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On the Existence of Constructions on Connections by Gauge Bundle Functors

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Abstract: We characterize gauge bundle functors $F : \mathcal{PB}_m(G) \to \mathcal{FM}$ which admit a construction of a classical linear connection $A(\Gamma, \nabla)$ on FP from a principal general connection Γ on $P \to M$ by means of a classical linear connection ∇ on M.

Key words: Principal general connection, classical linear connection, gauge bundle functor, natural (gauge) operator.

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0. Introduction

By [5], a general connection on a fibred manifold $p : Y \to M$ is a section $\Gamma : Y \to J^1Y$ of the first jet prolongation $J^1Y \to Y$ of $p : Y \to M$. If $P \to M$ is a principal G-bundle, where G is a Lie group, then a general connection $\Gamma : P \to J^1P$ is called principal if it is right G-invariant. Principal connections can be defined equivalently by many ways, e.g. by $Ad_{\xi^{-1}}$- right-invariant connection forms $\omega : TP \to \mathcal{L}ie(G)$, by right invariant horizontal distributions $H^\Gamma \subset TP$ complementing VP, by horizontal lifting maps $TM \times_M P \to TP$, e.t.c. If $E \to M$ is a vector bundle then a general connection $\Gamma : E \to J^1E$ is called linear if it is a vector bundle map. It is well-known that if $L(E) \to M$ is the frame $GL(n)$-bundle corresponding to $E \to M$ ($n =$ the dimension of the fibres of E), then linear connections on $E \to M$ correspond bijectively to principal connections on $L(E) \to M$. In particular if $E = TM$ is the tangent bundle of M, a linear connection $\Gamma : TM \to J^1TM$ is a classical linear connection on M (it can be equivalently defined by its covariant derivative ∇_XY on vector fields, or equivalently defined as the corresponding section of the affine bundle of connections $QM = \pi^{-1}(id_{TM}) \subset T^*M \otimes J^1TM$).

The theory of canonical constructions on connections has its origin in the works of C. Ehresmann, [3]. Some canonical constructions on connections have motivations in quantum mechanics, higher order dynamics, field theories and gauge theories of mathematical physics, [4]. That is why, canonical constructions on connections have been studied in many papers, see e.g. [5]. Roughly speaking, a canonical construction on connections is a rule A transforming given connections $\Gamma_1, \dots, \Gamma_k$ on Y (manifold, fibred manifold, vector bundle, principal bundle) into a connection $A(\Gamma_1, \dots, \Gamma_k)$ on a functor bundle FY of Y, which is well defined (i.e., the definition of $A(\Gamma_1, \dots, \Gamma_k)$ is independent of the choice of local coordinates on Y). Such constructions have reflection in the corresponding natural operators in the sense of Kolář-Michor-Slovák [5]. The theory and precise definitions of bundle functors and natural operators (canonical constructions) can be found in the fundamental monograph [5].

In the third part of [7] the second author solved the following problems.

Problem a. To characterize all gauge bundle functors F on vector bundles $E \to M$, which admit a canonical construction of a classical linear connection $A(\Gamma, \nabla)$ on FE from a linear general connection Γ on $E \to M$ by means of a classical linear connection ∇ on M.

Problem b. To give an example of a gauge bundle functor F on vector bundles $E \to M$ which does not admit any canonical construction of a classical linear connection $A(\Gamma, \nabla)$ on FE from a linear general connection Γ on $E \to M$ by means of a classical linear connection ∇ on M.

In the present note we study the following problems.

Problem A. To characterize all gauge bundle functors F on principal G-bundles $P \to M$, which admit a canonical construction of a classical linear connection $A(\Gamma, \nabla)$ on FP from a principal connection Γ on $P \to M$ by means of a classical linear connection ∇ on M.

Problem B. To give an example of a gauge bundle functor F on principal bundles $P \to M$ which does not admit any canonical construction of a classical linear connection $A(\Gamma, \nabla)$ on FP from a principal connection Γ on $P \to M$ by means a classical linear connection ∇ on M.

The problems A and B will be precise formulated in the next sections of the present note.

Clearly, by the bijection of principal connections on $L(E) \to M$ and linear connections on $E \to M$, Problems A and B for $G = GL(n)$ are exactly Problems a and b. Thus (roughly speaking) in the present note we extend the results of the third part of [7] for arbitrary Lie group G instead of the linear Lie group $GL(n)$.

