3) if and only if

$$\begin{cases} [(fX)_2, (f^2 X)_3] = \dfrac{a_3 - a_2 - a_1}{a_3 - a_2 + a_1} [(fX)_3, (f^2 X)_2], \\[4mm] [(fX)_3, (f^2 X)_2] = \dfrac{a_3 - a_2 - a_1}{a_3 - a_2 + a_1} [(fX)_2, (f^2 X)_3] \end{cases} \quad \text{for any } X \in \mathfrak{m}$$

(to obtain the second equation we substitute X for fX in the first one). It follows in the standard way that

$$
\begin{cases}
\left(1 - \left(\dfrac{a_3 - a_2 - a_1}{a_3 - a_2 + a_1}\right)^2\right)[(fX)_2, (f^2X)_3] = 0, \\[2mm]
[(fX)_2, (f^2X)_3] = \dfrac{a_3 - a_2 - a_1}{a_3 - a_2 + a_1}[(fX)_3, (f^2X)_2]
\end{cases}
\qquad \text{for any } X \in \mathfrak{m}.
$$

The first equation yields that $\frac{a_3 - a_2 - a_1}{a_3 - a_2 + a_1} = \pm 1$. As a_1, a_2 and a_3 are positive numbers, we have $a_2 = a_3$. Then

$$
[(fX)_2, (f^2X)_3] + [(fX)_3, (f^2X)_2] = 0.
$$

In the view of (17) and Assumption 1 this means that $[fX, f^2X]_{\mathfrak{m}} = 0$ for any X in $\mathfrak{m}$. Thus 2) is proved.

3) As (4) holds, here we consider f-structures satisfying (16) and invariant Riemannian metrics with characteristic numbers (a_1, a_2, a_2) $(a_1,\ a_2 > 0)$ only.

As above, we check that

$$
U(X, X) = \frac{a_2 - a_1}{a_2}[X_1, X_2 + X_3],
$$

$$
U(X, fX) = \frac{a_2 - a_1}{2a_2}[X_1, (fX)_2 + (fX)_3].
$$

Since (16) holds,

$$
\frac{1}{2}[X, fX]_{\mathfrak{m}} = \frac{1}{2}[X_1, (fX)_2 + (fX)_3] + \frac{1}{2}[X_2 + X_3, (fX)_2 + (fX)_3]_{\mathfrak{m}}
$$

$$
= \frac{1}{2}[X_1, (fX)_2 + (fX)_3].
$$

Thus (9) can be represented as follows:

$$
\frac{2a_2 - a_1}{2a_2}[X_1, (fX)_2 + (fX)_3] - \frac{a_2 - a_1}{a_2}f([X_1, X_2 + X_3]) = 0
$$

$$
\text{for any } X \in \mathfrak{m}.
$$

For convenience we shall rewrite it in this way:

$$
\frac{2a_2 - a_1}{2a_2}[Y, fZ] - \frac{a_2 - a_1}{a_2}f([Y, Z]) = 0 \ \text{ for any } \ Y \in \mathfrak{m}_1, \ Z \in \mathfrak{m}_2 \oplus \mathfrak{m}_3.
$$

Then it follows that

$$[Y, fZ] = \frac{2(a_2 - a_1)}{2a_2 - a_1} f([Y, Z]) \quad \text{for any} \ \ Y \in \mathfrak{m}_1, \ Z \in \mathfrak{m}_2 \oplus \mathfrak{m}_3 \qquad (21)$$

$(2a_2 - a_1 \neq 0$, because otherwise $a_1 = a_2 = 0)$. If we replace Z by fZ in (21) and then apply f to its both sides, we obtain

$$f([Y, Z]) = \frac{2(a_2 - a_1)}{2a_2 - a_1} [Y, fZ] \quad \text{for any} \ \ Y \in \mathfrak{m}_1, \ Z \in \mathfrak{m}_2 \oplus \mathfrak{m}_3. \qquad (22)$$

(21) and (22) produce the following system of equations

$$\begin{cases} \dfrac{4(a_2 - a_1)^2}{2a_2 - a_1} = 1, \\ [Y, fZ] = \dfrac{2(a_2 - a_1)}{2a_2 - a_1} f([Y, Z]) \end{cases} \quad \text{for any} \ \ Y \in \mathfrak{m}_1, \ Z \in \mathfrak{m}_2 \oplus \mathfrak{m}_3.$$

