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Range, Kernel Orthogonality and Operator Equations

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Abstract: Let $\mathcal{A}$ be a Banach algebra and $\mathcal{L}(\mathcal{A})$ the algebra of all bounded linear operators acting on $\mathcal{A}$. For $a, b \in \mathcal{A}$, the generalized derivation $\delta_{a,b} \in \mathcal{L}(\mathcal{A})$ and the elementary operator $\Delta_{a,b} \in \mathcal{L}(\mathcal{A})$ are defined by $\delta_{a,b}(x) = ax - xb$ and $\Delta_{a,b}(x) = axb - x$, $x \in \mathcal{A}$. Let $d_{a,b} = \delta_{a,b}$ or $\Delta_{a,b}$. In this note we give couples $(a, b) \in \mathcal{A}^2$ such that the range and the kernel of $d_{a,b}$ are orthogonal in the sense of Birkhoff. As application of this results we give consequences for certain operator equations inspired by earlier studies of the equation $\alpha + \alpha^{-1} = \beta + \beta^{-1}$ for automorphism α, β on Von Neuman algebras.

Key words: Elementary operators, orthogonality, operator equation.

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"to my wife Hasna"

1. Introduction

Let $\mathcal{A}$ be a Banach algebra and $\mathcal{L}(\mathcal{A})$ the algebra of all bounded linear operators acting on $\mathcal{A}$. For $a, b \in \mathcal{A}$, the generalized derivation $\delta_{a,b} \in \mathcal{L}(\mathcal{A})$ and the elementary operator $\Delta_{a,b} \in \mathcal{L}(\mathcal{A})$ are defined by $\delta_{a,b}(x) = ax - xb$ and $\Delta_{a,b}(x) = axb - x$, $x \in \mathcal{A}$. If L_a is the left multiplication by a and R_b is the right multiplication by b defined by $L_a(x) = ax$ and $R_b(x) = xb$, $x \in \mathcal{A}$, then $\delta_{a,b} = L_a - R_b$ and $\Delta_{a,b} = L_a R_b - I$.

Let $d_{a,b} = \delta_{a,b}$ or $\Delta_{a,b}$. Then the following implications hold for a general bounded linear operator T on a normed linear space V, in particular for $T = d_{a,b}$:

$$\begin{aligned} \operatorname{Ker} T \perp \operatorname{Im} T &\Rightarrow \operatorname{Ker} T \cap \overline{\operatorname{Im} T} = \{0\} \\ &\Rightarrow \operatorname{Ker} T \cap \operatorname{Im} T = \{0\}. \end{aligned}$$

Here $\overline{\operatorname{Im} T}$ denote the closure of the range of T and $\operatorname{Ker} T \perp \operatorname{Im} T$ denotes that the kernel of T is orthogonal to the range of T in the sense of Birkhoff.

Recall that if M, N are linear subspaces of a normed linear space V, then $M \perp N$ in the sense of Birkhoff [5] if

$$\|a+b\| \geq \|a\| \qquad \text{for all } a \in M \text{ and } b \in N\,.$$

Orthogonality of matrices (more generally bounded linear operators on an infinite dimensional Hilbert space), in particular orthogonality of the range and the kernel of elementary operators has been studied in recent years. In the first part of this work we study orthogonality of the range and the kernel of $d_{a,b}$, and we give couples $(a,b) \in \mathcal{A}^2$ such that

$$\overline{\operatorname{Im} d_{a,b}} \cap \operatorname{Ker} d_{a,b} = \{0\}\,.$$

Consider the equation

$$\alpha + \alpha^{-1} = \beta + \beta^{-1} \tag{*}$$

for automorphisms α, β on Von Neumann algebras. As application of results given in the first section for elementary operators we study the equation (*) in the setting of Banach algebra.

2. Elementary operators

We begin this section by the following lemma.