We inform that in [6], the second author proved that there is no canonical construction of a classical linear connection $A(\Gamma)$ on FP from a principal connection Γ on $P \to M$. So, the using of an auxiliary classical linear connection ∇ on M is unavoidable in Problem A.

All manifolds and maps are assumed to be of class $\mathbf{C}^\infty$.

1. Some definitions

We fix an arbitrary Lie group G. Let $\mathcal{PB}_m(G)$ be the category of all principal G-bundles with m-dimensional bases and their local principal bundle isomorphisms. Let $B' : \mathcal{PB}_m(G) \to \mathcal{M}f$ and $B : \mathcal{FM} \to \mathcal{M}f$ be the base functors, where $\mathcal{M}f$ is the category of all manifolds and all maps and $\mathcal{FM}$ is the category of all fibred manifolds and all fibred maps.

Definition 1. A gauge bundle functor on $\mathcal{PB}_m(G)$ is a covariant functor $F : \mathcal{PB}_m(G) \to \mathcal{FM}$ satisfying $B \circ F = B'$ and the localization property: for every $\mathcal{PB}_m(G)$-object $p : P \to M$ and every inclusion of an open sub-bundle $i_U : P|U \to P$, $F(P|U)$ is the restriction $p_P^{-1}(U)$ of $p_P : FP \to M$ over U and Fi_U is the inclusion $p_P^{-1}(U) \to FP$.

The most important example of a gauge bundle functor on $\mathcal{PB}_m(G)$ is the r-th order principal prolongation functor $W^r_m : \mathcal{PB}_m(G) \to \mathcal{PB}_m(W^r_m G)$ sending any $\mathcal{PB}_m(G)$-object $P \to M$ into its r-th order principal prolongation $W^r_m P = \{j^r_0\varphi \mid \varphi : \mathbf{R}^m \times G \to P \text{ is a } \mathcal{PB}_m(G)\text{−map}\}$ over M and any $\mathcal{PB}_m(G)$-map $\psi : P_1 \to P_2$ into the induced map $W^r_m\psi : W^r_m P_1 \to W^r_m P_2$ defined via composition of jets. It is clear that $W^r_m P \to M$ is a principal $W^r_m G$-bundle, where $W^r_m G$=the fiber of $W^r_m(\mathbf{R}^m \times G)$ over $0 \in \mathbf{R}^m$ is the so called r-th order principal prolongation of G. There is a canonical identification $W^r_m P = P^r(M) \times_M J^r P$ and $W^r_m G = G^r_m \times T^r_m G$ (semi-direct product), where $G^r_m = invJ^r_0(\mathbf{R}^m, \mathbf{R}^m)_0$, $T^r_m G = J^r_0(\mathbf{R}^m, G)$, see [5]. One can show that for any gauge bundle functor $F : \mathcal{PB}_m(G) \to \mathcal{FM}$ of order r it is $FP \tilde{=} W^r_m P \times_{W^r_m G} F_0$, where F_0 is the fiber of $F(\mathbf{R}^m \times G)$ over $0 \in \mathbf{R}^m$ with the induced left action of $W^r_m G$, see [5].

Let $F : \mathcal{PB}_m(G) \to \mathcal{FM}$ be a gauge bundle functor.

Definition 2. A $\mathcal{PB}_m(G)$-natural gauge operator transforming principal connections Γ on $\mathcal{PB}_m(G)$-objects $P \to M$ and classical linear connections ∇ on M into classical linear connections $A(\Gamma, \nabla)$ on FP is a family of $\mathcal{PB}_m(G)$-

invariant regular operators

$$A : Con_{princ}(P \to M) \times Con_{clas-lin}(M) \to Con_{clas-lin}(FP)$$

for any $\mathcal{PB}_m(G)$-object $p : P \to M$, where $Con_{princ}(P \to M)$ is the set of principal general connections on $P \to M$ and $Con_{clas-lin}(M)$ is the set of all classical linear connections on M. The invariance means that for any principal general connections Γ and Γ_1 on $\mathcal{PB}_m(G)$-objects $p : P \to M$ and $p_1 : P_1 \to M_1$ (respectively) and classical linear connections ∇ and ∇_1 on M and M_1 (respectively), if Γ and Γ_1 are f-related and ∇ and ∇_1 are $\underline{f}$-related for some $\mathcal{PB}_m(G)$-map $f : P \to P_1$ covering $\underline{f} : M \to M_1$, then $A(\Gamma, \nabla)$ and $A(\Gamma_1, \nabla_1)$ are Ff-related. The regularity means that A transforms smoothly parametrized families of pairs of connections into smoothly parametrized families of connections.