To conclude the proof, it remains to note that this system is equivalent to

$$\begin{cases} a_2 = \dfrac{4}{3} a_1, \\ [Y, fZ] + f([Y, Z]) = 0 \end{cases} \quad \text{for any} \ \ Y \in \mathfrak{m}_1, Z \in \mathfrak{m}_2 \oplus \mathfrak{m}_3. \qquad \blacksquare$$

3. Examples

3.1. The manifolds of oriented flags   In [10] we considered manifolds of oriented flags of the form

$$SO(n)/SO(2) \times SO(n - 3) \ (n \geq 4) \qquad (23)$$

as homogeneous Φ-spaces [11] of order 6. We proved that for any $n \geq 4$ the reductive complement $\mathfrak{m}$ of any such space is decomposed into the direct sum $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$ of irreducible $Ad(H)$-invariant summands. For the canonical f-structures on this homogeneous Φ-space of order 6 we obtained the following result (in the notations of [10]).

1) For $f_1(\theta) = \frac{1}{\sqrt{3}}(\theta - \theta^5)$

$$\text{Im} \, f_1 = \mathfrak{m}_1 \oplus \mathfrak{m}_2, \ \ \text{Ker} \, f_1 = \mathfrak{m}_3.$$

2) For $f_2(\theta) = \frac{1}{2\sqrt{3}}(\theta - \theta^2 + \theta^4 - \theta^5)$

$$\text{Im } f_2 = \mathfrak{m}_2, \ \text{Ker } f_2 = \mathfrak{m}_1 \oplus \mathfrak{m}_3.$$

3) For $f_3(\theta) = \frac{1}{2\sqrt{3}}(\theta + \theta^2 - \theta^4 - \theta^5)$

$$\text{Im } f_3 = \mathfrak{m}_1, \ \text{Ker } f_3 = \mathfrak{m}_2 \oplus \mathfrak{m}_3.$$

4) For $f_4(\theta) = \frac{1}{\sqrt{3}}(\theta^2 - \theta^4)$

$$\text{Im } f_4 = \mathfrak{m}_1 \oplus \mathfrak{m}_2, \ \text{Ker } f_4 = \mathfrak{m}_3.$$

In [10] it was checked that for any $i \in \{1, 2, 3, 4\}$ f_i is compatible with any invariant Riemannian metric (13), where $g_0 = -B(X, Y) = -(n-2)\,\text{Tr}(X{\cdot}Y)$.

The application of Theorem 2 immediately gives us that (f_2, g) and (f_3, g) are not Killing f-structures for any invariant Riemannian metric g. Nevertheless, (f_2, g) and (f_3, g) are nearly Kähler f-structures (and, hence, $G_1 f$-structures) with respect to any invariant Riemannian metric g.

Taking account of the facts that $[f_1 X, f_1^2 X] = 0$, $[f_4 X, f_4^2 X] \neq 0$, and $[Y, f_1 Z] + f_1([Y, Z]) = 0$ for any $X \in \mathfrak{m}$, $Y \in \mathfrak{m}_3$, and $Z \in \mathfrak{m}_1 \oplus \mathfrak{m}_2$, by Theorem 3, we obtain

1) (f_1, g) and (f_4, g) are $G_1 f$-structures for any invariant Riemannian metric g;

2) (f_1, g) belongs to **NKf** if and only if the characteristic numbers of g are (s, s, t) $(s, t > 0)$; (f_4, g) is not a nearly Kähler f-structure for any invariant Riemannian metric g;

3) (f_1, g) belongs to **Kill f** if and only if the characteristic numbers of g are $(3s, 3s, 4s)$, where $s > 0$. (f_4, g) is not a Killing f-structure for any invariant Riemannian metric g.

The same results where obtained in [10] by means of direct calculations.

3.2. THE COMPLEX FLAG MANIFOLD   All invariant metric f-structures on the complex flag manifold $SU(3)/T_{max}$ (T_{max} is a maximal torus of $SU(3)$) were considered in the view of generalized Hermitian geometry in [9]. Therefore, here we restrict ourselves to mentioning that $SU(3)/T_{max}$ satisfies the conditions of Assumption 1. Hence Theorems 2 and 3 are applicable in this case.