Lemma 2.1. *Let $a, b \in \mathcal{A}$ with $cb = e$. Then*

$$R_b(\overline{\operatorname{Im} \delta_{a,c}} \cap \operatorname{Ker} \delta_{a,c}) = \overline{\operatorname{Im} \Delta_{a,b}} \cap \operatorname{Ker} \Delta_{a,b}\,.$$

In particular if $\overline{\operatorname{Im} \delta_{a,c}} \cap \operatorname{Ker} \delta_{a,c} = \{0\}$ then $\overline{\operatorname{Im} \Delta_{a,b}} \cap \operatorname{Ker} \Delta_{a,b} = \{0\}$.

Proof. First, observe that if $cb = e$, then $R_b\delta_{a,c} = \Delta_{a,b}$. Indeed, for all $x \in \mathcal{A}$ we have that $R_b\delta_{a,c}(x) = axb - xcb = axb - x = \Delta_{a,b}(x)$. Assume that $t \in R_b(\overline{\operatorname{Im} \delta_{a,c}} \cap \operatorname{Ker} \delta_{a,c})$. Since $R_b\delta_{a,c} = \Delta_{a,c}$ and R_b is continuous for the uniform norm, then $t \in \overline{\operatorname{Im} \Delta_{a,b}} \cap \operatorname{Ker} \Delta_{a,b}$. Conversely, since R_c is continuous for the uniform norm, then by the same argument we prove that if $t \in \overline{\operatorname{Im} \Delta_{a,b}} \cap \operatorname{Ker} \Delta_{a,b}$, then $t \in R_b(\overline{\operatorname{Im} \delta_{a,c}} \cap \operatorname{Ker} \delta_{a,c})$. ∎

Remark 2.2. Let $a, b \in \mathcal{A}$ with $cb = e$. By an easy adaptation of the proof of Lemma 2.1, we get that

$$R_b(\operatorname{Im} \delta_{a,c} \cap \operatorname{Ker} \delta_{a,c}) = \operatorname{Im} \Delta_{a,b} \cap \operatorname{Ker} \Delta_{a,b}\,.$$

2.1. Normal elements. Let $a \in \mathcal{A}$. The algebraic numerical range $V(a)$ of a is defined by [2]:

$$V(a) = \left\{ f(a) \,:\, f \in \mathcal{A}' \text{ and } \|f\| = |f(e)| = 1 \right\}$$

where $\mathcal{A}'$ is the dual space of $\mathcal{A}$ and e is the identity on $\mathcal{A}$. If $V(a) \subseteq \mathbb{R}$, then a is called hermitian. If $n = h + \mathrm{i}k$ where h, k are hermitian elements and $hk = kh$, then n is called normal.

Define

$$D(\mathcal{A}) = \left\{ (a,b) \in \mathcal{A}^2 \,:\, \operatorname{Im} \delta_{a,b} \perp \operatorname{Ker} \delta_{a,b} \right\}.$$

In this section we show that normal elements are in $D(\mathcal{A})$.

Lemma 2.3. *If $h, k \in \mathcal{A}$ are hermitian elements, then so is $\delta_{h,k}$.*

Proof. Because $V(\delta_{h,k}) \subseteq V(L_h) - V(L_k) = V(h) - V(k) \subseteq \mathbb{R}$. ∎

Remark 2.4. If X is a Banach space, it is known [7] that

$$V(\delta_{h,k}) = V(h) - V(k) \qquad \text{for all } h \in \mathcal{L}(X).$$

So the converse of Lemma 2.3, is true if $\mathcal{A} = \mathcal{L}(X)$ where X is a Banach space.