We have an interesting and very important example of a $\mathcal{PB}_m(G)$-gauge natural operator in the sense of Definition 2 for $F = id_{\mathcal{PB}_m(G)}$.

Example 1. ([5]) Let Γ be a principal connection on a $\mathcal{PB}_m(G)$-object $p : P \to M$ and $\nabla : TM \to J^1TM$ be a classical linear connection on M. Let vA be the vertical component of a vector $A \in T_yP$ and bA be its projection to the base manifold M. Consider a vector field X on M such that $j^1_xX = \nabla(bA)$, $x = p(y)$. Construct the lift X^Γ of X and the fundamental vector field $\varphi(vA)$ determined by vA. An easy calculation shows that the rule

$$A \to j^1_y(X^\Gamma + \varphi(vA))$$

determines a classical linear connection $N_P(\Gamma, \nabla) : TP \to J^1(TP \to P)$ on P. One can easily see that this connection $N_P(\Gamma, \nabla)$ is p-related with ∇ and G-invariant.

2. Adapted trivialization

In this section, for a reader convenience, we cite from [2] some special trivialization on a principal G-bundle $P \to M$ which we need in the sequel.

Lemma 1. ([2]) *Let Γ be a principal connection on a principal G-bundle $\pi : P \to M$ and ∇ be a classical linear connection on M. If $p \in P_x$, $x \in M$, then on some neighborhood of x we can define a local section $\tilde{p} : M \to P$ such that for all $\xi \in G$*

$$\tilde{p}.\xi = \widetilde{p.\xi}\ . \tag{1}$$

Proof. ([2]) Let $N_P(\Gamma,\nabla)$ be the classical linear connection on P from Example 1. Denote by $exp_p^{N_P(\Gamma,\nabla)} : T_pP \to P$ the locally defined exponent of $N_P(\Gamma,\nabla)$ at p and $exp_x^{\nabla} : T_xM \to M$ the exponent of ∇ at x. Since $N_P(\Gamma,\nabla)$ is G-invariant and π-related with ∇ we have

$$exp_{p.\xi}^{N_P(\Gamma,\nabla)} \circ T_pR_\xi = R_\xi \circ exp_p^{N_P(\Gamma,\nabla)} \tag{2}$$

and

$$\pi \circ exp_p^{N_P(\Gamma,\nabla)} = exp_x^{\nabla} \circ T_p\pi \ . \tag{3}$$

We define

$$\tilde{p}(y) = exp_p^{N_P(\Gamma,\nabla)}(\Gamma(p,(exp_x^{\nabla})^{-1}(y))) \ ,$$

where $\Gamma : P \times_M TM \to TP$ is the lifting map (denoted by the same symbol) of Γ. By (3), $\tilde{p}$ is a section near x. Finally, (1) follows from (2). ∎

Definition 3. ([2]) The local section $\tilde{p}$ defined above is called the (Γ,∇)-horizontal extension of the point p.

Now let $P \to M$ be a $\mathcal{PB}_m(G)$-object. Let ∇ be a classical linear connection M and Γ be a principal connection on $P \to M$. Given a point $p \in P_x$ and a frame $l \in P_x^1M$, $x \in M$, we can define a local $\mathcal{PB}_m(G)$-map $\Phi^{p,l} : P \to \mathbf{R}^m \times G$ as follows. Choose a unique (more precisely a unique germ at x) ∇-normal coordinate system φ on M with center x sending the given frame l into the frame $l_o = (\frac{\partial}{\partial x^i}) \in P_0^1\mathbf{R}^m$. We define $\Phi^{p,l}$ to be the unique $\mathcal{PB}_m(G)$-map covering φ such that $\Phi^{p,l} \circ \tilde{p} \circ \varphi^{-1}$ is the constant section $x \to (x,e)$ of $\mathbf{R}^m \times G \to \mathbf{R}^m$, where $e \in G$ is the neutral element and $\tilde{p}$ is the (Γ,∇)-horizontal extension of the point p.

