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Addendum to "Interpolation of Banach Spaces by the γ-Method"

Jesús Suárez, Lutz Weis

Escuela Politécnica, Universidad de Extremadura, Avenida de la Universidad s/n,
10071 *Cáceres, Spain,* *jesus@unex.es*

Mathematisches Institut I, Universität Karlsruhe,
76128 *Karlsruhe, Germany,* *lutz.weis@math.uni-karlsruhe.de*

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Abstract: In this note we correct and simplify an interpolation theorem for radonifying operators in [7].

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1. Introduction

In [7] it was claimed that under B-convexity the radonifying operators $\gamma(H, X)$ form an interpolation scale for the complex method. However the proof given there used the implicit hypothesis of a compact embedding $H_0 \to H_1$. In this note we give a simpler proof which also eliminates the compactness condition and works in some important situation where H_0 is not embedded into H_1. Among examples are scales of Sobolev spaces and domains of selfadjoints operators. However the case of general interpolation pairs of Hilbert spaces remains open.

2. Interpolation of $\gamma(H, X)$

Given an interpolation couple (H_0, H_1) of two Hilbert spaces H_0, H_1 continuously embedded in some locally convex space H one considers the Hilbert space $H = H_0 \cap H_1$ with the norm $\|f\|_H^2 = \|f\|_{H_0}^2 + \|f\|_{H_1}^2$. Since $(x, y)_{H_j} \leq \|x\|_H \|y\|_H$ for $j = 0, 1$ the Lax-Milgram theorem gives us positive, selfadjoint operators B_j on H of norm 1 so that

$$(x, y)_{H_j} = (x, B_j y)_H \,.$$

The interpolation couple (H_0, H_1) of Hilbert spaces is called *commutative* if the two bounded operators B_j constructed above commute. The point of this definition is the following consequence of the spectral theorem for commuting selfadjoint operators which allows us to replace abstract Hilbert spaces by L_2-spaces with densities.

LEMMA 2.1. *The interpolation couple (H_0, H_1) is commutative if and only if there is a σ-finite measure space (Ω, μ), a unitary map $U : H_0 \cap H_1 \to L_2(\Omega, \mu)$ and densities $0 < g_j \le 1$ so that U extends to isometries $U_j : H_j \to L_2(\Omega, g_j \,\mathrm{d}\mu)$.*

Proof. The spectral theorem (see e.g. [6, p. 246, Problem 4]) gives (Ω, μ), U and g_j so that for $x, y \in H_0 \cap H_1$

$$\begin{aligned}(x, y)_{H_j} &= (x, B_j y)_H = \big(Ux, (UB_jU^{-1})Uy\big)\\ &= \int_\Omega \overline{f(\omega)} h(\omega) g_j(\omega) \,\mathrm{d}\mu(\omega) = (f, h)_{L_2(g_j \,\mathrm{d}\mu)}\end{aligned}$$

with $f = U(x)$, $h = U(y)$ from a dense subset of $L_2(\Omega, g_j \,\mathrm{d}\mu)$. Since the operators B_j have norm one and are positive and injective we obtain $0 < g_j \le 1$. ∎

There are some important interpolation couples covered by this situation.

EXAMPLES. (1) If H_0, H_1 with a continuous dense embedding $H_0 \cap H_1 = H_0$ and $B_0 = Id$, so that the pair (H_0, H_1) is commutative.

(2) Let A be a positive injective and selfadjoint operator on a Hilbert space G and $H_j = D(A^{\alpha_j})$ with $\|h\|_j = \|A^{\alpha_j} x\|$ for $\alpha_j \in \mathbb{R}$, $j = 0, 1$. Then the pair (H_0, H_1) is commutative. Indeed it is easy to check that $B_j = A^{2\alpha_j}\big(A^{2\alpha_0} + A^{2\alpha_1}\big)^{-1}$.

(3) As a particular case of (2) we obtain for $G = L_2(\mathbb{R}^n)$ and $A = \Delta$ the pairs of Sobolev spaces $H_2^{\alpha_0}(\mathbb{R}^n)$, $H_2^{\alpha_1}(\mathbb{R}^n)$ for $\alpha_j \in \mathbb{R}$.

(4) An interpolation pair which is not commutative can be obtained as follows:

Let $H_0 = L_2\big(\mathbb{R}, \langle t\rangle^{-2} dt\rangle\big)$ with $\langle t\rangle = \big(1 + t^2\big)^{1/2}$ and $H_1 = H^2(\mathbb{R})$. Then $L_2(\mathbb{R}) \subset H_0 \cap H_1$. However the multiplication operator $B_0 f(t) = \langle t\rangle^{-2} f(t)$ and $B_2 = (I - A)^{-1}$ do not commute.

Theorem 2.1. *Let H_0, H_1 be separable Hilbert spaces forming a commutative interpolation couple. If X_0, X_1 are B-convex, then*

$$\big(\gamma(H_0, X_0), \gamma(H_1, X_1)\big)_{[\theta]} = \gamma\left((H_0, H_1)_{[\theta]}, (X_0, X_1)_{[\theta]}\right)$$

with equivalent norms.