Lemma 2.5. *If $m, n \in \mathcal{A}$ are normal elements, then so is $\delta_{m,n}$.*

Proof. Assume that $m = h + \mathrm{i}k$ and $n = p + \mathrm{i}q$ where h, k, p and q are hermitian elements in $\mathcal{A}$ such that $hk = kh$ and $pq = qp$. Then $\delta_{m,n} = \delta_{h,p} + \mathrm{i}\delta_{k,q}$ with $\delta_{h,p}\delta_{k,q} = \delta_{k,q}\delta_{h,p}$. Since h, p and k, q are hermitian, then by Lemma 2.3, $\delta_{n,p}$ and $\delta_{k,q}$ are hermitian. So $\delta_{m,n}$ is normal. ∎

Theorem 2.6. ([3]) *Let E be a Banach space and $T \in \mathcal{L}(E)$. If T is a normal operator then*

$$\operatorname{Ker} T \perp \operatorname{Im} T.$$

Theorem 2.7. *If $m, n \in \mathcal{A}$ are normal elements, then*

$$\operatorname{Ker} \delta_{m,n} \perp \operatorname{Im} \delta_{m,n}.$$

Proof. Assume that $m, n \in \mathcal{A}$ are normal elements. Then by Lemma 2.5, $\delta_{m,n}$ is normal and by Theorem 2.6, $\operatorname{Ker} \delta_{m,n} \perp \operatorname{Im} \delta_{m,n}$. ∎

Corollary 2.8. *If $a, b \in \mathcal{A}$ are normal and there exist $c \in \mathcal{A}$ such that $bc = e$, then*

$$\operatorname{Ker}\Delta_{a,c} \cap \overline{\operatorname{Im}\Delta_{a,c}} = \{0\}\,.$$

Proof. If a, b are normal elements, then by Theorem 2.7 $\operatorname{Ker}\delta_{a,b} \perp \operatorname{Im}\delta_{a,b}$. This implies that $\operatorname{Ker}\delta_{a,b} \cap \overline{\operatorname{Im}\delta_{a,b}} = \{0\}$. Using Lemma 2.1, we conclude that $\operatorname{Ker}\Delta_{a,c} \cap \overline{\operatorname{Im}\Delta_{a,c}} = \{0\}$. ∎

In the following theorem we give other couples $(a, b) \in \mathcal{A}^2$ which are in $D(\mathcal{A})$. The technique used to prove this theorem was published first in [4].

Theorem 2.9. *Let a and b in $\mathcal{A}$ with $bc = e$ and $\|c\| \leq 1$ for some c in $\mathcal{A}$. If $\|a^n\| \leq 1$ and $\|b^n\| \leq 1$ for all $n \in \mathbb{N}$, then*

$$\operatorname{Ker}\delta_{a,b} \perp \operatorname{Im}\delta_{a,b}\,.$$

Proof. Since we have that

$$a^n x - x b^n = \sum_{i=0}^{n-1} a^{n-i-1}(ax - xb)b^i\,,$$

then

$$a^n x - x b^n - \sum_{i=0}^{n-1} a^{n-i-1}(ax - xb - y)b^i = n y b^{n-1}\,,$$

where $y \in \operatorname{Ker}\delta_{a,b}$. If we multiply this equality right by c^{n-1} we obtain

$$ny = a^n x c^{n-1} - xb - \sum_{i=0}^{n-1} a^{n-i-1}(ax - xb - y)b^i c^{n-1}\,,$$

so

$$\|y\| \leq \frac{1}{n}\Big\{\|a^n\|\|x\|\|c\|^{n-1} + \|x\|\|b\|\Big\} + \frac{1}{n}\sum_{i=0}^{n-1}\|a^{n-i-1}\|\|ax - xb - y\|\|b^i\|\|c\|^{n-1}\,,$$

hence

$$\|y\| \leq \frac{2}{n}\|x\| + \frac{1}{n}\sum_{i=0}^{n-1}\|ax - xb - y\|\,.$$

Passing to the limit $n \to \infty$ we obtain that

$$\|y\| \leq \|ax - xb - y\| .$$

Finally, $\operatorname{Im} \delta_{a,b} \perp \operatorname{Ker} \delta_{a,b}$. ∎