3.3. THE STIEFEL MANIFOLD   Let us consider $G/H = SO(4)/SO(2)$ (a Stiefel manifold). Then

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & a & b_1 & b_2 \\ -a & 0 & c_1 & c_2 \\ -b_1 & -c_1 & 0 & 0 \\ -b_2 & -c_2 & 0 & 0 \end{pmatrix} : a, b_1, b_2, c_1, c_2 \in \mathbb{R} \right\}.$$

It is not difficult to see that the manifold in question satisfies Assumption 1. Indeed, there is a decomposition of $\mathfrak{m}$ into the sum of three $Ad(H)$-invariant mutually inequivalent irreducible submodules $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$ (see [2]), where

$$\mathfrak{m}_1 = \left\{ \begin{pmatrix} 0 & a & 0 & 0 \\ -a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : a \in \mathbb{R} \right\},$$

$$\mathfrak{m}_2 = \left\{ \begin{pmatrix} 0 & 0 & b_1 & b_2 \\ 0 & 0 & 0 & 0 \\ -b_1 & 0 & 0 & 0 \\ -b_2 & 0 & 0 & 0 \end{pmatrix} : b_1, b_2 \in \mathbb{R} \right\},$$

$$\mathfrak{m}_3 = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & c_1 & c_2 \\ 0 & -c_1 & 0 & 0 \\ 0 & -c_2 & 0 & 0 \end{pmatrix} : c_1, c_2 \in \mathbb{R} \right\}.$$

The conditions A_3) and A_4) are easily checked by straightforward calculations.

Let us consider the following f-structures on this manifold:

$$f_1 : \begin{pmatrix} 0 & a & b_1 & b_2 \\ -a & 0 & c_1 & c_2 \\ -b_1 & -c_1 & 0 & 0 \\ -b_2 & -c_2 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 0 & b_2 & -b_1 \\ 0 & 0 & 0 & 0 \\ -b_2 & 0 & 0 & 0 \\ b_1 & 0 & 0 & 0 \end{pmatrix},$$

$$f_2 : \begin{pmatrix} 0 & a & b_1 & b_2 \\ -a & 0 & c_1 & c_2 \\ -b_1 & -c_1 & 0 & 0 \\ -b_2 & -c_2 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & c_2 & -c_1 \\ 0 & -c_2 & 0 & 0 \\ 0 & c_1 & 0 & 0 \end{pmatrix},$$

$$f_3 : \begin{pmatrix} 0 & a & b_1 & b_2 \\ -a & 0 & c_1 & c_2 \\ -b_1 & -c_1 & 0 & 0 \\ -b_2 & -c_2 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 0 & b_2 & -b_1 \\ 0 & 0 & c_2 & -c_1 \\ -b_2 & -c_2 & 0 & 0 \\ b_1 & c_1 & 0 & 0 \end{pmatrix},$$

$$f_4 : \begin{pmatrix} 0 & a & b_1 & b_2 \\ -a & 0 & c_1 & c_2 \\ -b_1 & -c_1 & 0 & 0 \\ -b_2 & -c_2 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 0 & b_2 & -b_1 \\ 0 & 0 & -c_2 & c_1 \\ -b_2 & c_2 & 0 & 0 \\ b_1 & -c_1 & 0 & 0 \end{pmatrix}.$$

There is no difficulty in checking that these f-structures are invariant and compatible with any invariant Riemannian metric (13), where $g_0 = -B(X,Y) = -2\operatorname{Tr}(X \cdot Y)$.

By Theorem 2, we obtain that both (f_1, g) and (f_2, g), where g is an arbitrary invariant Riemannian metric, belong to the classes **NKf** and **G_1f**, but they are not Killing f-structures.

By Theorem 3, we immediately see that (f_3, g) and (f_4, g) are $G_1 f$-structures for any invariant Riemannian metric.

As f_3 does not satisfy (16), (f_3, g) in not an NKf-structure, and, consequently, not a Killing f-structure with respect to any invariant Riemannian metric.

The verification of the respective conditions of Theorem 3 yields that (f_4, g) is an NKf-structure if and only if the characteristic numbers of g are (s, t, t), where $s, t > 0$. (f_4, g) belongs to **Kill f** if and only if the characteristic numbers of g are $(4s, 3s, 3s)$, where $s > 0$.