Proof. Since it is well known already that for $H = (H_0, H_1)_{[\theta]}$ we have, see [4],

$$\big(\gamma(H_0, X_0), \gamma(H_1, X_1)\big)_{[\theta]} = \gamma\big(H, (X_0, X_1)_{[\theta]}\big)$$

it only remains to show that

$$\big(\gamma(H_0, X_0), \gamma(H_1, X_1)\big)_{[\theta]} = \big(\gamma(H, X_0), \gamma(H, X_1)\big)_{[\theta]}\,.$$

By assumption and the lemma we may assume that $H_0 \cap H_1 = L_2(\Omega, \mu)$ and $H_j = L_2(\Omega, g_j d\mu)$, $j = 0, 1$, for a σ-finite measure space (Ω, μ) and densities $0 < g_j \leq 1$ on Ω. For convenience let us use the abbreviation $\gamma_j(h)$ for $L_2(\Omega, g_j\, \mathrm{d}\mu)$. Note the following continuous inclusion

$$\gamma\big(L_2(\Omega, \mathrm{d}\mu), X_0 \cap X_1\big) \;\subset\; \gamma_j(g_\theta) \;\subset\; \gamma\big(L_2(\Omega, g_0 \wedge g_1\, \mathrm{d}\mu), X_0 + X_1\big)$$

where $g_\theta = g_0^{1-\theta} g_1^{\theta}$ for $\theta \in [0, 1]$. Hence $(\gamma_0(g_0), \gamma_1(g_1))$ and $(\gamma_0(g_\theta), \gamma_1(g_\theta))$ are interpolation couples and, going back to the definition of the complex interpolation method, we can consider the following [1] Banach space $\mathcal{F}$ of all continuous

$$f : S = \big\{z \,:\, 0 \leq \operatorname{Re} z \leq 1\big\} \longrightarrow \gamma_0(g_0) + \gamma_1(g_1)$$

which are all continuous on S and analytic in its interior such that $f(j + \mathrm{i}t) \in \gamma_j(g_j)$, $j = 0, 1$, with $\|f(j + \mathrm{i}t)\| \to 0$ for $|t| \to \infty$ and

$$\|f\|_{\mathcal{F}} = \sup\Big\{\|f(j + \mathrm{i}t)\|_{\gamma_j(g_j)} \,:\, t \in \mathbb{R}\Big\}\,.$$

Then $(\gamma_0(g_0), \gamma_1(g_1))_{[\theta]} = \{f(\theta) : f \in \mathcal{F}\}$ endowed with the quotient norm

$$\|T\| = \inf\big\{\|f\| \,:\, f(\theta) = T\big\}\,.$$

In a similar way one defines $\mathcal{F}_\theta$ for the pair $(\gamma_0(g_\theta), \gamma_1(g_\theta))$. For the moment we asume in addition that

$$\delta < g_j \quad \text{on} \quad \Omega \quad \text{for} \quad j = 0, 1 \quad \text{and some fixed} \quad \delta > 0\,. \qquad (*)$$

To prove the inclusion

$$(\gamma_0(g_0), \gamma_1(g_1))_{[\theta]} \subset (\gamma_0(g_\theta), \gamma_1(g_\theta))_{[\theta]} \,,$$

we fix $\theta \in [0,1]$ and choose $f \in \mathcal{F}$ with $\|f\|_{\mathcal{F}} \leq 2\|f(\theta)\|_{(\gamma_0(g_0),\gamma_1(g_1))_{[\theta]}}$. Next we define a family of operators

$$T(z) : L_2(\Omega, g_\theta \, \mathrm{d}\mu) \longrightarrow L_2(\Omega, (g_0 \wedge g_1) \, \mathrm{d}\mu)$$

with $z \in S$ as the multiplication operators

$$T(z)f := \left(\frac{g_\theta}{g_0^{1-z} g_1^z} \right)^{1/2} f.$$

By our assumption we have bounded and invertible operators $T(z)$ for all z in the interior of S. However for $z = j + \mathrm{i}t$ these operators extend to isometries by [3]

$$T(j + \mathrm{i}t) : \gamma_j(g_\theta) \longrightarrow \gamma_j(g_j)$$

and $T(\theta) = Id$. It follows that $f \circ T$ belongs to $\mathcal{F}_\theta$ and $\|f \circ T\| \leq \|f\|$. Therefore

$$\|f \circ T(\theta)\| \leq \|f \circ T\| \leq 2\|f(\theta)\|$$

and the inclusion $\subset$ is continuous . To show the reverse we argue similarly now replacing $T(z)$ by $S(z) = T(z)^{-1}$.

It remains to remove the assumption (*). Note that the assumption $\delta < g_j$ was only used to bound the operators $T(z)$ and $T(z)^{-1}$ for z in the interior of S. For $z \in \partial S$ and $z = \theta$ the $T(z)$ were isometries independent of the condition $\delta < g_j$, which therefore does not enter the calculation of the norms $\|f\|$ or $\|f \circ T\|$. Now, for any $\delta > 0$, if we consider functions f which are zero on $\Omega_\delta = \{g_0 \leq \delta\} \cup \{g_1 \leq \delta\}$ the above norm estimate apply with constants independent of δ. However such functions f are dense in $\mathcal{F}$ or $\mathcal{F}_\theta$. ∎

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