Definition 4. ([2]) The map $\Phi^{p,l} : P \to \mathbf{R}^m \times G$ is called the (∇,Γ)-adapted trivialization corresponding to $p \in P_x$ and $l \in P_x^1M$.

Clearly, given $A \in GL(m)$ and $\xi \in G$ we have

$$\Phi^{p.\xi,l.A} = (A^{-1} \times L_{\xi^{-1}}) \circ \Phi^{p,l} \ . \tag{4}$$

3. Solution of Problems A and B

Let $F : \mathcal{PB}_m(G) \to \mathcal{FM}$ be a gauge bundle functor. On the standard fiber $F_0(\mathbf{R}^m \times G)$, $0 \in \mathbf{R}^m$, we have the left action of $GL(m) \times G$ by $(B,\xi).f = F(B \times L_\xi)(f)$, $f \in F_0(\mathbf{R}^m \times G)$. The following theorem is a solution of Problem A.

THEOREM 1. *Let $F : \mathcal{PB}_m(G) \to \mathcal{FM}$ be a gauge bundle functor. The following conditions are equivalent:*

(a) *There exists a canonical construction (a $\mathcal{PB}_m(G)$-natural gauge operator) of a classical linear connection $A(\Gamma, \nabla)$ from a principal general connection Γ on $P \to M$ by means of a classical linear connection ∇ on M.*

(b) *There exists a $GL(m) \times G$-invariant classical linear connection $\tilde{\nabla}$ on the standard fibre $F_0(\mathbf{R}^m \times G)$ of F.*

Proof. Suppose we have a $GL(m) \times G$-invariant classical linear connection $\tilde{\nabla}$ on $F_0(\mathbf{R}^m \times G)$. Let Γ be a principal general connection on an $\mathcal{PB}_m(G)$-object $p : P \to M$ and let ∇ be a classical linear connection on M. We are going to construct a classical linear connection $A(\Gamma, \nabla)$ on FP. Let $f \in F_x P$, $x \in M$. We choose $p \in P_x$ and $l \in P^1_x M$. Let $\Phi^{p,l}$ over φ^l be be the (∇, Γ)-adapted trivialization corresponding to p and l (see Definition 4). We have classical linear connection $\varphi^l_* \nabla \times \tilde{\nabla}$ on some neighborhood of the fibre over zero of $F(\mathbf{R}^m \times G) \tilde{=} \mathbf{R}^m \times F_0(\mathbf{R}^m \times G)$. We put

$$A(\Gamma, \nabla)_f = (QF\Phi^{p,l})^{-1}((\varphi^l)_* \nabla \times \tilde{\nabla})_{F\Phi^{p,l}(f)}) ,$$

where Q is the bundle functor of classical linear connections. Because of (4) and the $GL(m) \times G$-invariance of $\tilde{\nabla}$, the definition of $A(\Gamma, \nabla)_f$ is correct (it is independent of the choice of (p, l)).

Conversely, suppose we have a canonical construction ($\mathcal{PB}_m(G)$-natural gauge operator) A transforming principal general connections Γ on $P \to M$ and classical linear connections ∇ on M into classical linear connections $A(\Gamma, \nabla)$ on FP. Let ∇^o be the flat classical linear connection on $\mathbf{R}^m$ and Γ^o be the trivial principal general connection on $\mathbf{R}^m \times G \to \mathbf{R}^m$. Then we have the classical linear connection $A(\Gamma^o, \nabla^o)$ on $F(\mathbf{R}^m \times G) = \mathbf{R}^m \times F_0(\mathbf{R}^m \times G)$. Thus (by the Gauss formula) we have the classical linear connection $\tilde{\nabla}$ on $F_0(\mathbf{R}^m \times G)$. Since Γ^o is $GL(m) \times G$-invariant and ∇^o is $GL(m)$-invariant and A is invariant, then $\tilde{\nabla}$ is $GL(m) \times G$-invariant. ∎