3.4. THE QUATERNIONIC FLAG MANIFOLD To conclude this paper, we consider the example of the quaternionic flag manifold $G/H = Sp(3)/SU(2) \times SU(2) \times SU(2)$, which also satisfies Assumption 1 [20]. In this case

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & x & y \\ -\overline{x} & 0 & z \\ -\overline{y} & -\overline{z} & 0 \end{pmatrix} : x, y, z \in \mathbb{H} \right\}, \qquad \mathfrak{m}_1 = \left\{ \begin{pmatrix} 0 & x & 0 \\ -\overline{x} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : x \in \mathbb{H} \right\},$$

$$\mathfrak{m}_2 = \left\{ \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & 0 \\ -\overline{y} & 0 & 0 \end{pmatrix} : y \in \mathbb{H} \right\}, \qquad \mathfrak{m}_3 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & z \\ 0 & -\overline{z} & 0 \end{pmatrix} : z \in \mathbb{H} \right\}.$$

Let $\{p, q, r\} = \{1, 2, 3\}$. Suppose that $a_1, a_2, a_3 \in \mathbb{R}$ are such that $a_1^2 + a_2^2 + a_3^2 = 1$. It is straightforward to check that

$$f(X) = \begin{cases} (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})X, & X \in \mathfrak{m}_p, \\ 0, & X \in \mathfrak{m}_q \oplus \mathfrak{m}_r \end{cases}$$

is both invariant and compatible with any invariant Riemannian metric (13), where $g_0 = -\operatorname{Re}(B(X, Y)) = -8\operatorname{Re}\operatorname{Tr}(X \cdot Y)$. By Theorem 2, f is both NKf- and G_1f-structure. At the same time, it is not a Killing f-structure with respect to any invariant Riemannian metric.

Also invariant and compatible with any invariant Riemannian metric (13) are f-structures of the form

$$f_1 : \begin{pmatrix} 0 & x & y \\ -\overline{x} & 0 & z \\ -\overline{y} & -\overline{z} & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & h_1 x & h_2 y \\ -\overline{h_1 x} & 0 & 0 \\ -\overline{h_2 y} & 0 & 0 \end{pmatrix}, \qquad (24)$$

where $h_1, h_2 \in \mathbb{H}$ are such that $\operatorname{Re} h_1 = \operatorname{Re} h_2 = 0$, $|h_1| = |h_2| = 1$.

In this case we have

$$[f_1 X, f_1^2 X]_{\mathfrak{m}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \overline{h_1 x} y - \overline{x} h_2 y \\ 0 & \overline{h_2 y} x - \overline{y} h_1 x & 0 \end{pmatrix},$$

where

$$X = \begin{pmatrix} 0 & x & y \\ -\overline{x} & 0 & z \\ -\overline{y} & -\overline{z} & 0 \end{pmatrix} \in \mathfrak{m}.$$

For this reason, $[f_1 X, f_1^2 X]_{\mathfrak{m}} = 0$ for any $X \in \mathfrak{m}$ if and only if $h_1 = \overline{h_2} = -h_2$. At the same time, there exist such $Y \in \mathfrak{m}_3$, $Z \in \mathfrak{m}_1 \oplus \mathfrak{m}_2$ that, regardless of the choice of h_1 and h_2, $[Y, fZ] + f([Y, Z]) \neq 0$.

Thus, an invariant metric f-structure (f_1, g), where f_1 is of the form (24), g is an arbitrary Riemannian metric, belongs to the class $\mathbf{G_1 f}$ and does not belong to the class $\mathbf{Kill\, f}$. In this case (f_1, g) is an NKf-structure if and only if $h_1 = -h_2$ and the characteristic numbers of g are (λ, λ, μ), where $\lambda, \mu > 0$.

Arguing as above, we obtain that for any invariant Riemannian metric g (f_2, g) and (f_3, g) are $G_1 f$ structures and are not NKf-structures (and, consequently, not Killing f-structures). Here

$$f_2 : \begin{pmatrix} 0 & x & y \\ -\overline{x} & 0 & z \\ -\overline{y} & -\overline{z} & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & h_1 x & 0 \\ -\overline{h_1 x} & 0 & h_2 z \\ 0 & -\overline{h_2 z} & 0 \end{pmatrix},$$

$$f_3 : \begin{pmatrix} 0 & x & y \\ -\overline{x} & 0 & z \\ -\overline{y} & -\overline{z} & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 0 & h_1 y \\ 0 & 0 & h_2 z \\ -\overline{h_1 y} & -\overline{h_2 z} & 0 \end{pmatrix},$$

where h_1, $h_2 \in \mathbb{H}$ are such that $\operatorname{Re} h_1 = \operatorname{Re} h_2 = 0$, $|h_1| = |h_2| = 1$.

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