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Operator Theory and Complex Geometry†

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Abstract: One approach to the study of multivariate operator theory on Hilbert space involves the study of Hilbert spaces that are modules over natural function algebras or Hilbert modules. Techniques from complex and algebraic geometry have natural application in this setting. Many modules give rise to a canonical hermitian holomorphic bundle and part of the study involves relating the operator and geometric structures.

In these notes, an exposition is presented of work by several authors over the past two or three decades with an emphasis on some more recent work. In particular, concrete examples are drawn from algebras acting on classical Hilbert spaces of holomorphic functions. The characterization of reducing submodules in geometric terms is considered, particularly the relation to the curvature of the Chern connection on the associated bundle. An interpretation of the model theory of Sz.-Nagy and Foias in this context is given including possible generalizations to the several variable context. Recent results characterizing submodules isometrically isomorphic to the original are described. Many proofs are given especially when new insights are possible and references are provided for those readers interested in following up on these ideas.

Key words: Hilbert modules, Šilov modules, kernel Hilbert spaces, invariant subspaces, isometries, holomorphic structure, localization.

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