EXAMPLE 2. In the case of a vector gauge bundle functor $F : \mathcal{PB}_m(G) \to \mathcal{VB}$ (where $\mathcal{VB}$ is the category of all vector bundles and all vector bundle maps) we have the linear action of $GL(m) \times G$ on the vector space $F_0(\mathbf{R}^m \times G)$. Let $\tilde{\nabla} = \nabla^F$ be the usual flat connection on $F_0(\mathbf{R}^m \times G)$. It is $GL(m) \times G$-invariant. Therefore (because of Theorem 1) we have a $\mathcal{PB}_m(G)$-natural gauge operator A^F transforming principal general connections Γ on $\mathcal{PB}_m(G)$-objects $P \to M$ and classical linear connections ∇ on M into classical linear connections $A^F(\Gamma, \nabla)$ on FP.

EXAMPLE 3. Let $F = W^r_m : \mathcal{PB}_m(G) \to \mathcal{FM}$ be the r-th order principal prolongation functor. The fiber $W^r_m G$ over 0 of $W^r_m(\mathbf{R}^m \times G)$ is a Lie group and therefore there exists left $W^r_m G$-invariant classical linear connection $\tilde{\nabla}$ on $W^r_m G$. Since $Gl(m) \times G$ is a subgroup of $W^r_m G$, then this connection $\tilde{\nabla}$ is also $GL(m) \times G$ invariant. Consequently, by Theorem 1 we have a $\mathcal{PB}_m(G)$-natural gauge operator A transforming principal general connections Γ on $P \to M$ and classical linear connections ∇ on M into classical linear connections $A(\Gamma, \nabla)$ on $W^r_m P$.

Remark 1. In [1], M. Doupovec and the second author classified all $\mathcal{PB}_m(G)$-natural gauge operators A transforming principal connections Γ on $P \to M$ and r-th order linear connections $\Lambda : TM \to J^r TM$ on M into classical linear connections $A(\Gamma, \Lambda)$ on $W^r_m P$.

EXAMPLE 4. (*A solution of Problem* B) Let $\tilde{\mathbf{P}}(T) : \mathcal{PB}_m(G) \to \mathcal{FM}$ be the gauge bundle functor

$$\tilde{\mathbf{P}}(T)(P) = \bigcup_{x \in M} \mathbf{P}(T_x M) \ , \quad \tilde{\mathbf{P}}(T)(f) = \bigcup_{x \in M} \mathbf{P}(T_x \underline{f}) \ ,$$

where $\mathbf{P}(V)$ is the projective space determined by a vector space V. By Lemma 5 in [7] for $n = 0$ we have that there is no $GL(m)$-invariant classical linear connection on $\mathbf{P}(\mathbf{R}^m)$ for $m \geq 2$. That is why, there is no $GL(m) \times G$-invariant classical linear connection on $\tilde{\mathbf{P}}(T)_0(\mathbf{R}^m \times G) \tilde{=} \mathbf{P}(\mathbf{R}^m)$. By Theorem 1, there is no canonical construction of a classical linear connection $A(\Gamma, \nabla)$ on $\tilde{\mathbf{P}}(T)(P)$ from a principal general connection Γ on $P \to M$ by means of a classical linear connection ∇ on M.

References

[1] M. Doupovec, W.M. Mikulski, Gauge natural constructions on higher order principal prolongations, *Ann. Polon. Math.* **92** (1) (2007), 87–97.

[2] M. Doupovec, W.M. Mikulski, Reduction theorem for classical and principal connections, *Acta Math. Sinica (English-Series)* **25** (2009), to appear.

[3] C. Ehresmann, Sur les connections d'ordre supérieur, in "Atti del C. Cang. del'Unione Mat. Italiana 1955, Roma Cremonese (1956)", 344–346.

[4] L. Fatibene, M. Francaviglia, "Natural and Gauge Formalisms for Classical Field Theories", Kluwer Academic Publishers, Dordrecht, 2003.

[5] I. Kolář, P.W. Michor, J. Slovák, "Natural Operations in Differential Geometry", Springer-Verlag, Berlin, 1993.

[6] W.M. Mikulski, Negative answers to some questions about constructions on connections, *Demonstratio Math.* **39** (3)(2006), 685 – 689.

[7] W.M. Mikulski, On the existence of prolongation of connections by bundle functors, *Extracta Math.* **22** (3)(2007), 297 – 314.