THEOREM 2.10. *Let a and b in $\mathcal{A}$ with $\|a\| \leq 1$ and $\|b\| \leq 1$. Then*

$$\operatorname{Im} \Delta_{a,b} \perp \operatorname{Ker} \Delta_{a,b} .$$

Proof. If $\|a\| \leq 1$ and $\|b\| \leq 1$, then $\|M_{a,b}\| \leq 1$. So $V(M_{a,b}) \subseteq \mathbb{D}$ where $\mathbb{D}$ is the closed unit disk. Since $\Delta_{a,b} = M_{a,b} - 1$, then

$$V(\Delta_{a,b}) \subseteq \{\lambda \in \mathbb{C} \,:\, |\lambda + 1| \leq 1\} .$$

If 0 is an eigenvalue of $\Delta_{a,b}$, then $0 \in V(\Delta_{a,b})$. Hence 0 is in the boundary of $V(\Delta_{a,b})$. It follows from a result of Sinclair [8, Proposition 1] that $\operatorname{Im} \Delta_{a,b} \perp \operatorname{Ker} \Delta_{a,b}$. If 0 is not an eigenvalue of $\Delta_{a,b}$ then $\operatorname{Ker} \Delta_{a,b} = \{0\}$ and the result follows because all linear subspaces are orthogonal to $\{0\}$. ∎

2.2. NORMAL-EQUIVALENT ELEMENTS.

Recall that an element $a \in \mathcal{A}$ is hermitian-equivalent if

$$\sup\{\|\exp(\mathrm{i}at)\| \,:\, t \in \mathbb{R}\} < \infty ,$$

and that n is normal-equivalent if $n = r + \mathrm{i}s$ where $rs = sr$ and r, s are hermitian-equivalent: we write $n^* = r - \mathrm{i}s$ for such an n.

Define

$$B(\mathcal{A}) = \big\{(a,b) \in \mathcal{A}^2 \,:\, \operatorname{Im} \delta_{a,b} \cap \operatorname{Ker} \delta_{a,b} = \{0\}\big\} .$$

Note that every element of $D(\mathcal{A})$ is in $B(\mathcal{A})$. Hence normal elements are in $B(\mathcal{A})$. In this section we show that normal-equivalent elements are in $B(\mathcal{A})$.

DEFINITION 2.11. ([1]) Let $\mathcal{A}$ be an algebra with involution and $a, b \in \mathcal{A}$. (a,b) is said to be a couple of Fuglede if $ax = xb$ implies $a^*x = xb^*$ for every $x \in \mathcal{A}$.

THEOREM 2.12. *Let $\mathcal{A}$ be an algebra with involution and $a, b \in \mathcal{A}$. If (a,b) is a couple of Fuglede, then*

$$\operatorname{Im} \delta_{a,b} \cap \operatorname{Ker} \delta_{a,b} = \{0\} .$$

Proof. Let $x \in \operatorname{Im}\delta_{a,b} \cap \operatorname{Ker}\delta_{a,b} = \{0\}$, then $x = ay - yb$ for $y \in \mathcal{A}$ and $ax = xb$. If (a,b) is a couple of Fuglede, then

$$\begin{aligned} x^*x &= x^*ay - x^*yb \\ &= b(x^*y) - (x^*y)b \end{aligned}$$

and $b(x^*x) = x^*ax = (x^*x)b$. By Kleinecke-Shirokov's Theorem [6] x^*x is quasi-nilpotent. Since x^*x is positive, then $x = 0$. ∎

The following result which is proved in [1] is an extension of Fuglede's Theorem to normal-equivalent elements:

THEOREM 2.13. *Let A be a complex unital Banach algebra. If $a, b \in \mathcal{A}$ are normal-equivalent, then (a,b) is a couple of Fuglede.*

As a consequence of Theorem 2.12 and Theorem 2.13, we get the following corollaries:

COROLLARY 2.14. *Let $a, b \in \mathcal{A}$. If a and b are normal-equivalent, then*

$$\operatorname{Im}\delta_{a,b} \cap \operatorname{Ker}\delta_{a,b} = \{0\}\,.$$

Proof. If a and b are normal-equivalent, then by Theorem 2.13 (a,b) is a couple of Fuglede and by Theorem 2.12 we have

$$\operatorname{Im}\delta_{a,b} \cap \operatorname{Ker}\delta_{a,b} = \{0\}\,.$$

∎

COROLLARY 2.15. *Let $a, b \in \mathcal{A}$ with $cb = e$. If a, c are normal equivalent, then*

$$\operatorname{Im}\Delta_{a,b} \cap \operatorname{Ker}\Delta_{a,b} = \{0\}\,.$$

Proof. If $cb = e$, then by Remark 2.2

$$R_b(\operatorname{Im}\delta_{a,c} \cap \operatorname{Ker}\delta_{a,c}) = \operatorname{Im}\Delta_{a,b} \cap \operatorname{Ker}\Delta_{a,b}\,.$$

If a, c are normal equivalent, then by Corollary 2.14 we have

$$\operatorname{Im}\delta_{a,c} \cap \operatorname{Ker}\delta_{a,c} = \{0\}\,.$$

So

$$\operatorname{Im}\Delta_{a,b} \cap \operatorname{Ker}\Delta_{a,b} = \{0\}\,.$$

∎

3. Application

The following lemma will be used later.

Lemma 3.1. *Let $u, v \in \mathcal{A}$ invertible elements. Then the following statements are equivalent:*

(i) $uxv^{-1} + u^{-1}xv = 2x \;\Rightarrow\; ux = xv$;

(ii) $x \in \operatorname{Ker}(\Delta_{u,v^{-1}})^2 \;\Rightarrow\; x \in \operatorname{Ker}(\Delta_{u,v^{-1}})$.

Proof. The proof of this lemma follows from the fact that $x \in \operatorname{Ker}(\Delta_{u,v^{-1}})^2$ is equivalent to $uxv^{-1} + u^{-1}xv = 2x$ and $x \in \operatorname{Ker}\Delta_{u,v^{-1}}$ is equivalent to $ux = xv$. ∎

If u and v are invertible elements such that

$$\operatorname{Im}\Delta_{u,v^{-1}} \cap \operatorname{Ker}\Delta_{u,v^{-1}} = \{0\}\,,$$

then

$$\operatorname{Ker}(\Delta_{u,v^{-1}})^2 = \operatorname{Ker}\Delta_{u,v^{-1}}\,.$$

From Lemma 3.1, it follow that

$$uxv^{-1} + u^{-1}xv = 2x \;\Rightarrow\; ux = xv \qquad \text{for all } x \in \mathcal{A}\,.$$

In particular, if u and v are normal invertible elements, then by Corollary 2.8, $\operatorname{Im}\Delta_{u,v^{-1}} \cap \operatorname{Ker}\Delta_{u,v^{-1}} = \{0\}$.

The following corollary gives more.

Corollary 3.2. *Let $u, v \in \mathcal{A}$ invertible elements such that $\|u\| \leq 1$ and $\|v^{-1}\| \leq 1$. If $uxv^{-1} + u^{-1}xv = 2x$ for all $x \in \mathcal{A}$, then $ux = xv$.*

Proof. Assume that u and v are invertible such that $\|u\| \leq 1$ and $\|v^{-1}\| \leq 1$, then from Theorem 2.10 we have that $\operatorname{Im}\Delta_{u,v^{-1}} \perp \operatorname{Ker}\Delta_{u,v^{-1}}$. This implies that $\operatorname{Ker}\Delta^2_{u,v^{-1}} = \operatorname{Ker}\Delta_{u,v^{-1}}$. We conclude by Lemma 3.1. ∎

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